Extremal and Probabilistic Graph Theory

May 16th, Tuesday

Def. BC_k = Berge cycle of length k.

$\mathscr{C}_k$ = minimum cycle of length k that is a collect of k edges $e_1, \cdots, e_k$ s.t. $e_i \cap e_j \neq \emptyset$ iff $j = i+1$ or $\{i,j\} = \{1,k\}$.

C_k = linear cycle of length k.

Thm.(Furedi-Tao, Kostochka-Mubayi-Verstraete). $ex_r(n,P_k) = ex_r(n,C_k) = \binom{n}{r} - \binom{n-(\frac{k-1}{2})}{2}$

$$+\begin{cases} 0 & ,if\ k\ is\ odd \\ \binom{n-(\frac{k-1}{2})-2}{r-2} & ,if\ k\ is\ even \end{cases}.$$

Thm.(Asymptotic form). For $r \geq 3$, $k \geq 4$, $ex_r(n,P_k) \sim ex_r(n.C_k) \sim l \cdot \binom{n}{r-1}$, where $l = \lfloor \frac{k-1}{2} \rfloor$.

Def. Given an r-graph H, the shadow ∂H of H is an $(r-1)$-graph with the same vertex set $V(H)$ and $e \in \partial H$ iff $|e| = r-1$ and $e \in f$ for some$f \in H$. And $\forall e \in \partial H$ is called a sub-edge in H.

The co-degree of a set $S = \{x_1, \cdots, x_s\}$ is $d_H(S) = |\{e \in H : S \subset e\}|$ where $|S| = r-1$, the neighborhood of S in H is $N_H(S) = \{x : S \cup \{x\} \in H\}$, so $|N_H(S)| = d_H(S)$.

We say an (x,y)-path is a linear path $P = e_1 e_2 \cdots e_k$, where $x \in e_1 \backslash e_2$, $y \in e_k \backslash e_{k-1}$.

Def. For $G \subset \partial H$ and $e \in G$, the last of e is $L_G(e) = N_H(e) - V(G)$. And the elements of $L_G(e)$ are called "colors".

Let $L_G = \bigcup_{e \in G} L_G(e)$ and $\hat{G} = \{e \cup \{x\} : e \in G,\ x \in L_G(e)\}$.

Def. An r-graph H is d-full if any sub-edge of H has co-degree at least d.

Lemma 1. For $r \geq 2$, $d \geq 1$, every n-regular r-regular H has a $(d+1)$-full subgraph F with $|F| \geq |H| - d|H|$.

Proof. Greedily delete those $e \in \partial H$ with co-degree at most d.

Lemma 2. Let $r \geq 3$, $k \geq 3$ and H be a rk-full r-graph (non-empty). Then C_k, $P_{k-1} \subset H$.

Proof. Fact. $\partial^i H$ is rk-full for $1 \leq i \leq r-2$.

Claim. If $C_k \subset \partial^{i+1} H$, then $\exists C_k \subset \partial^i H$. This can be done in a greed method.

Lemma 3. For $r, t \geq 2$, there exists $n_0 = n_0(r,t)$, s.t. if an n-vertex r-graph H has $|H| > n^{r-t^{1-r}}$, then the complete r-partite r-graph $K^r_{t,\cdots,t} \subset H$.

Def. Fix $c > 0$, $r, k \geq 3$. An r-graph H is called (t,c)-sparse if every t-set of $V(H)$ lies in at most c edges of H.
If $c = 1$, then it is t-linear.

Lemma 4.(Sarkozy-Selkow) If H is an n-vertex $(r-1,c)$-sparse r-graph which contains no C_k nor P_k, then $|H| = o(n^{r-3})$.

Lemma 5. Let $k \geq 3$, let H be an r-graph and let $P = e_0 e_1 \cdots e_{2^{2l+1}-1}$ be a 2^{2l+1}-path in ∂H. If $|L_P(e)| \geq l+1$ for $\forall e \in P$, then $\hat{P}$ contains P_k whose first edge contains e_0. ($l = \lfloor \frac{k-1}{2} \rfloor$.)
Proof. Since $\lfloor \frac{k-1}{2} \rfloor = \lfloor \frac{k-2}{2} \rfloor$ for even k, it sufficec to consider even k.

We prove by induction on k.

Base case: $k = 4 (l = 1)$.

We want: 8-path $P \subset \partial H \Rightarrow P_4 \subset \hat{P}$.

Case 1. $L_P(e_0) \cap L_P(e_i) \neq \emptyset$ for some $i \geq 2$.

Let $\alpha \in L_P(e_0) \cap L_P(e_i)$. Let e_i, f, g, $h \in P$ be a subpath of P.

Def $L^{'}(e) = L_P(e) - \{\alpha\}$ for $\forall e \in P$. If $\exists$distinct $\beta \in L^{'}(f)$, $\gamma \in L^{'}(g)$, then we are done.

Otherwise $L_P(f) = L_P(g) = \{\alpha, \beta\}$. Now replace e_i by f, and repeat the argument, we have to set $L_P(f) = L_P(g) = L_P(h) = \{\alpha, \beta\}$. Then we have a $P_4 \subset \hat{P}$ as $e_0 \cup \{\alpha\}$, $e_i \cup \{\alpha\}$, $f \cup \{\beta\}$, $g \cup \{\beta\}$.

Case 2. $L_P(e_0) \cap L_P(e_i) = \emptyset$ for $\forall i \geq 2$.

Let $L_P(e_0) = \{\alpha, \beta\}$. If $L_P(e_0) \cap L_P(e_1) \neq \emptyset$, say $\beta \in L_P(e_0) \cap L_P(e_1)$, then we may pick distinct $\gamma \in L_P(e_2)$ and $\delta \in L_P(e_3)$ so that $e_0 \cup \{\alpha\}$, $e_1 \cup \{\beta\}$, $e_2 \cup \{\gamma\}$, $e_3 \cup \{\delta\}$ is a 4-path. Suppose $L_P(e_0) \cap L_P(e_1) = \emptyset$. If there is $\gamma \in L_P(e_1) \cap L_P(e_3)$, then choose any $\lambda \in L_P(e_4) - \gamma$, and the edges $e_0 \cup \{\alpha\}$, $e_1 \cup \{\gamma\}$, $e_3 \cup \{\gamma\}$, $e_4 \cup \{\lambda\}$ form a 4-path. Otherwise, as $|L_P(e_i)| \geq 2$ for $i \geq 1$, we can choose all distinct $\alpha_1 \in L_P(e_1)$, $\alpha_2 \in L_P(e_2)$, $\alpha_3 \in L_P(e_3)$, and the edges in the set $\{e_i \cup \{\alpha_i\} : i = 1,2,3\}$ together with $e_0 \cup \{\alpha\}$ form a 4-path.

Now suppose $k \geq 6$. If for some $i > 1$ we have $\beta \in L_P(e_0) \cap L_P(e_i)$, let $P^{'} = \{e_{i+1}, e_{i+2}, \cdots, e_{i+2^{k-3}}\}$ if $i \leq 2^{k-3}+1$ and $P^{'} = \{e_{i-1}, e_{i-2}, \cdots, e_{i-2^{k-3}}\}$ if $i > 2^{k-3}+1$(note that $i - 2^{k-3} \geq 2$). Let $e^{'}_0 = e_{i+1}$ if $i \leq 2^{k-3}+1$ and $e^{'}_0 = e_{i-1}$ if $i > 2^{k-3}+1$. Let us remove β from all lists of edges of $P^{'}$. Then $P^{'}$ is a 2^{k-3}-path all of whose lists have size at least l. So by induction on k, $\hat{P} - \beta$ has a $(k-2)$-path $\{f_2, f_3, \cdots, f_{k-1}\}$ where $e^{'}_0 \subset f_2$. Set $f_0 = e_0 \cup \{\beta\}$, $f_1 = e_i \cup \{\beta\}$. Then $\{f_0, f_1, \cdots, f_{k-1}\}$ is the required path. So we may assume for all $i > 1$, $L_P(e_0) \cap L_P(e_i) = \emptyset$. If we find $\gamma \in L_P(e_1) - L_P(e_0)$, then remove γ from all lists $L_P(e_i)$ where $i \geq 2$. Let $\hat{P}^{'} = \hat{P} - L_P(e_0) - \{\gamma\}$ if γ exists and $\hat{P}^{'} = \hat{P} - L_P(e_0)$ otherwise(in this case $L_P(e_1) \subset L_P(e_0)$). By induction, $\hat{P}^{'}$ contains a $(k-2)$-path $\{f_2, f_3, \cdots, f_{k-1}\}$ with $e_2 \subset f_2$ as the lists sizes have reduced by at most one. Set $f_0 = e_0 \cup \{\alpha\}$, $f_1 = e_1 \cup \{\beta\}$ with $\alpha \neq \beta$, $\alpha \in L_P(e_0)$, and $\beta \in L_P(e_1) \cup \{\gamma\}$(if γ exists we may choose $\beta = \gamma$). This works since $|L_P(e)| \geq 2$ for $e \in P$. Now $\{f_0, f_1, \cdots, f_{k-1}\} \subset \hat{P}$ is a k-path.