Extremal and Probabilistic Graph Theory

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1 Lecture 9.

1.1 Induced Embedding Lemma

Recall the Embedding lemma as follows.

Theorem 1.1 (Embedding Lemma). *For every $d \in (0,1]$ and Δ, there exists a constant $\varepsilon = \varepsilon(d,\Delta)$ such that the following holds: for any graph H with $\Delta(H) \leq \Delta$, if $V_1, V_2, \cdots, V_{\chi(H)}$ are disjoint sets of size at least $|V(H)|/\varepsilon$ and each (V_i, V_j) is ε-regular with $d(V_i, V_j) \geq d$, then one can find a copy of H in $(V_1, V_2, \cdots, V_{\chi(H)})$.*

Now we state an induced version of Embedding lemma .

Theorem 1.2 (Induced Embedding Lemma). *For every $c \in (0, 1/2]$ and h, there is a constant $\varepsilon = \varepsilon(c,h)$ such that the following holds: for any graph H with h vertices, if $V_1, V_2, \cdots, V_h$ are disjoint sets of size at least $(1/\varepsilon)^2$ and each (V_i, V_j) is ε-regular with density in $[c, 1-c]$, then one can find an induced copy of H in $(V_1, V_2, \cdots, V_h)$.*

Proof (hint). If (V_i, V_j) is ε-regular with density in $[c, 1-c]$, then the complement of (V_i, V_j) is ε-regular with density in $[c, 1-c]$ too. So it suffices to show that a proper designed graph on $(V_1, V_2, \cdots, V_h)$ has a copy of K_h, which is implied by the Embedding Lemma. ∎

1.2 Induced Ramsey Theorem

We are familar with Ramsey Theorem. One may ask that if there is a graph R such that any 2-edge-coloring of it contains a monochromatic induced copy of H for a fixed graph H. Specially, when $H = K_h$, it is the same as before and we know such R can be chosen as K_{4^h}. What if H is not a clique? Then R can't be a clique.

Theorem 1.3 (Induced Ramsey Theorem). *For any graph H, there exists a host graph $R = R(H)$ such that any 2-edge-coloring of R contains a monochromatic induced copy of H.*

Proof. Our proof will show that for all graphs H on h vertices, there exists a common host graph $R \sim G(n, 1/2)$ for $n \gg h$. To begin with, let us introduce some notation. Set $\varepsilon = \varepsilon(0.2, h)$ from Induced Embedding Lemma with $c = 0.2$ and h. Let $m = 4^h$ and $\tilde{\varepsilon} = \min\{\frac{\varepsilon}{3}, \frac{1}{4m}\}$. Let $T = T(\tilde{\varepsilon})$ where $T(\cdot)$ is from regularity lemma.

We will prove the following statement: if G is an n-vertex graph such that

(1) $n/T \geq (1/\varepsilon)^2$.

(2) For any disjoint pair (A, B) with $|A|, |B| \geq \frac{\varepsilon n}{T}$, we have $|d(A,B) - \frac{1}{2}| \leq \frac{\varepsilon}{3}$,

then G can be taken as the host graph $R(H)$.

First, we remark that random graphs $G \sim G(n, 1/2)$ for some $n \gg h$ satisfy (1) and (2). One can check it by applying Chernoff's inequality and union bound.

Then we return to the statement. Consider any 2-edge-coloring of G. Let G_0 be the subgraph of all blue edges and G_1 be that of red edges. Applying regularity lemma for $\tilde{\varepsilon}$ to G_0, we get an $\tilde{\varepsilon}$-regular partition $V_1, \cdots, V_k$ with $|V_i| = \frac{n}{k}$, where $4m \leq 1/\tilde{\varepsilon} \leq k \leq T(\tilde{\varepsilon}) \triangleq T$, such that at most $\varepsilon k^2 \leq \frac{k^2}{4m}$ pairs (V_i, V_j) are not $\tilde{\varepsilon}$-regular in G_0. By Turán's Theorem, there are m sets say $V_1, \cdots, V_m$ such that any pair (V_i, V_j) is $\tilde{\varepsilon}$-regular in G_0 for $1 \leq i < j \leq m$, and of course they are $\varepsilon/3$-regular.

Now we claim that any pair (V_i, V_j) in G_1 is ε-regular for $1 \leq i \leq j \leq m$. If $U_i \subseteq V_i$ and $U_j \subseteq V_j$ satisfy $|U_i| \geq \varepsilon |V_i| \geq \varepsilon n/T$ and $|U_j| \geq \varepsilon |V_j| \geq \varepsilon n/T$, then

$$\begin{aligned} &|d_{G_1}(U_i, U_j) - d_{G_1}(V_i, V_j)| = |d_G(U_i, U_j) - d_{G_0}(U_i, U_j) - d_G(V_i, V_j) + d_{G_0}(V_i, V_j)| \\ \leq &|d_G(U_i, U_j) - 1/2| + |d_G(V_i, V_j) - 1/2| + |d_{G_0}(U_i, U_j) - d_{G_0}(V_i, V_j)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \leq \varepsilon, \end{aligned}$$

which confirms the claim.

Note tha ramsey $R(K_h, K_h) \leq 4^h \leq m$. Applying this on the m sets $V_1, \cdots, V_m$, there exist h sets, say $V_1, \cdots, V_h$, such that one of the following holds:

(a) each $d_{G_0}(V_i, V_j) \in [1/4, 0.51]$ for $1 \leq i < j \leq h$.

(b) each $d_{G_0}(V_i, V_j) \in [0, 1/4]$ for $1 \leq i < j \leq h$.

Assume case (a) holds. We now pick some edges between the clusters $V_1, \cdots, V_h$ to form a new graph G^* as follows. For $1 \leq i < j \leq h$, take all edges in $G_0(V_i, V_j)$ if $(i, j) \in H$; otherwise, take all edges in $G(V_i, V_j)$. Now G^* satisfies the condition in the Induced Embedding Lemma (for $c = 0.2$ and h). So there is an induced copy of H in G^*, and this is also an induced blue copy of H. Next we assume case (b) occurs. Then each $d_{G_1}(V_i, V_j) \in [1/4, 0.51]$ for $1 \leq i < j \leq h$. Similarly, we can define a new graph G^* on $V_1, \cdots, V_h$ such that it contains all edges in $G_1(V_i, V_j)$ if $(i, j) \in H$, and all edges in $G(V_i, V_j)$ when $(i, j) \notin H$. Therefore we can get an induced red copy of H in G. This finishes the proof of the statement and the theorem. ∎