Extremal and Probabilistic Graph Theory

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1 Lecture 8. Erdős-Simonovits Stability Theorem

1.1 Stability

For two graphs G and H with the same number of vertices, the *edit-distance* $d(G, H)$ is the minimum integer k such that G can be obtained from H by adding or deleting k edges.

Theorem 1.1 (Füredi, 2015)**.** *If G is an n-vertex K_{r+1}-free graph with $e(G) = e(T_r(n)) - t$, then there exists an r-partite subgraph H of G such that $e(H) \geq e(G) - t$.*

Proof. We use the so-called Erdős degree majorization algorithm, which will find a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ such that $\sum_{i=1}^{r} e(G[V_i]) \leq t$. Let $x_1 \in V(G)$ be a vertex of maximum degree in G, and let $V_1^+ = N(x_1)$ and $V_1 = V(G) \backslash V_1^+$. So

$$2e(G[V_1]) + e(V_1, V_1^+) = \sum_{u \in V_1} d_G(u) \leq |V_1| \cdot |V_1^+|.$$

Suppose we have defined V_{i-1}^+. Let $x_i \in V_{i-1}^+$ be a vertex of maximum degree in $G[V_{i-1}^+]$. Let $V_i^+ = V_{i-1}^+ \cap N(x_i)$ and $V_i = V_{i-1}^+ \backslash V_i^+$, so $V_{i-1}^+ = V_i \cup V_i^+$ and $x_i \in V_i$. Note that $V(G) = V_1 \cup V_2, \cup \cdots \cup V_i \cup V_i^+$ and $x_j \in V_j$ for any $j \in [i]$. Also

$$2e(G[V_i]) + e(V_i, V_i^+) = \sum_{u \in V_i} d_{G[V_{i-1}^+]}(u) \leq |V_i| \cdot |V_i^+|. \tag{1.1}$$

Observe that $G[\{x_1, x_2, \cdots, x_i\}]$ is a clique. So this procedure will stop in s steps (until $V_s^+ = \emptyset$) for some integer $s \leq r$.

Summing up (1.1) for all $i \in [s]$. Since $V_i^+ = V_{i+1} \cup V_{i+2} \cup \cdots \cup V_s$, we have

$$\begin{aligned} e(G) + \sum_{i=1}^{s} e(G[V_i]) &= \sum_{i=1}^{s} \left(2e(G[V_i]) + e(V_i, V_i^+)\right) \leq \sum_{i=1}^{s} |V_i| \cdot |V_i^+| \\ &= e(K(V_1, V_2, \cdots, V_s)) \leq e(T_s(n)) \leq e(T_r(n)). \end{aligned}$$

Thus, $\sum_{i=1}^{s} e(G[V_i]) \leq t$. ∎

Corollary 1.2. *Suppose G is K_{r+1}-free with $e(G) \geq e(T_r(n)) - t$. Then there is a complete r-partite graph $K = K(V_1, V_2, \cdots, V_r)$ with $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ satisfying $d(G, K) \leq 3t$.*

Proof. By Theorem 1.1, there is an r-partite subgraph H of G with parts $V_1, V_2, \cdots, V_r$ such that $d(H, G) \leq t$. Let $K = K(V_1, V_2, \cdots, V_r)$. Then $e(H) \geq e(T_r(n)) - 2t \geq e(K) - 2t$. This means by adding at most $2t$ edges, one can turn H into a complete r-partite K. Since $d(H, G) \leq t$ and $d(H, K) \leq 2t$, we see $d(G, K) \leq d(G, H) + d(H, K) \leq 3t$. ∎

Theorem 1.3 (Erdős-Simonovits Stability Theorem)**.** *For every $\varepsilon > 0$ and every graph F with $\chi(F) = r+1$, there exists a real $\delta > 0$ and n_0 such that if G is F-free with $n \geq n_0$ vertices and $e(G) \geq (1-\frac{1}{r})\binom{n}{2} - \delta n^2$, then $d(G, T_r(n)) \leq \varepsilon n^2$.*

Proof Sketch. The proof can be completed in the following 5 steps:

1. Use regularity lemma to get a regularity graph R of size k.
2. Observe that $e(R) \geq (1-1/r)\binom{k}{2} - o(k^2)$.
3. Use embedding lemma to show that R is K_{r+1}-free.
4. By the stability for K_{r+1}, we get the structure of R.
5. Finally, we obtain the structure of G.

∎

1.2 Ramsey Number

Given two graphs K and H, the *Ramsey number* $R(K, H)$ is the smallest integer N so that every 2-edge-coloring (blue/red) of K_N contains either a blue K or a red H.

Theorem 1.4 (Ramsey Theorem)**.** *Every 2-edge-coloring of K_{4^n} contains a monochromatic copy of K_n.*

In fact, it is not hard to prove the following slightly stronger result, using induction.

Proposition 1.5.

$$R(K_s, K_t) \leq \binom{s+t-2}{s-1}.$$

Since $R(K, H) \leq R(K_{|V(K)|}, K_{|V(H)|})$ holds for any graphs K, H, we see that $R(K, H)$ is always finite.

For graphs K, H with bounded maximum degree, we can have a much better upper bound on $R(K, H)$ as follows.

Theorem 1.6. *If K, H are graphs on n vertices with maximum degree Δ, then there is a constant $d = d(\Delta)$ such that $R(K, H) \leq dn$.*

Proof. Consider any 2-edge-coloring of K_{dn}. Let K_B be the subgraph of K_{dn} consisting of blue edges. Let $\varepsilon = \varepsilon(1/2, \Delta)$ and $c = c(1/2, \Delta)$ be from the embedding lemma. Let $m = 4^{\Delta+1}$ and $\varepsilon' = \min\{\varepsilon, 1/(4m)\}$. Applying the regularity lemma on K_B with ε', we get a ε'-regular partition $V_1, V_2, \cdots, V_k$ with $1/\varepsilon' \leq k \leq T(\varepsilon')$. There are $k \geq 1/\varepsilon' \geq 4m$ parts and at most $\varepsilon' k^2 \leq k^2/(4m)$ pairs are not ε'-regular. Therefore, at least $\frac{k^2}{2}(1-\frac{1}{2m})$ pairs are ε'-regular. Let R be the regularity graph. So $e(R) \geq \binom{k}{2} - \frac{k^2}{4m} \geq \frac{k^2}{2}(1-\frac{1}{m})$. By Turán theorem, R contains a K_m, say $V_1, V_2, \cdots, V_m$, which are pairwisely ε'-regular. Considering the following 2-edge-coloring on this K_m: color (i, j) by black if $d_B(V_i, V_j) \geq 1/2$, otherwise color it by white. Note that if (i, j) is white, the red edges in (V_i, V_j) is ε'-regular with density at least $1/2$. Since $m = 4^{\Delta+1}$, by the Ramsey theorem, we get a monochromatic (black/white) $K_{\Delta+1}$, say $V_1, \cdots, V_{\Delta+1}$. That says, either all the density of blue edges in (V_i, V_j) are at least $1/2$ or all the density of red edges in (V_i, V_j) are at least $1/2$. By the embedding lemma, each case would give us a monochromatic (blue/red) $K_{\Delta+1}(s)$ in G, which implies a blue H or red K. ∎