Extremal and Probabilistic Graph Theory

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1 Lecture 5

In this lecture, we are going to prove triangle removal lemma and Roth's theorem as the applications of regularity lemma.

1.1 Triangle Removal Lemma

Lemma 1.1. *Let (A, B) be an ε-regular pair with $d = d(A, B)$, and let $Y \subseteq B$ have size $|Y| \geq \varepsilon|B|$. Then, all but fewer than $\varepsilon|A|$ of the vertices in A have at least $(d - \varepsilon)|Y|$ neighbors in Y.*

Proof. Let $X \subseteq A$ be the set of vertices with fewer than $(d - \varepsilon)|Y|$ neighbors in Y. Then $d(X, Y) < d - \varepsilon$. So $|d(X, Y) - d(A, B)| > \varepsilon$. Since (A, B) is ε-regular and $|Y| \geq \varepsilon|B|$, this implies that $|X| < \varepsilon|A|$. ∎

Applying the above lemma, we can easily get the following one.

Lemma 1.2. *Assume A, B, C are disjoint vertex sets such that $d(A, B) = c$, $d(A, C) = b$, $d(B, C) = a$, where $a, b, c \geq 2\varepsilon$, and the three bipartite graphs are ε-regular. Then the tripartite graph (A, B, C) has at least $(1 - 2\varepsilon)(a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|A||B||C|$ triangles.*

Theorem 1.3 (Triangle removal lemma)**.** *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if one has to remove at least εn^2 edges from an n-vertex graph G in order to make it triangle-free, then G has at least δn^3 triangles.*

Proof. Apply regularity lemma to G with $\varepsilon/3$, we get an $(\varepsilon/3)$-regular partition $V_1, \cdots, V_k$, where $\varepsilon/3 \leq k \leq T(\varepsilon/3)$. Now we remove from G the following 3 types of edges:

(1) Edges that belongs to some V_i.

(2) Edges between non-$(\varepsilon/3)$-regular pairs (V_i, V_j).

(3) Edges between $(\varepsilon/3)$-regular pairs (V_i, V_j) with $d(V_i, V_j) \leq 2\varepsilon/3$.

So the total number of edges we removed is at most

$$k\binom{n/k}{2} + \frac{\varepsilon}{3}\left(\frac{n}{k}\right)^2\binom{k}{2} + \frac{2\varepsilon}{3}\left(\frac{n}{k}\right)^2\binom{k}{2} \leq \frac{n^2}{2k} + \frac{\varepsilon n^2}{3} + \frac{\varepsilon n^2}{3} \leq \frac{5\varepsilon}{6}n^2.$$

By the condiction, the new graph, say G', still contains some triangles. Let xyz be such a triangle where $x \in V_1$, $y \in V_2$ and $z \in V_3$. So we see each of (V_1, V_2), (V_1, V_3), (V_2, V_3) is an $(\varepsilon/3)$-regular pair with density at least $2\varepsilon/3$. By Lemma 1.2, we see the number of triangles in G is at least

$$\left(1 - \frac{2\varepsilon}{3}\right)\left(\frac{\varepsilon}{3}\right)^3 |V_1||V_2||V_3| \geq \frac{1}{2}\left(\frac{\varepsilon}{3}\right)^3\left(\frac{n}{k}\right)^3 \geq \frac{1}{2}\left(\frac{\varepsilon}{3T(\varepsilon/3)}\right)^3 n^3 \triangleq \delta(\varepsilon)n^3.$$

∎

1.2 Roth's Theorem

Theorem 1.4 (Roth's theorem, 1953)**.** *If $S \subseteq [n]$ does not contain any non-trivial 3-term arithmetic progression,* [1] *then $|S| = o(n)$.*

Proof (by Ruzsa-Szemerédi). We will show that for any $\varepsilon > 0$, such S has size at most εn for $n \geq n(\varepsilon)$. Therefore, $|S| = o(n)$.

Given any $S \subseteq [n]$ with $|S| \geq \varepsilon n$ (for large n), we will show that S contains some non-trivial 3-term arithmetic progression. To see this, we construct a 3-partite graph $G = (A, B, C)$ with $A = [n]$, $B = [2n]$ and $C = [3n]$. For any $s \in S$ and $x \in [n]$, we put a triangle $T_{x,s}$ in G on the vertices $x \in A, x + s \in B$ and $x + 2s \in C$. So we put $|S|n \geq \varepsilon n^2$ triangles $T_{x,s}$ and we claim that they are all edge-disjoint. It is because given any edge, we can uniquely recover the corresponding x, s and thus the triangle $T_{x,s}$. Therefore, one needs to delete at least εn^2 edges to make G triangle-free. By triangle removal lemma, G has at least δn^3 triangles. Since $n \geq n_0$ is sufficiently large, $\delta n^3 \gg |S|n$, thus we have a triangle in G, say $xyzx$, which is not of the type $T_{x,s}$. The 3 edges of this triangle must come from 3 different $T_{x,s}$. Suppose $x \in A$, $y = x + s_1 \in B$ and $z = y + s_2 \in C$ such that $z = x + 2s_3$, where s_1, s_2, s_3 are distinct. Then we have $x + s_1 + s_2 = z = x + 2s_3$, implying that $s_1 + s_2 = 2s_3$. So this (s_1, s_2, s_3) forms a non-trival 3-term arithmetic progression in S. ∎

[1] A 3-term arithmetic progression is *non-trivial* if the common difference is at least one.