Extremal and Probabilistic Graph Theory

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1 Lecture 7. Graph counting lemma and removal lemma.

Theorem 1.1 (Graph Counting Lemma). *Let H be a graph with $V(H) = [h]$. For $\varepsilon > 0$, let G be a graph with verex subsets $V_1, ..., V_h \subseteq V(G)$ such that (V_i, V_j) is ε-regular whenever $ij \in E(H)$. Then the number of tuples $(v_1, ..., v_k) \in V_1 \times ... \times V_k$ such that $v_i v_j \in E(G)$ whenever $ij \in E(H)$ is*

$$\left(\prod_{ij \in E(H)} d(v_i, v_j) \pm \varepsilon \cdot e(H) \right) \prod_{i \in [h]} |V_i|.$$

Proof 1: We skip the detail of the proof, instead leaving a sketch of it. Use induction on number of vertices of H. It is trival for $h = 2$. So assume $h > 2$. Fix an $x \in V_1$. We will define the following subset $U_i \subseteq V_i$ for $j \neq 1$. If $j \in N(1)$, then $U_j = V_j \cap N(x)$. Otherwise, let $U_j = V_j$. Then by induction, we get the number of tuples $(u_2, u_3, ..., u_k) \in U_2 \times ... \times U_k$ such that $u_i u_j \in E(G)$ whenever $ij \in e(H \backslash \{1\})$. Note that, these $(x, u_2, u_3, ..., u_k)$ are tuples we wanted. ∎

Proof 2: This can be rephrased into the following probabilistic form. Choose $v \in V_1, \cdots, v_k \in V_k$ uniformly and independently at random. If we let $p = \mathbb{P}(v_i v_j \in E(G)$ for all $ij \in E(H))$, then the conclusion in the theorem is the same as $|p - \prod_{ij \in E(H)} d(V_i, V_j)| \leq \varepsilon \cdot e(H)$.

Assume $12 \in E(H)$. Let $p' = \mathbb{P}\left(v_i v_j \in E(G) \text{ for all } ij \in E(H) \backslash \{12\}\right)$. Here we claim that $|p - d(V_1, V_2)p'| \leq \varepsilon$. If it holds, then we can do induction on $e(H)$ as follows. Let $H' = H \backslash \{12\}$. We have

$$\begin{aligned} \left| p - \prod_{ij \in E(H)} d(V_i, V_j) \right| &\leq \left| p - d(V_i, V_j) p' \right| + d(V_1, V_2) \left| p' - \prod_{ij \in E(H')} d(V_i, V_j) \right| \\ &\leq \varepsilon + d(V_1, V_2)(\varepsilon \cdot e(H')) \leq \varepsilon \cdot e(H), \end{aligned}$$

as desired.

So it suffices to show the claim holds whenever $v_3, \cdots, v_k$ are fixed arbitrarily and only v_1, v_2 are random. Define

$$A_1 = \{v_1 \in V_1 : v_1 v_j \in E(G) \text{ whenever } j \in N_H(1) \backslash \{2\}\},$$

$$A_2 = \{v_2 \in V_2 : v_2 v_j \in E(G) \text{ whenever } j \in N_H(2) \backslash \{1\}\}.$$

Therefore, for us it suffices to show

$$\left| \frac{e(A_1, A_2)}{|V_1||V_2|} - d(V_1, V_2) \frac{|A_1||A_2|}{|V_1||V_2|} \right| \leq \varepsilon.$$

If $|A_1| \le \varepsilon|V_1|$ or $|A_2| \le \varepsilon|V_2|$, then $\frac{e(A_1,A_2)}{|V_1||V_2|} \le \frac{|A_1||A_2|}{|V_1||V_2|} \le \varepsilon$ and $d(V_1,V_2)\frac{|A_1||A_2|)}{|V_1||V_2|} \le \varepsilon$. Thus the inequality holds. Otherwise, we have $|A_1| \ge \varepsilon|V_1|$ and $|A_2| \ge \varepsilon|V_2|$. By ε-regularity of (V_1, V_2),

$$\left|\frac{e(A_1,A_2)}{|V_1||V_2|} - d(V_1,V_2)\frac{|A_1||A_2|}{|V_1||V_2|}\right| = \left|\frac{e(A_1,A_2)}{|A_1||A_2|} - d(V_1,V_2)\right| \frac{|A_1||A_2|}{|V_1||V_2|} \le \varepsilon.$$

This completes the proof. ∎

Theorem 1.2 (Graph Removal Lemma)**.** *For every graph H and $\varepsilon > 0$, there exists a constant $\delta = \delta(H,\varepsilon) > 0$ such that any n-vertex graph with less then $\delta n^{|V(H)|}$ copies of H can be made H-free by deleting at most εn^2 edges.*

The proof is similar to the triangle removal lemma (one can use the graph counting lemma to prove the graph removal lemma). We leave this proof to the reader.