Extremal and Probabilistic Graph Theory

May 9th, Tuesday

Def. $ex(n,H)$=max # edges in an n-vertex H-free graph.

Def. $ex(n,n,H)$=max # edges in a bipartite H-free graph $G \subset K_{n,n}$.

Def. The Zarankiewicz function $Z(m,n,s,t)$= max # edges in a bipartite graph G, $G \subset K_{m.n}$, s.t. G has no $K_{s,t}$ where the s-part of $K_{s,t}$ is in the m-part of G and the t-part is in the n-part of G.

Fact. $2ex(n,K_{s,t}) \leq ex(n,n,K_{s,t}) \leq Z(n,n,s,t)$.

Kovari-Sos-Turan
(1) $Z(m,n,s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$.
(2) $ex(n,K_{s,t}) \leq \frac{1}{2}(s-1)^{\frac{1}{t}}n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$.
Proof. (1) Count the number of $K_{1,t}$ where the center of $K_{1,t}$ is in the m-part. Call this number as S, and we have:

$$\sum_{i\in[m]} \binom{d_i}{t} = S \leq \binom{n}{t}(s-1), \text{ where } \binom{x}{t} = \begin{cases} \dfrac{x(x-1)\cdots(x-t+1)}{t!} & ,x \geq t \\ 0 & ,x < t \end{cases}.$$

As the function $\binom{x}{t}$ is convex, $\frac{1}{m}\sum_{i\in[m]}\binom{d_i}{t} \geq \binom{\sum_i d_i/m}{t} = \binom{e/m}{t}$.

$\Rightarrow (s-1)\frac{n^t}{t!} \geq (s-1)\binom{n}{t} \geq m\binom{e/m}{t} \geq n\frac{(\frac{e}{m}-t+1)^t}{t!} \Rightarrow (s-1)n^t \geq m(\frac{e}{m}-t+1)^t$.
So $e(G) = e \leq (s-1)^{\frac{1}{t}}nm^{1-\frac{1}{t}} + (t-1)m$.
(2) Prove in the same way by counting the number of $K_{1,t}$ in the whole graph G.

Brown. $ex(n,K_{3,3}) \geq \frac{1}{2}n^{\frac{5}{3}} - \frac{1}{2}n^{\frac{4}{3}}$.

Alon-Romyar-Szabv. $ex(n,K_{3,3}) \geq \frac{1}{2}n^{\frac{5}{3}} + \frac{1}{3}n^{\frac{4}{3}} + C$.

Furedi. $ex(n,K_{3,3}) \leq \frac{1}{2}n^{\frac{5}{3}} + n^{\frac{4}{3}} + 3n$.

Therefore $ex(n.K_{3,3}) = \frac{1}{2}n^{\frac{5}{3}} + \Theta(n^{\frac{4}{3}})$.

Thm 1. (Furedi) $Z(m,n,s,t) \leq (s-t+1)n \cdot m^{1-\frac{1}{t}} + t \cdot n + t \cdot m^{2-\frac{2}{t}}$ for $m \geq s$, $n \geq t$ and $s \geq t \geq 1$.

This is the best bound for $ex(n, K_{3,3})$.

(Behmun-Keerash) $\exists\ c_1,\ c_2,\ c,\ c_1(\log n)^c n^{2-\frac{2}{r}+\frac{2}{r^2}} \leq ex(n, K_{r,r}) \leq c_2 \cdot n^{2-\frac{1}{r}}$.

Def.

$$\binom{x}{k} = \begin{cases} \dfrac{x(x-1)\cdots(x-k+1)}{k!} & ,x > k-1 \\ 0 & ,x \leq k-1 \end{cases},$$

k is an integer, so $\binom{x}{k}$ is convex.

Lemma 1. Let $v,\ k \geq 1$ be integers and $c,\ x_0,\ \cdots,\ x_v \geq 0$. If $\sum\limits_{1\leq i\leq v} \binom{x_i}{k} \leq c\binom{x_0}{k}$, then

$$\sum_{1\leq i\leq v} x_i \leq x_0 \cdot c^{\frac{1}{k}} \cdot v^{1-\frac{1}{k}} + (k-1)\cdot v.$$

Proof. Just by Jensen's inequality.

Lemma 2. Let $t \geq 2$, $v \geq 1$ be integers and $y_1,\ \cdots,\ y_v \geq t-2$. Then

$$[\sum_{1\leq i\leq v} \tbinom{y_i}{t-2}]\cdot[\sum_{1\leq i\leq v} (y_i-(t-2))] \leq v(t-1)\sum_{1\leq i\leq v}\tbinom{y_i}{ti1}.$$

Proof. The case $t = 2$ is trivial. So let $t \geq 3$.

Let $(a)_k = a(a-1)\cdots(a-k+1)$. Then for $\forall a,\ b \in \mathbb{R}$, $[(a)_{t-2}-(b)_{t-2}]\cdot(a-b) \geq 0$.

This implies $\binom{a}{t-2}(b-(t-2)) + \binom{b}{t-2}(a-(t-2)) \leq (t-1)(\binom{a}{t-1}+\binom{b}{t-1})$.

Adding up the above inequality with $(a,b) = (y_i, y_j)$ for all $1 \leq i,\ j \leq v$, we have the desired inequality.

Proof of Thm 1. We may integrate the function $Z(m,n,s,t)$ in another way, i.e. the maximum number of 1's in a 0-1 matrix M with m-rows and n-columns, containing NO submatrix with s rows and t columns consisting of 1's.

By induction on t.

It is trivial when $t = 1$. And when $t = 2$, we also know this by Kovari-Sos-Turan. So assume $t \geq 3$.

Let $R_i = \{j : M_{ij} = 1\}$, and $C_j = \{i : M_{ij} = 1\}$.

We may assume $|R_i|$, $|C_j| \geq t$ (otherwise we can induction on $m+n$).

We count the number of $(t-2)\times t$ submatrixes of all 1's.

Fix $t-2$ rows, say $1 \leq i_1 < \cdots < i_{t-2} \leq m$, consider all t-sets $T \subset R_{i_1}\cap\cdots\cap R_{i_{t-1}}$.

Let us consider the number N^* of pairs (T, R_x), where $R \subset R_i \cap\cdots\cap R_{i_{t-2}}$, $|T| = t$, and $x \notin \{i_i, \cdots, i_{t-2}\}$ but $T \subset T_x$.

Since M has no $s\times t$ submatrix of all 1's, for fixed T, there are at most $s-t+1$ many such pairs (T, R_x), so

$$\sum_{\substack{x\notin\{i_1,\cdots,i_{t-2}\}\\ x\in[m]}} \binom{|R_{i_1}\cap\cdots\cap R_{i_{t-2}}\cap R_x|}{t} = N_{i_1,\cdots,i_{t-2}} \leq (s-t+1)\binom{|R_{i_1}\cap\cdots\cap R_{i_{t-2}}|}{t}$$

By Lemma 1,

$$\sum_{\substack{x \notin \{i_1, \cdots, i_{t-2}\} \\ x \in [m]}} |R_{i_1} \cap \cdots \cap R_{i_{t-2}} \cap R_x| \leq (s-t+1)^{\frac{1}{t}} (m-t+2) |R_{i_1} \cap \cdots \cap T_{i_{t-2}}| + (t-2)(m-t+2).$$

Add up for all $1 \leq i_1 < \cdots < i_{t-2} \leq m$, then RHS counts the number of $(t-2) \times 1$ submatrixs with all 1 entries.

So $(t-1) \sum_{j \in [n]} \binom{|C_j|}{t} \leq (s-t+1)^{\frac{1}{t}} (m-t+2)^{1-\frac{1}{t}} \sum_{j \in [n]} \binom{|C_j|}{t-2} + (t-1)(m-t+2)\binom{m}{t-2}$.

By Lemma 2, the LHS is at least $\frac{1}{n}[\sum_j \binom{C_j}{t-2}] \cdot [\sum_j (|C_j| - (t-2))]$

$\Rightarrow (\sum_{j \in [n]} (|C_j| - n(t-2))) \leq (s-t+1)^{\frac{1}{t}} (m-t+2)^{1-\frac{1}{t}} n + (t-1)(m-t+2) \frac{\binom{m}{t-2}}{\sum_j \binom{|C_j|}{t-2}}$.

Case 1. $\frac{\binom{m}{t-2}}{\sum_j \binom{|C_j|}{t-2}} \leq m^{1-\frac{2}{t}}$.

Then easily we have what we want.

Case 2. $\frac{\binom{m}{t-2}}{\sum_j \binom{|C_j|}{t-2}} > m^{1-\frac{2}{t}}$.

So $\sum_j \binom{|C_j|}{t-2} \leq \frac{n}{m^{1-\frac{2}{t}}} \binom{m}{t-2}$. By Lemma 1, $\sum_j |C_j| \leq n(\frac{n}{m^{1-\frac{2}{t}}})^{\frac{1}{t-2}} \cdot n^{1-\frac{1}{t-2}} + (t-3)n = n \cdot m^{1-\frac{1}{t}} + (t-3) \cdot n < (s-t+1) n \cdot m^{1-\frac{1}{t}} + t \cdot n + t \cdot m^{2-\frac{2}{t}}$