Extremal and Probabilistic Graph Theory

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1 Lecture 6. Graph Embedding Lemma.

Let G be a graph and $\{V_1, V_2, ..., V_k\}$ be an ε-regular equipartition with $|V_i| \in \{\ell, \ell+1\}$. Given $d \in [0,1]$, let R be the graph with the vertex set $\{V_1, V_2, ..., V_k\}$ where $V_iV_j \in E(R)$ if and only if (V_i, V_j) is an ε-regular pair in G with $d(V_i, V_j) \geq d$. We call such R a *regularity graph* (or *reduce graph*) of G with parameters ε, ℓ and d.

The following lemma says that if some blowup $R(S)$ of R contains a given graph H, then assuming each V_i is large enough, G also contains a copy of H.

Theorem 1.1 (Graph Embedding Lemma)**.** *For every $d \in (0,1]$ and $\Delta \geq 1$, there exists an ε_0 such that the following holds: if G is any graph, H is a graph with $\Delta(H) \leq \Delta$, and R is a regularity graph of G with parameter $\varepsilon \leq \varepsilon_0$, $\ell \geq 2s/d^\Delta$ and d, then*

$$H \subseteq R(s) \ \Rightarrow \ H \subseteq G.$$

Proof. Given d and Δ, we choose $\varepsilon_0 > 0$ small enough such that $\varepsilon_0 < d$ and

$$(d-\varepsilon_0)^\Delta - \Delta\varepsilon_0 \geq \frac{1}{2}d^\Delta.$$

Note that such ε_0 exists, as $(d-\varepsilon_0)^\Delta - \Delta\varepsilon_0 \to d^\Delta$ as $\varepsilon_0 \to 0$. Let $\{V_1, V_2, ..., V_k\}$ be the ε-regular partition of G with $\varepsilon \leq \varepsilon_0$. Let R be the corresponding regularity graph of G with the above parameters ε, ℓ, d. By condition, H is contained in some blowup $R(s)$ of R, where each vertex V_i in R is replaced by an s-set V_i^s. Suppose H has vertices $u_1, u_2, ..., u_h$ where each u_i lies in the s-set $V_{\sigma(i)}^s$ of $R(s)$. In this way, we define a mapping $\sigma : [h] \to [k]$.

Our goal is to define an embedding $u_i \mapsto v_i \in V_{\sigma(i)}$ such that $v_1, v_2, ..., v_h$ are distinct, and $v_iv_j \in E(G)$ when $u_iu_j \in E(H)$. We will choose the vertices $v_1, v_2, ..., v_h$ in a recursive process, where we pick one vertex v_i in the i^{th} iteration. Throughout this process, we shall maintain a "target set" $Y_i \subseteq V_{\sigma(i)}$ assigned to each u_i: this contains the vertices that are still candidates for the choice of v_i. Initially, we let $Y_i = V_{\sigma(i)}$. The set Y_i will evolve as

$$V_{\sigma(i)} = Y_i^0 \supseteq Y_i^1 \supseteq ... \supseteq Y_i^i = \{v_i\}.$$

As the embedding proceeds, $\{Y_i^j\}_{i \geq j}$ will be updated simultaneously for each j. After we choose v_j at the end of the j^{th} iteration, we update $Y_i^{j-1} \to Y_i^j$ for all $i \geq j$ as follows.

$$Y_i^j = \begin{cases} \{v_j\} & \text{if } i = j \\ N(v_j) \cap Y_i^{j-1} & \text{if } i > j \text{ and } u_ju_i \in E(H) \\ Y_i^{j-1} & \text{otherwise} \end{cases}$$

To make this approach work, we have to choose v_j carefully to make sure that the target sets Y_i^j are not small. This can be solved by using Lemma 1.1 from the last lecture. Provided that

$|Y_i^{j-1}| \geq \varepsilon\ell$, $i > j$ and $u_iu_j \in E(H)$, applying this lemma for $A = V_{\sigma(j)}$, $B = V_{\sigma(i)}$ and $Y = Y_i^{j-1}$, we see that all but at most $\varepsilon\ell$ of vertices x in $V_{\sigma(j)}$ satisfy $|N(x) \cap Y_i^{j-1}| \geq (d-\varepsilon)|Y_i^{j-1}|$. Since there are at most Δ such i, we know all but $\Delta\varepsilon\ell$ of vertices v_j in $V_{\sigma(j)}$ satisfy $|N(v_j) \cap Y_i^{j-1}| \geq (d-\varepsilon)|Y_i^{j-1}|$ for all such i. Therefore, as long as $|Y_j^{j-1}| - \Delta\varepsilon\ell \geq s$, we have a good choice of v_j in Y_j^{j-1} such that v_j is distinct from $\{v_1, ..., v_{j-1}\}$, and $|Y_i^j| = |N(v_j) \cap Y_i^{j-1}| \geq (d-\varepsilon)|Y_i^{j-1}|$ for all $i > j$ with $u_iu_j \in E(H)$. We have $|Y_i^0| = |V_{\sigma(i)}| \geq \ell$ and each Y_i shrinks by at most Δ times by a factor of $(d-\varepsilon)$. Thus

$$|Y_i^{j-1}| - \Delta\varepsilon\ell \geq \ell(d-\varepsilon)^\Delta - \Delta\varepsilon\ell \geq \ell(d-\varepsilon_0)^\Delta - \Delta\varepsilon_0\ell \geq \frac{\ell}{2}d^\Delta \geq s.$$

for all $j \leq i$. In particular, we have $|Y_i^{j-1}| \geq \varepsilon\ell$ for all $i \geq j$ and $|Y_j^{j-1}| - \Delta\varepsilon\ell \geq s$ as desired. ∎

Now we can give a new proof of Erdős-Stone-Simonivits Theorem.

Theorem 1.2 (Erdős-Stone-Simonivits)**.** *For any graph H with $\chi(H) = r+1$,*

$$\text{ex}(n, H) = (1 - \frac{1}{r} + o(1))\binom{n}{2}.$$

Proof. For a graph H with $\chi(H) = r+1$, there exists an integer h such that $H \subseteq K_{r+1}(h)$. So it suffices to show for every $\delta > 0$, there exists $n_0 = n_0(\delta, r, h)$ such that if G has $n \geq n_0$ vertices and at least $(1 - \frac{1}{r} + o(1))\binom{n}{2}$ edges, it contains a $K_{r+1}(h)$.

Let $d = \delta/4$ and $\Delta = r$. Then let ε_0 be from the embedding lemma for d and Δ. Let $\varepsilon = \min\{\varepsilon_0, \delta/4\}$. Applying regularity lemma to G with ε, we get an ε-regular partition $(V_1, ..., V_k)$ with $1/\varepsilon \leq k \leq T(\varepsilon)$ and $|V_i| = n/k \triangleq \ell \geq n/T(\varepsilon)$. Remove the following 3 types of edges:

(1) Edges inside each V_i.

(2) Edges between non-ε-regular pairs (V_i, V_j).

(3) Edges between ε-regular pairs (V_i, V_j) with $d(V_i, V_j) \leq \frac{\delta}{4}$.

So the total number of edges deleted is at most

$$k\binom{n/k}{2} + \varepsilon\binom{k}{2}\left(\frac{n}{k}\right)^2 + \frac{\delta}{4}\binom{k}{2}\left(\frac{n}{k}\right)^2 \leq \frac{n^2}{2k} + \varepsilon n^2 + \frac{\delta}{8}n^2 \leq \frac{\delta}{2}n^2.$$

Thus the remaining graph G' still has at least $(1 - \frac{1}{r})\binom{n}{2}$ edges. By Turán's Theorem, G' contains a copy of K_{r+1}. We may assume K_{r+1} has vertices $x_1, ..., x_{r+1}$, where $x_i \in V_i$. Then all $1 \leq i, j \leq r+1$, (V_i, V_j) is ε-regular with $d(V_i, V_j) \geq \delta/4$. Apply the embedding lemma with $d = \delta/4$ and $\Delta = r$. If every $|V_i|$ is sufficiently large, then we get $K_{r+1}(h) \subseteq R(h) \subseteq G$. Now all we need is to make sure the size of V_i is large as

$$\varepsilon n \geq \frac{n}{k} = \ell \geq \frac{2h}{d^\Delta} \Leftrightarrow n \geq \frac{2h}{\varepsilon d^\Delta} \geq \frac{2h}{\min\{\varepsilon_0, \delta/4\}(\delta/4)^r}.$$

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