Extremal and Probabilistic Graph Theory

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1 Lecture 11. Tao's Spectral Proof of Regularity Lemma.

Definition 1.1. Given a graph G with $V(G) = [n]$, its *adjacency matrix* $A = A_G$ is an $n \times n$ matrix such that $A(i,j) = 1$ if $ij \in E(G)$ and $A(i,j) = 0$ otherwise.

Theorem 1.2 (Spectral Theorem)**.** *If G is simple, then A_G has n orthononal eigenvectors $\vec{u}_1, \cdots, \vec{u}_n$ with real eigenvalues $\lambda_1, \cdots, \lambda_n$*

Proof. This is because A_G is symmetric real $n \times n$ matrix. ∎

Definition 1.3. the *spectral radius* of G is the maximum eigenvalue of A_G.

Proposition 1.4. *(1) $tr(A_G) = \sum_{i\in[n]} \lambda_i = 0$ and*

$$\sum_{i\in[n]} \lambda_i^2 tr(A^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} A(i,j)^2 = 2e(G) \le n^2.$$

(2) If we let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then we have $\lambda_1 \ge -\lambda_n$ and

$$\max_i |\lambda_i| = \lambda_1 = \max_{\vec{x}\neq\vec{0}} \frac{|\vec{x}^T A\vec{x}|}{\vec{x}^T\vec{x}}.$$

(3)

$$\max_i |\lambda_i| = \max_{\vec{y}\neq\vec{0}} \frac{|A\vec{y}|}{|\vec{y}|} = \max_{\vec{x},\vec{y}\neq\vec{0}} \frac{|\vec{x}^T A\vec{y}|}{|\vec{x}||\vec{y}|}.$$

Theorem 1.5 (Spectral version of regularity lemma; Tao's proof)**.** *For any $\varepsilon > 0$, there exists a constant $M(\varepsilon) > 0$ such that the following holds for any n-vertex graph G: There exists a partition $V_1 \cup \cdots \cup V_M = V(G)$ where $M \le M(\varepsilon)$ and a subset $\Sigma \subseteq [M]^2$ such that*

- *$\sum_{(i,j)\in\Sigma} |V_i||V_j| \le \varepsilon n^2$, and*
- *for all $(i,j) \in [M]^2 \backslash \Sigma$ and for all $A \subseteq V_i$ and $B \subseteq V_j$, $\big|e(A,B) - d_{ij}|A||B|\big| \le \varepsilon |V_i||V_j|$.*

Proof. Let $\vec{u}_1, \cdots, \vec{u}_n \in \mathbb{R}^n$ be orthonornal eigenvectors of A_G with real eigenvalues $\lambda_1, \cdots, \lambda_n$. We arrange them such that $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$. So $A_G = \sum_{i\in[n]} \lambda_i \vec{u}_i \vec{u}_i^T$. We have the following lemmas.

Lemma 1.6. $|\lambda_i| \le n/\sqrt{i}$ *for any $i \in [n]$.*

Proof. We have $i\lambda_i^2 \le \sum_{j=1}^{i} \lambda_j^2 \le tr(A^2) \le n^2$ and thus $\lambda_i \le n/\sqrt{i}$. ∎

Lemma 1.7. *Fix a function* $F : \mathbb{N} \to \mathbb{N}$ *such that* $F(x) \geq \frac{1}{\varepsilon^6}\left(\frac{2x^2}{\varepsilon^2}\right)^{4x}$. *Then there exists* $c = c(\varepsilon, F) = c(\varepsilon)$ *such that there is some integer* $J < c$ *satisfying* $\sum_{J \leq i < F(J)} \lambda_i^2 \leq \varepsilon^3 n^2$.

Proof. Let $J_1 = 1$ and $J_{i+1} = F(J_i)$. We cannot have $\sum_{J_k \leq i < J_{k+1}} \lambda_i^2 > \varepsilon^3 n^2$ for all $k \leq 1/\varepsilon^3$ (otherwise $\sum \lambda_i^2 > n^2$, a contradiction). Therefore, there exists $J = J_k$ for some $k \leq 1/\varepsilon^3$ with the desired inequality. Note that J is bounded by $c = F(F(\cdots F(1)))$. ∎

Now we partition $A_G = \sum_{i \in [n]} \lambda_i \vec{u}_i \vec{u}_i^T$ into the following 3 matrices $A_G = A_1 + A_2 + A_3$, where A_1 is the "structured" component with $A_1 = \sum_{i<J} \lambda_i \vec{u}_i \vec{u}_i^T$, A_2 is the "error" component with $A_2 = \sum_{J \leq i < F(J)} \lambda_i \vec{u}_i \vec{u}_i^T$, and A_3 is the "pseudorandom" component with $A_3 = \sum_{i \geq F(J)} \lambda_i \vec{u}_i \vec{u}_i^T$.

Step 1. Partitioning $V(G)$ by considering A_1.

Goal: We will partition vertices of G into a bounded number of parts V_i's such that any $(a,b) \in V_i \times V_j$, $A(a,b) \sim d_{ij}$ is approximately a constant. We will achieve this by giving a partition based on each $\vec{u}_i$ with $i < J$ as following.

Let $\vec{u}_i = (x_1, x_2, ..., x_n)^T$ and let $L^{(i)} = \left\{k \in [n] : |x_k| > \sqrt{\frac{J}{\varepsilon}} n^{-\frac{1}{2}}\right\}$ be an exceptional set of $\vec{u}_i$. Note that $1 = |\vec{u}_i|^2 = \sum_{i \in [n]} x_i^2$. So $|L^{(i)}| \leq \frac{\varepsilon n}{J}$. The other $x_k \in [-\sqrt{\frac{J}{\varepsilon}} n^{-\frac{1}{2}}, \sqrt{\frac{J}{\varepsilon}} n^{-\frac{1}{2}}] = I$. Partition this interval I into subintervals of length $\left(\frac{\varepsilon}{J}\right)^{\frac{3}{2}} n^{-\frac{1}{2}}$. This gives $\frac{2J^2}{\varepsilon^2}$ many subintervals. Partition $[n] \setminus L^{(i)}$ into $\frac{2J^2}{\varepsilon^2}$ subsets according to which subinterval x_k belongs to. For each $i < J$, let $\mathcal{U}_i$ be the partition of $[n] = V(G)$ formed by $L^{(i)}$ and the above $\frac{2J^2}{\varepsilon^2}$ subsets.

Let $V_0 = \bigcup_{i<J} L^{(i)}$ be the set of all exceptional subsets. Let $\{V_1, V_2, ..., V_M\}$ be the unique minimal common refinement of $\mathcal{U}_i \setminus V_0 = \{F \setminus V_0 : F \in \mathcal{U}_i\}$ for all $i < J$. Then we have a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_M$ where

$$|V_0| \leq (J-1)\frac{\varepsilon n}{J} \leq \varepsilon n \text{ and } M \leq (2J^2/\varepsilon^2)^J. \tag{1.1}$$

Note that the upper bound of M only depends on ε. Further, consider any $j < J$, $i \in [M]$ and any $a, b \in V_i$. We have

$$|\vec{u}_j(a)| \leq \sqrt{\frac{J}{\varepsilon}} n^{-\frac{1}{2}} \text{ and } |\vec{u}_j(a) - \vec{u}_j(b)| \leq \left(\frac{\varepsilon}{J}\right)^{\frac{3}{2}} n^{-\frac{1}{2}}.$$

For any pairs $(a,b), (c,d) \in V_i \times V_j$ where $i, j \in [M]$, we have

$$\begin{aligned}
|A_1(a,b) - A_1(c,d)| &= |\sum_{k<J} \lambda_k \vec{u}_k(a)\vec{u}_k(b) - \sum_{k<J} \lambda_k \vec{u}_k(c)\vec{u}_k(d)| \\
&\leq \sum_{k<J} |\lambda_k||\vec{u}_k(a)\vec{u}_k(b) - \vec{u}_k(c)\vec{u}_k(b) + \vec{u}_k(c)\vec{u}_k(b) - \vec{u}_k(c)\vec{u}_k(d)| \\
&\leq Jn\left(|\vec{u}_k(a) - \vec{u}_k(c)||\vec{u}_k(b)| + |\vec{u}_k(b) - \vec{u}_k(d)||\vec{u}_k(c)|\right) \\
&\leq Jn\left(2\left(\frac{\varepsilon}{J}\right)^{\frac{3}{2}} n^{-\frac{1}{2}} \sqrt{\frac{J}{\varepsilon}} n^{-\frac{1}{2}}\right) = 2\varepsilon.
\end{aligned}$$

Therefore, if we let d_{ij} be the average of the entries in the block $V_i \times V_j$ of the matrix A_1, then by triangle inequality, for any $A \subseteq V_i$, $B \subseteq V_j$ where $i, j \in [M]$, we have

$$\left|\mathbb{1}_A^T A_1 \mathbb{1}_B - d_{ij}|A||B|\right| = \Big|\sum_{a\in A}\sum_{b\in B}(A_1(a,b) - d_{ij})\Big| \le \sum_{a\in A}\sum_{b\in B}|A_1(a,b) - d_{ij}| \le 2\varepsilon|A||B| \le 2\varepsilon|V_i||V_j|. \tag{1.2}$$

Step 2. Consider A_2 and A_3.

Consider A_2. By Lemma 1.7, we have

$$tr(A_2^2) = \sum_{J\le i<F(j)} \lambda_i^2 \le \varepsilon^3 n^2.$$

Let $\Sigma_1 = \{(i,j) \in [M]^2 : \sum_{a\in V_i}\sum_{b\in V_j}|A_2(a,b)|^2 > \varepsilon^2|V_i||V_j|\}$. Then

$$\varepsilon^2 \sum_{(i,j)\in\Sigma_1}|V_i||V_j| < \sum_{(i,j)\in\Sigma_1}\sum_{a\in V_i}\sum_{b\in V_j}|A_2(a,b)|^2 \le tr(A_2^2) \le \varepsilon^3 n^2.$$

Thus

$$\sum_{(i,j)\in\Sigma_1}|V_i||V_j| \le \varepsilon n^2. \tag{1.3}$$

For any $(i,j) \notin \Sigma_1$, $\sum_{a\in V_i}\sum_{b\in V_j} A_2(a,b)^2 \le \varepsilon^2|V_i||V_j|$. By Cauchy-Schwarz, for any $A \subseteq V_i$, $B \subseteq V_j$,

$$\left(\mathbb{1}_A^T A_2 \mathbb{1}_B\right)^2 = (\sum_{a\in V_i}\sum_{b\in V_j} A_2(a,b))^2 \le |A||B|\sum_{a\in V_i}\sum_{b\in V_j} A_2(a,b)^2 \le \varepsilon^2|A||B||V_i||V_j| \le \varepsilon^2|V_i|^2|V_j|^2.$$

Thus for any $A \subseteq V_i$, $B \subseteq V_j$ where $(i,j) \notin \Sigma_1$,

$$\left|\mathbb{1}_A^T A_2 \mathbb{1}_B\right| \le \varepsilon|V_i||V_j|. \tag{1.4}$$

Consider A_3. By Lemma 1.6, the spectral radius of A_3 is $|\lambda_{F(J)}| \le n/\sqrt{F(J)}$. For any $A \subseteq V_i$ and $B \subseteq V_j$, since $M \le (2J^2/\varepsilon^2)^J$ (by (1.1)) and $F(J) \ge \frac{1}{\varepsilon^6}(2J^2/\varepsilon^2)^{4J} \ge \frac{M^4}{\varepsilon^6}$, by proposition 1.4 we have

$$\left|\mathbb{1}_A^T A_3 \mathbb{1}_B\right| \le \left|\lambda_{F(J)}\right| |\mathbb{1}_A||\mathbb{1}_B| \le \frac{n^2}{\sqrt{F(J)}} \le \frac{\varepsilon^3 n^2}{M^2}. \tag{1.5}$$

Step 3. Combining all together. Let Σ be the set consisting of pairs $(i,j) \in \{0,1,...,M\}^2$ such that $(i,j) \in \Sigma_1$, or $i = 0$, or $j = 0$, or $\min\{|V_i|,|V_j|\} \le \frac{\varepsilon n}{M}$. Then by (1.1) and (1.3),

$$\sum_{(i,j)\in\Sigma}|V_i||V_j| \le \sum_{(i,j)\in\Sigma_1}|V_i||V_j| + 2|V_0||V| + 2\sum_{i\ne 0,|V_i|\le\frac{\varepsilon n}{M}}|V_i||V_j| \le \varepsilon n^2 + 2\varepsilon n^2 + 2M\left(\frac{\varepsilon n}{M}\right)n = 5\varepsilon n^2.$$

For $(i,j) \notin \Sigma$, we have $|V_i|, |V_j| \ge \frac{\varepsilon n}{M}$. So by (1.2),(1.4) and(1.5), for any $A \subseteq V_i$, $B \subseteq V_j$ where $(i,j) \notin \Sigma$, we have

$$\begin{aligned}|e(A,B) - d_{ij}|A||B|| = \left|\mathbb{1}_A^T A_G \mathbb{1}_B - d_{ij}|A||B|\right| &\le \left|\mathbb{1}_A^T A_1 \mathbb{1}_B - d_{ij}|A||B|\right| + \left|\mathbb{1}_A^T A_2 \mathbb{1}_B\right| + \left|\mathbb{1}_A^T A_3 \mathbb{1}_B\right| \\ &\le 2\varepsilon|V_i||V_j| + \varepsilon|V_i||V_j| + \varepsilon^3 n^2/M^2 \le 4\varepsilon|V_i||V_j|.\end{aligned}$$

This completes the proof of the theorem. ∎

Remark. One can turn this spectral version of regularity lemma into the normal version of regularity lemma, but we will not pursue here.