Extremal and Probabilistic Graph Theory

Instructor: Jie Ma, Scribed by Tianchi Yang

Feb 26th, 2020, Wednesday

1 Second Lecture

1.1 Erdös-Moon Theorem

Theorem 1.1 (Erdös-Moon)**.** *Let G be an n-vertex graph with more than $\frac{1}{2}s^{1+1/s}n^{2-1/s}+2sn$ edges. Then it has at least $\Omega(p^{s^2}n^{2s})$ copies of $K_{s,s}$, where $p=e(G)/\binom{n}{2}$ is the edge-density of G.*

Proof. Let M denote the number of stars $K_{1,s}$ in G. We see $M=\sum_{v\in V}\binom{d(v)}{s}$. For a subset $S\subseteq V(G)$ of size s, let $f(S)$ be the number of vertices adjacent to all vertices of S. Then we see $M=\sum_{S\in\binom{V}{s}}f(S)$. And G has $\frac{1}{2}\sum_{S\in\binom{V}{s}}\binom{f(S)}{s}$ copies of $K_{s,s}$. By Jensen's inequality, we also get

$$\frac{1}{2}\sum_{S\in\binom{V}{s}}\binom{f(S)}{s}\geq\frac{1}{2}\binom{n}{s}\binom{\sum_{S\in\binom{V}{s}}f(S)/\binom{n}{s}}{s}\geq\frac{1}{2}\binom{n}{s}\binom{M/\binom{n}{s}}{s}=\Omega(n^{s-s^2})M^s$$
$$=\Omega(n^{s-s^2})\left(\sum_{v\in V}\binom{d(v)}{s}\right)^s\geq\Omega(n^{s-s^2})\left(n\binom{\sum_{v\in V}d(v)/n}{s}\right)^s=\Omega(p^{s^2}n^{2s}).$$

∎

1.2 Hypergraph Kövari-Sós-Turán Theorem

Let $K^{(k)}_{(t_1,t_2,\dots,t_k)}$ be the complete k-partite k-graph. For convenience, we denote $K_{t:k}=K^{(k)}_{(t,t,\dots,t)}$. For a k-graph G and $v\in V(G)$, the *link-hypergraph* G_v is a $(k-1)$-graph on the vertex set $V(G)-\{v\}$ where $e\in E(G_v)$ if and only if $e\cup\{v\}\in E(G)$.

Theorem 1.2 (Erdős)**.** *Let $k,t\geq 2$ be integers. Then there exists a constant $c=c(k,t)$ such that any k-graph G with $e(G)=p\binom{n}{k}\geq cn^{k-(1/t)^{k-1}}$ has at least $\Omega(p^{t^k}n^{tk})$ copies of $K_{t:k}$.*

Proof. We prove this by induction on k. It is clear that the Erdös-Moon theorem gives the case for $k=2$. We may assume it holds for $k-1$ with $k\geq 3$. Suppose G is a k-graph with $cn^{k-(1/t)^{k-1}}$ edges. Let $V_1=\{v\in V(G):d(v)\geq cn^{(k-1)-(1/t)^{k-2}}\}$ and $V_2=V(G)\setminus V_1$. Then $\sum_{v\in V_2}d(v)\leq n\cdot cn^{(k-1)-(1/t)^{k-2}}\ll e(G)$, implying that $\sum_{v\in V_1}d(v)=(k-o(1))e(G)$.

For each $v\in V_1$, the link-hypergraph G_v is a $(k-1)$-graph with $e(G_v)=d(v)$. By induction, G_v has at least $\Omega\left(\left(\frac{e(G_v)}{\binom{n-1}{k-1}}\right)^{t^{k-1}}\cdot(n-1)^{t(k-1)}\right)=\Omega(n^{t(k-1)-(k-1)t^{k-1}})d(v)^{t^{k-1}}$ copies of $K_{t:(k-1)}$.

Let $\vec{S}=(S_1,\dots,S_{k-1})$, where $|S_i|=t$. We denote by $f(\vec{S})$ the number of vertices v such that $v\cup S_1\cup\cdots\cup S_{k-1}$ induces a copy of $K^{(k)}_{(1,t,\dots,t)}$ in G. Clearly each copy of $K^{(k)}_{(1,t,\dots,t)}$ in G associates with a $K_{t:(k-1)}$ in a unique G_v for $v \in V$. Thus the number of $K^{(k)}_{(1,t,\dots,t)}$ in G is

$$\sum_{\vec{S}} f(\vec{S}) \geq \Omega(n^{t(k-1)-(k-1)t^{k-1}}) \sum_{v \in V_1} d(v)^{t^{k-1}}.$$

We already have $\sum_{v \in V_1} d(v) = (k - o(1))e(G)$ and $|V_1| \leq n$. By Jensen's inequality, we can get

$$\sum_{\vec{S}} f(\vec{S}) \geq \Omega(n^{t(k-1)-(k-1)t^{k-1}}) \cdot n \left(\frac{d(v)}{n}\right)^{t^{k-1}} = \Omega(p^{t^{k-1}} n^{t(k-1)+1}).$$

Finally, the number of $K_{t:k}$ in G is equal to

$$\frac{1}{k} \sum_{\vec{S}} \binom{f(\vec{S})}{t} \geq \Omega(n^{t(k-1)}) \left(\frac{\sum_{\vec{S}} f(\vec{S})}{n^{t(k-1)}}\right)^t \geq \Omega(n^{t(k-1)})(p^{t^{k-1}} n)^t = \Omega(p^{t^k} n^{tk}).$$

∎

This theorem gives us an upper bound for the extremal number of $K_{t:k}$ as follows, and also implies that $\pi(K_{t:k}) = 0$.

Theorem 1.3 (Hypergraph Kövari-Sós-Turán).

$$\mathrm{ex}_k(n, K_{t:k}) = O(n^{k-(1/t)^{k-1}}).$$

1.3 Supersaturation lemma

Theorem 1.4 (Supersaturation lemma). *Let F be a k-graph with $k \geq 2$. For any $\epsilon > 0$, there exist positive constants $\delta = \delta(F, \epsilon)$ and $n_0 = n_0(F, \epsilon)$ such that for any n-vertex k-graph G with $n > n_0$, if G has at least $ex(n, F) + \epsilon \cdot n^k$ edges, then it contains at least δn^v copies of F, where $v = |V(F)|$.*

Proof. By the definition of $\pi(F)$, we can find an integer m such that $ex(m', F) < (\pi(F) + \frac{\epsilon}{2})\binom{m'}{k}$ for any $m' \geq m$. Let $n > n_0 \gg m$. Assume the n-vertex k-graph G has $(\pi(F) + \epsilon)\binom{n}{k}$ edges. We use T to denote the number of pairs (e, M), where $M \in \binom{V(G)}{m}$ and $e \in G[M]$. On the one hand, we have

$$T = \sum_{e \in E(G)} \binom{n-k}{m-k} = e(G)\binom{n-k}{m-k} = (\pi(F) + \epsilon)\binom{n}{k}\binom{n-k}{m-k} = (\pi(F) + \epsilon)\binom{n}{m}\binom{m}{k}.$$

On the other hand, if we let $\mathcal{A} = \{M \in \binom{V(G)}{m} : e(G[M]) > (\pi(F) + \frac{\epsilon}{2})\binom{m}{k}\}$, then we get

$$T = \sum_{M \in \binom{V(G)}{m}} e(G[M]) = \sum_{M \in \mathcal{A}} e(G[M]) + \sum_{M \notin \mathcal{A}} e(G[M]) \leq |\mathcal{A}|\binom{m}{k} + (\binom{n}{m} - |\mathcal{A}|)(\pi(F) + \frac{\epsilon}{2})\binom{m}{k}.$$

The above two inequalities indicate that $(\pi(F) + \epsilon)\binom{n}{m} \leq |\mathcal{A}| + (\binom{n}{m} - |\mathcal{A}|)(\pi(F) + \frac{\epsilon}{2})$. So $|\mathcal{A}| \geq \frac{\epsilon}{2}\binom{n}{m}/(1 - \frac{\epsilon}{2} - \pi(F))$. Since each $M \in \mathcal{A}$ satisfies $e(G[M]) > (\pi(F) + \frac{\epsilon}{2})\binom{m}{k} > ex(m, F)$, we know $G[M]$ has at least one copy of F. As each F can be contained in at most $\binom{n-v}{m-v}$ choices of $M \subset \mathcal{A}$, we finally get the number of F-copies in G is at least $\frac{|\mathcal{A}|}{\binom{n-v}{m-v}} = \frac{\frac{\epsilon}{2}\binom{n}{m}}{(1-\frac{\epsilon}{2}-\pi(F))\binom{n-v}{m-v}} = \delta n^v$.

∎