Extremal and Probabilistic Graph Theory
Lecture 6
March 28th, Tuesday

- **Recall:** A graph G is k-critical, if $\chi(G) = k$ but $\chi(G') < k$ for $\forall G' \subsetneq G$.
Fact1: Such G is 2-connected.
Fact2: For any 2-cut, $\exists$ a structure.
Hajos's Construction

- **Definition1**(*The general Hajos's Construction*)**:**

  Let G, H be two k-critical graphs. Suppose $u_1, u_2, \cdots, u_k$ generate a complete subgraph of G and $v_1, v_2, \cdots, v_k$ generate a complete subgraph of H. Let $u_i u' \in E(G)$ and $v_i v' \in E(H)$, the graph $G \nabla H$ is obtained from G and H by identifying each u_j with v_j, for $\forall j \in [k]$, deleting the edges $u_i u'$ and $v_i v'$, and adding a new edge $u'v'$.

  Exercise: $G \nabla H$ is k-critical.

- Let T be a tree and $x, y \in V(T)$, then there exists a unique path between x and y, which is denoted by xTy.

  Definition2: A tree T of a graph G is call a *depth-first search(DFS) tree* rooted at a vertex v, if for $\forall xy \in E(G)$ with $x, y \in V(T)$, either $x \in V(vTy)$ or $y \in V(vTx)$.

- **Fact1:** For any connected graph G and $\forall v \in V(G)$, $\exists$ a DFS spanning tree in G rooted with v.

  Proof: By DFS-algorithm.

- **Lemma1:** Let G be an n-vertex k-critical graph, and let T be a DFS spanning tree of G rooted at r. Then for $\forall j \geq 1$, the number of vertices of T at distance exactly j for r is at most

$$S(j,k) = \begin{cases} j!, & \text{if } j \leq k-2; \\ (k-2)!(k-2)^{j-k+2}, & \text{if } j \geq k-1. \end{cases}$$

  Proof: Call an edge uv in T, where u is the parent of v, as an edge of type$j$, if the path uTr has length j. It suffices to show: number of edges in T of type j is $\leq S(j,k)$.

  Let $e = uv$ be an edge of type$j$.Assign a word S_e of length $j+1$ over the alphabet $K = \{0, 1, \cdots, k-2\}$ to the edge e as following: Let $v_0 = r, v_1, \cdots, v_j = u$ be the path rTu, let F_e be a proper $k-1$ colouring of $G - e$ by using the $k-1$ colours in K, s.t. $F_e(v_i) \leq i$, for $\forall i \in \{0, 1, \cdots, k-2\}$, then $S_e = (F_e(v_0), F_e(v_1), \cdots, F_e(v_j))$.

  Claim: For any two distinct tree-edges, $e = uv, e' = u'v'$ of type$j$, we have $S_e \neq S_{e'}$.

  Proof of claim: Suppose $S_e = S_{e'}$ and u, u' be the parent of v, v'. Let w be the lowest common ancestor of u and u'(note that possibly $u = u' = w$). As $S_e = S_{e'}$, the two colourings F_e and $F_{e'}$ coincide on rTw. Let y be the vertex following w on the path wTv. Let Ty be the subtree of T with root y and consisting of all offsprings of y in T.

Define a colouring φ of G as following:

$$\forall z \in V(G), \varphi(Z) = \begin{cases} F_e(Z), & \text{if } z \notin V(Ty); \\ F_{e'}(Z), & \text{if } z \in V(Ty). \end{cases}$$

Since T is a DFS-tree, the only edges of G connecting Ty and $V(G)-V(Ty)$ are these edges from $V(Ty)$ to $V(rTw)$. By the two colourings F_e and $F_{e'}$ coinciding on rTw, we see φ is a proper $k-1$ colouring of G, contradicting to $\chi(G) = k$, this proves the claim. ∎

Therefore, the number of edges of type$j \leq$ the number of $S_e{}'s$ for edges e of type$j \leq S(j, k)$.

- **Theorem2(Alon-Krivelevich-Seymour):** Let G be an n-vertex k-critical graph, and let t be the length of a longest path in G. Then $n \leq 1 + \sum_{j=1}^{t-1} S(j, k)$, thus any such G has a path of length at least $\frac{\log(n-1)}{\log(k-2)}$

  Proof: Using the lemma1.

- **Lemma2:** Any 2-connected graph G containing a path P of length at least $2s^2$ contains a cycle pf length at least s.

  Proof: Homework.

- **Definition3:** Let $L_k(n)$ be the maximum integer l s.t. any n-vertex k-critical graph has a cycle of length at least l.

- **Theorem3(Alon-Krivelevich-Seymour):**

$$L_k(n) \geq \sqrt{\frac{\log(n-1)}{2\log(k-2)}}.$$

- **Theorem4(Thomas-shapria, 2011):** For $k \geq 4$ and $n \leq k+2$,

$$\frac{\log n}{100 \log k} \leq L_k(n) \leq \frac{2\log(k-1)}{\log(k-2)} \log n.$$

  Here, the upper bound is due to Gallai in 1963, using the Hajos's Construction.

- **Lemma3:** Any 3-connected graph G containing a path of length t contains a cycle of length $\frac{2}{5}t$.

  Proof: Homework.