Extremal and Probabilistic Graph Theory

Instructor: Jie Ma, Scribed by Tianchi Yang

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1 First Lecture

Let us begin this course by introducing some basic notations in graph theory. Let $G = (V, E)$ be a graph. The *degree* $d(v)$ of a vertex v is the number of neighbors of v. Let $\Delta(G) := \max\{d(v)|v \in V\}$ be the *maximum degree* of G and $\delta(G) := \min\{d(v)|v \in V\}$ be the *minimum degree*. The complete graph on n vertices is denoted by K_n, while the complete r-partite graph with parts of sizes $n_1, n_2, ..., n_r$ is denoted by $K_{n_1,n_2,...,n_r}$.

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G contains none of $\mathcal{F}$ as a subgraph. Let $\mathrm{ex}(n, \mathcal{F})$ denote the largest possible number of edges in an n-vertex $\mathcal{F}$-free graph, and call it the *Turan number* or *extremal number* of $\mathcal{F}$.

1.1 Turán Density

If $\mathcal{F}$ be a family of graphs, the *Turán density* of $\mathcal{F}$ is denoted by $\pi(\mathcal{F}) = \lim\limits_{n\to+\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{2}}$. We can prove the following.

Theorem 1.1. *$\pi(\mathcal{F})$ exists for any family $\mathcal{F}$.*

Proof. Let $\pi_n = \mathrm{ex}(n, \mathcal{F})/\binom{n}{2}$, then $\pi_n \in [0, 1]$. It suffices to show that $\{\pi_n\}$ is non-increasing. Let G be an n-vertex $\mathcal{F}$-free graph with $\mathrm{ex}(n, \mathcal{F})$ edges. By double-counting the number of pairs (e, T) where $e \in G[T]$ and $T \subseteq \binom{G}{n-1}$, we can get

$$\#(e, T) = \sum_{e\in E(G)} \binom{n-2}{n-3} = (n-2)\mathrm{ex}(n, \mathcal{F})$$

and

$$\#(e, T) = \sum_{T\subseteq\binom{V(G)}{n-1}} e(G[T]) \leq n \cdot \mathrm{ex}(n-1, \mathcal{F}).$$

Together we have $(n-2)\mathrm{ex}(n, \mathcal{F}) \leq n \cdot \mathrm{ex}(n-1, \mathcal{F})$, implying that $\pi_n \leq \pi_{n-1}$, as desired. ∎

1.2 Mantel's Theorem

Theorem 1.2 (Mantel). *If G is an n-vertex K_3-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if $G = K_{\lfloor \frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$.*

Proof. We will prove it by induction on n. It is trivial for $n \leq 3$. So assume $n \geq 4$. By deleting some edges, we can also assume $e(G) = \lfloor \frac{n^2}{4} \rfloor$. Then there exists a vertex v with $d(v) \leq \lfloor \frac{n}{2} \rfloor$. Let $G' = G - \{v\}$. Clearly G' is an $(n-1)$-vertex K_3-free graph with $e(G') \geq \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{(n-1)^2}{4} \rfloor$. By induction, we know $G' = K_{\lfloor \frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil}$ with two parts A, B. Also it is easy to see that $N(v)$

is a subset of A or B. Otherwise, there exists a K_3 in G. Then one can verify that $N(v) = A$ or $N(v) = B$ and thus $G = K_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$. ∎

1.3 Turán's Theorem

Let *Turán graph* $T_r(n)$ be the complete balanced r-partite graph on $n \geq r$ vertices. That is $V = V_1 \cup V_2 ... \cup V_r$ and $|V_i| = \lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, such that all pairs in $V_i \times V_j$ form edges.

Before proving the Turán's Theorem, let us show three easy observations on $T_r(n)$ as follows.

(i) $e(T_r(n)) = \sum\limits_{0 \leq i < j < r} \lfloor \frac{n+i}{r} \rfloor \lfloor \frac{n+j}{r} \rfloor$ achieves the unique maximum in all n-vertex r-paritite graphs.

(ii) $T_r(n-1) = T_r(n) - \{v\}$, where $d(v) = \delta(T_r(n)) = n - \lceil \frac{n}{r} \rceil$.

(iii) $T_r(n)$ has the highest minimum degree among all n-vertex graphs with the same number of edges.

Next, we will give two different proofs based on above observations.

Theorem 1.3 (Tuŕan)**.** *Let G be an n-vertex K_{r+1}-free graph. Then $e(G) \leq e(T_r(n))$ with equality holds if and only if $G = T_r(n)$.*

Proof. (first): We prove it by induction on n. The base case $n = r$ is clear. Let $n \geq r+1$. By observation (iii), there exists a vertex v with $d(v) \leq \delta(T_r(n))$. Let $G' = G - \{v\}$. We see $e(G') = e(G) - d(v) \geq e(T_r(n)) - \delta(T_r(n)) = e(T_r(n-1))$. By induction, we know $G' = T_r(n-1)$. Then we claim that G is a r-partite graph. As otherwies, each part of G' contains a neighbor of v, implying that these r vetices together with v form a K_{r+1}. Hence by (i) we get $G = T_r(n)$. ∎

Proof. (second): Let us prove it by induction on r. It is clear that Mantel's Theorem gives the case for $r = 2$. Assume $r \geq 3$. Let $u \in V(G)$ with $d(u) = \Delta(G)$. Let $S = N(u)$ and $T = V \backslash S$. We see $G[S]$ is K_r-free. Now let G' be obtained from G by deleting all edges in $G[T]$ and adding all missing edges in (S, T). Then we see G' is K_{r+1}-free. And the number of missing edges in (S, T) is $|S||T| - e_G(S, T)$. Now we claim that $e(G') = e(G)$ and $e(G[T]) = 0$. Since

$$2e(G[T]) + e_G(S,T) = \sum_{x \in T} d_G(x) \leq \Delta(G)|T| = |S||T|,$$

we have

$$e(G') = e(G) - e(G[T]) + (|S||T| - e_G(S,T)) \geq e(G) + e(G[T]).$$

This confirms the claim. Clearly $G = G'$. We see $G[S]$ is K_r-free and contain the maximum number of edges on $|S|$ vertices. By induction we know $G[S]$ must be $(r-1)$-partite. Thus G is r-partite. Finally, by (i) we get $G = T_r(n)$. ∎

Next, we consider the Turán number of complete bipartite graphs.

1.4 Kövari-Sós-Turán Theorem

Theorem 1.4 (Kövari-Sós-Turán)**.** *For any integers $t \geq s \geq 2$, we have*

$$\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n.$$

Proof. Let G be any n-vertex $K_{s,t}$-free graph. Let the number of stars $K_{1,s}$ in G be T. On the one hand, for a fixed vertex v, there are $\binom{d_G(v)}{s}$ many $K_{1,s}$ rooted at it. Then $T = \sum\limits_{v \in V(G)} \binom{d_G(v)}{s}$. Here we define

$$\binom{x}{s} = \begin{cases} \frac{x(x-1)\cdots(x-s+1)}{s!}, & x \geq s \\ 0, & \text{otherwise} \end{cases}.$$

On the other hand, for any fixed s vertices, there are at most $t-1$ vertices which are adjacent to all these s vertices. Thus we have $T \leq (t-1)\binom{n}{s}$.

Combining them and using Jensen's inequality, we get

$$(t-1)\frac{n^s}{s!} \geq (t-1)\binom{n}{s} \geq \sum_{v \in V(G)} \binom{d_G(v)}{s} \geq n \binom{\sum_{v \in V(G)} d_G(v)/n}{s} \geq n \cdot \frac{(2e(G)/n - s + 1)^s}{s!}.$$

Thus we have $e(G) \leq \frac{1}{2}(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n$. ∎

These theorems also tell us that $\pi(K_{r+1}) = 1 - \frac{1}{r}$ and $\pi(K_{s,t}) = 0$ for any integers $r, s, t \geq 1$.