Extremal and Probabilistic Graph Theory
May 5th, Tuesday

Recall the **Definition** of the general projective norm-graphs $H = H(q,t)$. Note that the degree of each vertex in H is either $q^{t-1} - 1$ or $q^{t-1} - 2$.

Use **Lemma 3** instead of **Lemma 1**, we have:

Theorem 4: The graph $H = H(q,t)$ contains no subgraph isomorphic to $K_{t,(t-1)!+1}$.

Corollary 1: For every fixed $t \geqslant 2$ and $s \geqslant (t-1)! + 1$, we have:
$ex(n, K_{t,s}) \geqslant \frac{1}{2} n^{2-\frac{1}{t}} - O(n^{2-\frac{1}{t}-c})$, where $c > 0$ is an absolute constant.

Corollary 2: $ex(n, K_{4,7}) = \Theta(n^{\frac{7}{4}})$

Remark: Upper bound: Double counting.

$\lim\limits_{s\to\infty} (\liminf\limits_{n\to\infty} ex(n, K_{t,s}) n^{-(2-\frac{1}{t})}) = \infty$, which is a stronger version of Erdös's conjecture.

Next, we will construct another graph, using these techniques, to show the above conjecture to be right.

Theorem 5: Let $t \geqslant 2$ be fixed, there is a constant c_t such that for any $s \geqslant (t-1)! + 1$, we have $ex(n, K_{t,s}) \geqslant (1 + o(1)) \frac{c_t}{2} (s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}}$.

Proof: Let r be a positive integer which divides $q - 1$. Let $\mathbb{Q}_r$ denote the subgroup of $\mathbb{F}_q^*$ of order r.

Define $H_r(q,t)$ as follows: $V(H_r) = \mathbb{F}_{q^{t-1}} \times (\mathbb{F}_q/\mathbb{Q}_r)$; $(A, a\mathbb{Q}_r)$ is adjacent to $(B, b\mathbb{Q}_r)$ iff $N(A+B) \in ab\mathbb{Q}_r$.

Then, H_r has $(q^t - q^{t-1})/r$ vertices and each vertex has degree $q^{t-1} - 1$ or $q^{t-1} - 2$.

It suffices to prove that H_r is $K_{t,(t-1)!r^{t-1}+1}$-free.

Similar to the proof of **Theorem 1**, the problem can be reduced to bounding the number of solutions of the following system of equations:

$$N(Y + A_1) \in b_1\mathbb{Q}_r$$
$$\vdots$$
$$N(Y + A_{t-1}) \in b_{t-1}\mathbb{Q}_r$$

For any choice of elements from $b_1\mathbb{Q}_r, \cdots, b_{t-1}\mathbb{Q}_r$, there are at most $(t-1)!$ solutions.

Since we have r^{t-1} choices on the right hand side, the total number of solutions is not more than $(t-1)!r^{t-1}$. $\square$

Zarankiewicz Problem

$z(n,m,s,t)$ $(m \geqslant t \geqslant 1, n \geqslant s \geqslant 1)$ denotes the maximumu possible number of 1 entries in an $n \times m$ matrix M with 0-1 entries such that M does not contain an $s \times t$ submatrix consisting entirely of entries 1.

Proposition: $z(n,n,s,t) \geqslant 2ex(n, K_{t,s})$

Proof: Let G be a graph with vertex set $[n]$ and $ex(n, K_{t,s})$ edges containing no copy of $K_{t,s}$.

Consider $n \times n$ 0-1 matrix M such that $M_{ij} = 1$ iff i is adjacent to j in G.

Then, the number of '1' in M is $2ex(n, K_{t,s})$, and M doesn't contain as $s \times t$ all 1 submatrix. □

Theorem 6: $z(n,m,s,t) \leqslant (s-1)^{\frac{1}{t}} mn^{1-\frac{1}{t}} + (t-1)n$

Proof: Double counting the number of 's-star's. □

Remark: See Note on Mar,3,2016.

Theorem 7: Let $t \geqslant 2$ and $s > t!$ be fixed. If $n^{\frac{1}{t}} \leqslant m \leqslant n^{1+\frac{1}{t}}$, then $z(n,m,s,t) = \Theta(mn^{1-\frac{1}{t}})$.

Proof: First we prove the lower bound for $n = q^t$, where q is a prime power, and for $m = (1 + o(1)n^{1+\frac{1}{t}})$.

Label the rows of the matrix with the element of $\mathbb{F}_{q^t}$ and the columns with the element of $\mathbb{F}_{q^t} \times \mathbb{F}_q^*$.

Let the entry at $(A, (B, b))$ be 1 iff $N(A + B) = b$.

In this construction, every row contains $q^t - 1$ entries 1 and every column contains $q^{t-1} + q^{t-2} + \cdots + q + 1$ entries 1.

It suffices to show this matrix doesn't contain a $(t! + 1) \times t$ matrix all of whose entries are 1.

Choose t distinct columns $(D_1, d_1), \cdots, (D_t, d_t)$. If they have a row where each of their entries is a 1, then all the D_is must be distinct.

By **Lemma 3**, the number of solutions X of the equation system $N(X + D_i) = d_i$ $(i = 1, \cdots, t)$ is at most $t!$.

Since each row contains the same number of 1 entries for $m \leqslant (1+o(1))n^{1+\frac{1}{t}}$, we just choose a submatrix of the above construction.

The upper bound is from **Theorem 6**, where $mn^{1-\frac{1}{t}}$ is the dominating term for $m \geqslant n^{\frac{1}{t}}$. □

Remark: The construction provides us with a family $\mathscr{F}_t$ of $\Theta(n^{1+\frac{1}{t}})$ subsets of an n element set X, where each subset is of size $\Theta(n^{1-\frac{1}{t}})$ and no t subsets have intersection of cardinality exceeding $t!$.

Matousěk's Question

Theorem 8: Let $t \geqslant 2$ and $s \geqslant (t-1)! + 1$ be fixed integers, then $R_k(K_{t,s}) = \Theta(k^t)$.

Theorem 9 (Furedi) : $z(m,n,s,t) \leqslant (s-t+1)^{\frac{1}{t}} n m^{1-\frac{1}{t}} + tn + tn^{2-\frac{2}{t}}$ holds for all $m \geqslant s, n \geqslant t, s \geqslant t \geqslant 1$.