Extremal and Probabilistic Graph Theory

May 18th, Thursday

Lemma 5. Let $k \geq 3$, let H be an r-graph and let $P = \{e_1, \cdots, e_{2^{2l+1}-1}\}$ be a 2^{2l+1}-path in ∂H. If $|L_P(e)| \geq l+1$ for $\forall e \subset P$, then $\hat{P}$ has a P_k.

Def. Let H be a r-graph($r \geq 3$). Let $\psi_t(H)$ be the set of complete $(r-1)$-partite $(r-1)$-graphs $G \subset \partial H$ with parts of size t and $\forall e \in G$, $|L_G(e)| \geq l+1$, and if $r = 3$ and k is odd, then in addition for $xy \in G$, there is $xt\alpha \in \hat{G}$ s.t.
(a) $\min\{d_H(x\alpha), d_H(y\alpha)\} \geq 2$ and
(b) $\max\{d_H(x\alpha), d_H(y\alpha)\} \geq 3k+1$.

Def. Given sets $S_1, S_2, \cdots, S_p$, an SDR is a choice of $s_i \in S_i$, $\forall i \in [p]$ s.t. $s_1, s_2, \cdots, s_p$ are distinct.

Lemma 6. Let $q \in \{2p, 2p+1\}$ and let $S_1, \cdots, S_q$ be sets such that $S_i \cap S_j \neq \emptyset$ for $i \leq p$, $j \geq p+2$ and $|S_i| \geq p+1$ for $i \leq p$, and $|S_i| \geq p$ for $i \geq p+1$. Then $\{S_1, \cdots, S_q\}$ has an SDR, unless $q = 2p+1$ and all $S_j (j \geq p+1)$ are all identical and of size p.
Proof. To have an SDR for $\{S_1, \cdots, S_q\}$, it is equivalent to have an SDR for $\{S_{p+1}, S_{p+2}, \cdots, S_q\}$. Assume no SDR for $\{S_{p+1}, S_{p+2}, \cdots, S_q\}$. Then $q = 2p+1$, moreover, $S_{p+1} = S_{p+2} = \cdots = S_q$ is of size p.

Lemma 7. Let $r \geq 3$, $k \geq 4$. Then there exists a $t_0 = t(r,k)$ s.t. for all $t \geq t_0$ and for all C_k-free r=graph H, $\psi_t(H) = \emptyset$.
Proof. Suppose $G \in \psi_t(H)$, we want $C_k \in H$.

Let $s = 2^{k-2}(r-1)$. Let M be a set of s pairwise disjoint edges of G.

Case 0. Suppose $\exists \alpha \in L_G(e)$, $\forall e \in M$.

Let $F \subset G$ be a complete $(r-1)$-partite $(r-1)$-graph with parts of size 2^{k-2} s.t.

- $V(F) \subset V(M)$.
- $\forall f \in F$, $e \in M$, $|f \cap e| \leq 1$.
- $\forall f \in F$, $\exists r-1$ distinct $e \in M$ s.t. $|f \cap e| = 1$.

We will show $\hat{F}$ contains a $(k-2)$-path avoiding α.

If so, using the color α plus the conditions, we can find a $C_k \subset H$.

If $k \geq 5$, by lemma 5, as F contains a 2^{k-2}-path($|L_F(e) \geq l|$, $\forall e \in F$), we can find a P_{k-2} avoiding α.

So $k = 4$. Find a 3-path $f_1 f_2 f_3$ in F and $L_P(f_i) \backslash \{\alpha\} = \{\beta_i\}$. If $\beta_1 = \beta_3$, then $\{f_1 \cup \{\beta_1\}, f_3 \cup \{\beta_1\}\}$ is a 2-path. Otherwise $\{f_1 \cup \{\beta_1\}, f_2 \cup \{\beta_2\}\}$ or $\{f_2 \cup \{\beta_2\}, f_3 \cup \{\beta_3\}\}$ is a 2-path.

Therefore, no color appears in the list of s pairwise disjoint edges of G.$(*)$

For $\forall e \in G$, $L'_G(e) \subset L_G(e)$ with $|L'_G(e)| = l+1$. Let $m = \lfloor \frac{t}{s+2} \rfloor \gg \Omega(1)$. Partiting G into m vertex-disjoint complete $(r-1)$-partite $(r-1)$-subgraphs F_i, $i \in [m]$, s.t. each part of each F_i is of size at least $s+2$ and $L'_i = \cup\{L'_G(e) : e \in F_i\}$. So $|L'_i| \leq (l+1)(s+2)^{r-1} \leq (s+2)^r$. For $\forall \alpha \in L'_1$, by $(*)$, there are at most s different i for which $\alpha \in L'_1 \cap L'_i$. So $L'_1 \cap L'_i \neq \emptyset$ for at most $(s+2)^{r+1}$ values $i \in [m]$. Choose t to be big enough s.t. $t \gg (s+2)^{r+1}$. Thus we may assume $L'_2 \cap L'_1 = \emptyset$.

Let $F = F_1 \cup F_2$ and let X, Y be two parts of F. Let $x \in X \cap F_1$, $y \in Y \cap F_2$ and $e \in G$ s.t. $\{x, y\} \subset e$.

Case 1. $r > 3$, and $r = 3$ if k is even.

Let $e \cup \{\alpha\} \subset \hat{G}$. Assume $\alpha \notin L'_1$. Let $U = F_1 \cap X$, $V = F_2 \cap Y$. Let $f \in G$ be any edge with $|f \cap U| = 1 = |f \cap V|$ and $|f \cap F| = 2$. There is a q-path $Q = \{f_1, f_2, \cdots, f_q\}$ from x to y s.t. $f_i \subset F_1$ for $i \leq p$ and $f_j \subset F_2$ for $j \geq p+2$ and $f_{p+1} = f$. Let $S_i = L'_G(f_i) - \{\alpha\}$. So $S_1, \cdots, S_q$ satisfy lemma 6.

If $\exists$ SDR of $\{S_1, \cdots, \S_q\}$, then plus $e \cup \{\alpha\}$, f_1, $\cdots$, f_q will extend to a $C_k \subset H$. So NO SDR of $\{S_1, \cdots, S_q\}$. By lemma 6, $S_{p+1} = S_{p+2} = \cdots = S_q$ is of size $p \Rightarrow |L'_G(f) = p+1|$ and $\alpha \in L'_G(f)$. But f is arbitrary, so that gives us a matching of size s, whose lists contain the color α, which is a contradiction to $(*)$.

Case 2. r=3 and k is odd.

Let $q = k-2$ and $p = l-1$, so $q = 2p+1$. Since $G \in \psi_t(H)$, $\exists xy\alpha \in \hat{G}$ satisfies (a) and (b) in the definition of $\psi_t(H)$. Since $L'_1 \cap L'_2 = \emptyset$, we may suppose $\alpha \notin L'_1$. By symmetry we may assume $d_H(x\alpha) > 3k$ and $d_H(y\alpha) > 1$. Choose an edge $y\alpha\beta \in H$ with $\beta \neq x$. Note that possibly $\beta \in V(G)$. For $i = 1$, 2, let $X_i = X \cap V(F_i) - \{\alpha, \beta\}$ and $Y_i = Y \cap V(F_i) - \{\alpha, \beta\}$. Let $f \in G$ be such that

$$|f \cap X_1| = 1 = |f \cap Y_2| \text{ if } q \equiv 1 (\text{mod } 4),$$
$$|f \cap X_2| = 1 = |f \cap Y_1| \text{ if } q \equiv 3 (\text{mod } 4).$$

Since q is odd, there is a q-path $Q = \{f_1, f_2, \cdots, f_q\}$ from x to y in G with $f_i \subset F_1$ for $i \leq p, f_i \subset F_2$ for $i > p+1$ and $f_{p+1} = f$. If Q expands to a q-path $\hat{Q} \subset \hat{G} - \{\alpha, \beta\}$, then select $\gamma \in V(H) - V(\hat{Q}) - \{\alpha, \beta\}$ so that $x\alpha\beta \in H$ and then $\hat{G} \cup \{x\alpha\gamma, y\alpha\beta\}$ is a k-cycle in $\hat{G}$. So Q does not expand to a q-path in $\hat{G} - \{\alpha, \beta\}$.

Let $S_i = L'_G(f_i) - \{\alpha, \beta\}$. Since $L'_1 \cap L'_2 = \emptyset$, we have $S_i \cap S_j = \emptyset$ for $i \leq p$ and $j > p+1$, and since $\alpha \notin L'_1$, $|S_i| = |L'_G(f_i) - \{\beta\}| \geq l \geq p+1$ for $i \leq p$ and $|S_i| \geq |L'_G(f_i)| - 2 \geq p$ for $i > p$. So $S_1, \cdots, S_q$ satisfy lemma 6. The same as before, $S_{p+1} = \cdots = S_q$ and $\forall i \in [q]$, S_i is of size p, so $\alpha \in S_i$ and then $\alpha \in L'_G f$. Since f is arbitrary, this gives us a matching of size s, whose lists contain the color α, which is a contradiction to $(*)$

Lemma 8. Let $\delta > 0$, $r \geq 3$ and $k \geq 4$. Let H be an r-graph and $E \subset \partial H$ with $|E| > \delta \cdot n^{r-1}$. Suppose $d_H(f) > l+1$ for $\forall f \in E$ and if $r = 3$ and k is odd, then in addition, for $\forall f = xy \in E$, there is $e_f = xy\alpha \in H$ s.t. $\min\{d_H(x\alpha), d_H(y\alpha)\} \geq 2$ and $\max\{d_H(x\alpha), d_H(y\alpha)\} \geq 3k+1$.