Extremal and Probabilistic Graph Theory
Lecture 5
March 23rd, Thursday

Structure of graphs with high chromatic number

- **Theorem1(Erdös)**: $\forall k, l \geq 0$, $\exists$ a graph G with $\chi(G) \geq k$ and girth$(G) \geq l$.

- **Definition1:** We say a graph G is *k-critical* if $\chi(G) = k$ but for any proper subgraph G' of G, $\chi(G') \leq \chi(G)$.

- **Fact1:** For any k-critical graph G, $\delta(G) \geq k-1$.

  Proof: Suppose $\exists v \in V(G)$, with $d_G(v) \leq k-2$, consider $G-v$. Since G is k-critical, $G-v$ has a $k-1$ colouring, and there is a colouring at most $k-2$ colours in $d_G(v)$, assuming these colours are $1, 2, \cdots, j (j \in [k-2])$. We may colour v the $(k-1)$th colour, then there is a $k-1$ colouring in G, a contradiction. ∎

- **Fact2:** Any k-critical graph is 2-connected.

  Proof: By Menger Theorem, if G is not 2-connected, there is a cut-vertex say v, s.t. $G = G_1 \bigcup G_2$, $V(G_1) \bigcap V(G_2) = v$, and there is no edge between $G_1 - v$ and $G_2 - v$. Each $G_i \subset G$ has a $k-1$ colouring, φ_i: $V(G_i) \to [k-1]$. By exchanging the colours, we may assume φ_1, φ_2 assign the same colour to v. Then $\varphi_1 \bigcup \varphi_2$ gives a $k-1$ colouring of G, a contradiction. ∎

- **Definition2:** A *k-cut* in a graph is a subset of $V(G)$ of size k, s.t. $G - S$ is disconnected.

- **Definition3:** Let $\{u, v\}$ be a 2-cut of a k-critical graph G, we say G' is *$\{u,v\}$-component*, if there is a component D in $G - \{u, v\}$ such that $G' = G[D \bigcup \{u, v\}]$.

  We say an $\{u, v\}$-component G' is of *type1*, if any $k-1$-colouring of G' will assign the same colour to u and v, and is of *type2*, if any $k-1$-colouring of G' will assign different colours to u and v.

- **Theorem2:** Let G be k-critical $k \geq 3$ and $\{u, v\}$ be a 2-cut of G. Then,
  (i)$G = G_1 \bigcup G_2$, where G_i is an u, v-component of G of type i, for $i \in 1, 2$;
  (ii)$H_1 \triangleq G_1 + uv$ and $H_2 \triangleq G_2/\{u, v\}$ both are k-critical.

  Proof: Exercise.

- **Definition4:**(*The Hajos's Construction*) Let G and H be two graphs. Let uu' be an edge in G and vv' be an edge in H. The graph $G \triangle H$ is obtained by identifying u and v, delting uu' and vv', and adding a new edge $u'v'$.

- **Theorem3(Hajos):** If G and H are k-critical, then $G \triangle H$ is k-critical.

  Proof: Claim1: $\chi(G \triangle H) \leq k$

  Proof of claim1: Since G and H are k-critical, $\exists$ a $k-1$ colouring φ in $G - uu'$ where $\varphi(u) = \varphi(u')$, and a $k-1$ colouring ϕ in $H - vv'$ where $\phi(v) = \phi(v')$. By exchanging the colours, we may assume φ and ϕ assign the same colour to $u(v)$, and give v' a new colour. Then we find a k colouring of G. ∎

Claim2: $\chi(G \triangle H) \geq k$

Proof of claim2: Suppose $\exists$ a $k-1$ colouring f of $G \triangle H$, then one of the vertices u', v'(say v') has a distinct colour from $f(v)$, this means $f|_H$ is a proper $k-1$ colouring of H, a contradiction. ∎

Claim3: Any proper subgraph M of $\chi(G \triangle H)$ is $k-1$ colourable.

Proof of claim3: Exercise.

By claim1, 2 and 3, $G \triangle H$ is k-critical. ∎

- **Problem(open)** Determine the longest cycle length (and path length) in an n-vertex k-critical graph G.

- **Theorem4(Alon-Krivelevich-Seymour):** Any n-vertex k-critical graph G has a path of length at least $C \cdot \frac{\log n}{\log k}$, for some absolute constant $C > 0$.

- **DFS=depth-first-search tree**

  Any non-tree-edge xy must belong to a branch where x is in the path of T from y to the root.