Extremal and Probabilistic Graph Theory

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1 Lecture 3

1.1 Blowup Lemma

Given a k-graph F, the *t-blowup* $F(t)$ is a k-graph obtained from F by replacing each $v \in V(F)$ by an independent subset $I(v)$ of size t and replacing every edge $\{v_1, v_2, ..., v_k\} \in E(F)$ by a complete k-partite k-graph $I(v_1), I(v_2), ..., I(v_k)$. For example, the t-blowup of K_r is $K_r(t) = T_r(rt)$.

Theorem 1.1 (Blowup Lemma)**.** *For any k-graph F and $t \geq 1$, we have $\pi(F(t)) = \pi(F)$. In other words,* $\text{ex}(n, F) \leq \text{ex}(n, F(t)) \leq \text{ex}(n, F) + \varepsilon n^k$ *for any $t, \varepsilon > 0$ and $n \geq n(t, \varepsilon)$.*

Proof. Let $f = |V(F)|$. First, clearly we have $\text{ex}(n, F) \leq \text{ex}(n, F(t))$ since $F \subseteq F(t)$. Then we can assume that for sufficiently large n, there exist $\varepsilon > 0$ and an $F(t)$-free k-graph G on n vertices, with $e(G) > \text{ex}(n, F) + \varepsilon n^k$. By supersaturation lemma, G contains at least $\delta\binom{n}{f}$ copies of F where $\delta = \delta(\varepsilon, F)$.

Next we define an anxiliary f-graph G^* on $V(G)$ where $X \in \binom{n}{f}$ is an edge of G^* if and only if $G[X]$ contains a copy of F. So $e(G^*) \geq \frac{\delta n^f}{f!} = \Omega(n^f)$. Take an integer T such that $n \gg T \gg t, f$. As $e(G^*) = \Omega(n^f) > \text{ex}(n, K_{T:f})$, by hypergraph Kövari-Sós-Turán theorem, we see G^* has at least one copy of $K_{T:f}$.

There are $f!$ possible ways to embed a copy of F in an edge of $K_{T:f}$. Now we give $f!$ colores to the edges of $K_{T:f}$. Notice that one color stands for one possible embedding. By pigeonhole priciple, there is a color whose number of edges is at least $\frac{T^f}{f!} > \text{ex}(vT, K_{t:f})$. So $K_{T:f}$ contains a monochromatic copy of $K_{t:f}$. This copy of $K_{t:f}$ gives a copy of $F(t)$ in G. ∎

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k such that $V(G)$ can be partitioned into k independent sets.

Fact 1.2. *$\chi(G) \leq k \Leftrightarrow G$ is is k-partite.*

Fact 1.3. *From the blowup lemma, for any $m \geq 1$,*

$$\pi(T_r(rm)) = \pi(K_r) = 1 - \frac{1}{r-1}.$$

1.2 Erdős-Stone-Simonovits Theorem

Theorem 1.4 (Erdős-Stone)**.** *If $\chi(F) = r$, then $\pi(F) = \pi(K_r) = 1 - \frac{1}{r-1}$.*

Proof. There exists an integer m such that $F \subseteq T_r(rm)$. So $\text{ex}(n, F) \leq \text{ex}(n, T_r(rm))$, implying that $\pi(F) \leq \pi(T_r(rm)) = \pi(K_r) = 1 - \frac{1}{r-1}$. For the lower bound, since $\chi(F) = r$, we see $T_{r-1}(n)$ is F-free. Then $\text{ex}(n, F) \geq e(T_{r-1}(n))$. Combining them, we get

$$\pi(F) = \pi(K_r) = 1 - \frac{1}{r-1} = 1 - \frac{1}{\chi(F) - 1}$$

∎

Obviously, the above theorem has the following version.

Theorem 1.5 (Erdős-Stone). *For any graph F and n,* $\mathrm{ex}(n,F) = (1 - \frac{1}{\chi(F)-1} + o(1))\binom{n}{2}$ *where* $o(1) \to 0$ *as* $n \to +\infty$.

Let $\mathcal{F}$ be a family of graphs. Let the chromatic number $\chi(\mathcal{F}) = \min_{F\in\mathcal{F}} \chi(F)$.

Theorem 1.6 (Erdős-Stone; observed by Simonovits). *For any family $\mathcal{F}$ of graphs,*

$$\pi(\mathcal{F}) = 1 - \frac{1}{\chi(\mathcal{F}) - 1}.$$

Let us see some easy observations.

1. For any family $\mathcal{F}$ of graphs, $\pi(\mathcal{F}) \in \{0, \frac{1}{2}, \frac{2}{3}, ..., \frac{r-1}{r}, ...\}$.
2. For any graph F, $\mathrm{ex}(n,F) = (1 - \frac{1}{\chi(F)-1} + o(1))\binom{n}{2}$. When $\chi(F) = 2$(i.e. F is bipartite), this becomes $\mathrm{ex}(n,F) = o(n^2)$.

The problem of finding $\mathrm{ex}(n,F)$ for bipartite graphs F is call *degenerate Tuán problem.*

1.3 Quantitative Version of Erdős-Stone-Simonovits Theroem

Erdős-Stone-Simonovits theorem says $\mathrm{ex}(n, T_r(rm)) \leq (1 - \frac{1}{r-1} + \varepsilon)\binom{n}{2}$. That means for fixed m, ϵ, it holds for large n. Now we consider the ounterpart that for fixed n, ε, how large m can be?

For convenince, let us define a function

$$f_r(n,\varepsilon) = \max\left\{m : \mathrm{ex}(n, T_r(rm)) \leq \mathrm{ex}(n, K_r) + \varepsilon\binom{n}{2} - 1\right\}.$$

For functions f, g, we write $f \lesssim g$ if $\lim_{n\to+\infty} \frac{f(n)}{g(n)} \leq 1$ and $f \sim g$ if $\lim_{n\to+\infty} \frac{f(n)}{g(n)} = 1$.

The Erdős-Renyí random graph $G(n,p)$ for $0 \leq p \leq 1$ is a graph on n vertices, where each pair of vertices forms an edge with probability p, independently at random. In particular, $G(n, \frac{1}{2})$ can be viewed as an equally distributed probability space which consists of all labelled n-vertex graphs.

Theorem 1.7 (upper bound proved by Bollobás-Erdős).

$$\log_{1/\varepsilon} n \lesssim f_2(n,\varepsilon) \lesssim 2\log_{1/\varepsilon} n$$

Proof. First consider the lower bound. Let $m = \log_{1/\varepsilon} n$, so $n^{-1/m} = \varepsilon$. Thus

$$\mathrm{ex}(n, T_2(2m)) = \mathrm{ex}(n, K_{m,m}) \leq \frac{1}{2}(m-1)^{1/m} n^{2-1/m} + \frac{1}{2}(m-1)n \leq \frac{\varepsilon n^2}{2}.$$

This proves $\log_{1/\varepsilon} n \lesssim f_2(n,\varepsilon)$.

Second, let $t = 2\log_{1/\varepsilon} n$. To show $f_2(n,\varepsilon) < t$, we need to prove $\mathrm{ex}(n, K_{t,t}) > \varepsilon\binom{n}{2} - 1$. It suffices to construct a n-vertex $K_{t,t}$-free graph G with at least $\varepsilon\binom{n}{2} - 1$ edges. Consider Erdős-Renyí random graph $G(n,\varepsilon)$. Let X be the number of $K_{t,t}$ in $G(n,\varepsilon)$. We have

$$\mathbb{E}[X] = \frac{1}{2}\binom{n}{2t}\binom{2t}{t}\varepsilon^{t^2} < n^{2t}\varepsilon^{t^2} = (n^2\varepsilon^t)^2.$$

Since $\varepsilon^t = \varepsilon^{2\log_{1/\varepsilon} n} = n^{-2}$, we see $\mathbb{E}[X] < 1$. By average, there exists a graph G such that $e(G) - X \geq \mathbb{E}[e(G) - X] > \varepsilon\binom{n}{2} - 1$. Let G' be obtained from G by deleting one edge for each copy of $K_{t,t}$ in G. Then G' is $K_{t,t}$-free with $e(G') \geq e(G) - X > \varepsilon\binom{n}{2} - 1$. ∎

The best general bound is used by Szemeredi's regularity lemma.

Theorem 1.8 (Ishigami)**.** *For any $r \geq 2$ and $\varepsilon = o(1)$, we have $f_r(n,\varepsilon) \sim f_2(n,\varepsilon)$. Thus*

$$\log_{1/\varepsilon} n \lesssim f_r(n,\varepsilon) \lesssim 2\log_{1/\varepsilon} n.$$

Theorem 1.9. *For any $\varepsilon \in (0, \frac{1}{r(r+1)})$ where r is fixed,*

$$f_{r+1}(n,\varepsilon) \leq f_2\left(\lceil \frac{n}{r} \rceil, r(r+1)\varepsilon\right).$$

Proof. Assume not. Let $t = f_2(\lceil \frac{n}{r} \rceil, r(r+1)\varepsilon) + 1$, then there exists a $K_{t,t}$-free $\lceil \frac{n}{r} \rceil$-vertex graph H with $e(H) > r(r+1)\varepsilon\binom{\lceil \frac{n}{r} \rceil}{2} - 1$. Let G be obtained from $T_r(n)$ by adding H into a part of size $\lceil \frac{n}{r} \rceil$. We claim that G is $T_{r+1}((r+1)t)$-free (prove it as an exercise). Thus we have

$$\mathrm{ex}(n, T_{r+1}((r+1)t)) \geq e(G) = e(T_r(n)) + e(H) \geq \mathrm{ex}(n, K_{r+1}) + \varepsilon\binom{n}{2}.$$

By definition, $f_{r+1}(n,\varepsilon) \leq t - 1 = f_2(\lceil \frac{n}{r} \rceil, r(r+1)\varepsilon)$. ∎