Extremal and Probabilistic Graph Theory

Instructor: Jie Ma, Scribed by Yanzhi Li and Tianchi Yang

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1 Lecture 12. Extremal Bounds on Paths and Posa's Rotation.

We now turn to extremal problems (mostly on Turán numbers) for bipartiate graphs. Let us start by considering trees, paths and cycles.

Proposition 1.1. *For $n \geq k+2$, any n-vertex graph G with at least $kn - \frac{k^2+k-2}{2}$ edges contains a subgraph of minimum degree at least $k+1$. (If $e(G) \geqslant kn$, then there is such a subgraph.)*

Proposition 1.2. *For any tree T on $t+1$ vertices, $ex(n,T) < (t-1)n$.*

Proof. Let G be an n-vertex graph with at least $(t-1)n$ edges. By Proposition 1.1, there is a subgraph G' of G with $\delta(G') \geq \leftrightarrow \forall x \in St$. Using a greedy algorithm, one can find any tree on $t+1$ vertices. ∎

Conjecture 1.3 (Erdős-Sós)**.** *Let T be any tree on $t+1$ vertices. Then $ex(n,T) \leq (t-1)n/2$.*

Here are two remarks towards the conjecture. Note that a vertex-disjoint union of cliques K_t show that the inequality is tight for some n if the conjecture is ture. An approximate version of this conjecture was confirmed by Ajtai-Komlós-Simonovits-Szemerédi.

Definition 1.4. Denote the *length* of a path P by $|P|$, which is the number of edges in P. Let P_t be the path of length t.

Proposition 1.5. *Any graph G has a path of length at least $\delta(G)$.*

Definition 1.6. Let P be a path in G from u to v. For $x \in V(P)$, denote x^- and x^+ to be the immediate predecessor and immediate successor of x on P. For $S \subseteq V(P)$ let $S^+ = \{x^+ : x \in S\}$ and $S^- = \{x^- : x \in S\}$.

Definition 1.7. Let P be a longest path in G from u to v. For $w \in N(v)$, the path $P' = P - \{ww^+\} + \{wv\}$ is also a longest path in G. This transformation from P to P' is called a *Pósa rotation.*

Definition 1.8. A path or a cycle is called *Hamiltonian* if it contains all vertices of the graph G. And G is *Hamiltonian* if G has such a cycle.

Proposition 1.9. *Let G be connected and P be a longest path in G. If there exists a cycle C with $V(C) = V(P)$, then G is Hamiltonian.*

Proof. Suppose $V(C) \neq V(G)$. As G is connected, there exists a vertex $a \notin V(C)$ and a path Q from a to $b \in V(C)$ internally disjoint from $V(C)$. Then we can find a longer path than P, a contradiction. ∎

Theorem 1.10. *If G is connected, then G has a path with at least* $\min\{n, 2\delta(G)+1\}$ *vertices.*

Proof. Let $P = x_0x_1 \cdots x_n$ be a longest path in G. So $N(x_0), N(x_m) \subseteq V(P)$. Suppose $|V(P)| < \min\{n, 2\delta(G)+1\}$. We claim that there is an $i \in \{0, 1, ..., m-1\}$ such that $x_0x_{i+1}, x_mx_i \in E(G)$. Suppose not. Then $N(x_0) \cap N(x_m)^+ = \emptyset$. Also $x_0 \notin N(x_0) \cup N(x_m)^+$, we have $|V(P)| \geq 1 + |N(x_0) \cup N(x_m)^+| \geq 1 + d(x_0) + d(x_m) \geq 1 + 2\delta(G)$, a contradiction. So we can find a cycle $C = P - \{x_ix_{i+1}\} + \{x_0x_{i+1}, x_mx_i\}$ with $V(C) = V(P)$. By Proposition 1.9, G is hamiltonian. ∎

Now we consider a special case of Erdős-Sós Conjecture.

Theorem 1.11 (Erdős-Gallai)**.** *For $n \geq t$, $ex(n, P_t) \leq (t-1)n/2$.*

Proof. We prove by induction on n. It's trivial for $n \leq t$. For $n \geq t+1$, let G be a P_t-free graph on n vertices. We want to show $e(G) \leq (t-1)n/2$. We may assume that G is connected. If $\delta(G) \geq t/2$, by Theorem 1.10, G has a path with at least $\min\{n, 2\delta(G)+1\} \geq \min\{n, t+1\} \geq t+1$ vertices, a contradiction. So there is a vertex v of degree at most $(t-1)/2$. Let $G' = G - \{v\}$. By inducion, we know $e(G') \leq (t-1)(n-1)/2$, thus $e(G) = e(G') + d(v) \leq (t-1)n/2$. ∎

Note that the unique extremal graph of P_t is a disjoint union of K_t. (exercise)

Theorem 1.12 (Ore's Theorem)**.** *Let G be an n-vertex graph such that for any non-adjacent vertices u and v, $d(u) + d(v) \geq n$. Then G is Hamiltonian.*

Proof. First, we see G is connected. Let P be a longest path in G with endpoints u, v. By Proposition 1.9, we may assume $uv \notin E(G)$. So $d(u) + d(v) \geq n$ implies that $N(u) \cap N(v)^+ \neq \emptyset$, then there exists a cycle C with $V(C) = V(P)$. Again by Proposition 1.9, G is Hamiltonian. ∎

Corollary 1.13 (Dirac's Theorem)**.** *If $\delta(G) \geq |V(G)|/2$, then G is Hamiltonian.*

Definition 1.14. The *closure* of a graph G is the graph obtained from G by recursively joining pairs of non-adjacent vertices u, v whose degree sum is at least n until no such pairs exist.

Proposition 1.15. *The closure of G is well defined, that is, the ordering of adding edges will not affect the final graph.*

Theorem 1.16 (Bondy-chvátal)**.** *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

Proof. It suffices to show: For any non-adjacent $\{u, v\}$ with $d_G(u) + d_G(v) \geq n$, G is hamiltonian if and only if $G + \{uv\}$ is Hamiltonian. (exercise) ∎

This theorem implies Theorem 1.12 directly.

Definition 1.17. For $S \subseteq V(G)$, define $N(S) = \{v \notin S : v \in N(w) \text{ for some } w \in S\}$ to be the neighborhood of S in G.

Lemma 1.18 (Pośa's lemma)**.** *Let P be a longest path in graph G with endpoints u, v. Let S be the set of all endpoints of paths obtained by repeatedly applying Posa's rotations from P, while preserving u as an endpoint. Clearly $S \subseteq V(P)$ and $N(S) \subseteq V(P)$. Let S^+ and S^- be subsets of $V(P)$ as before. Then $N(S) \subseteq S^+ \cup S^-$.*

Proof. It suffices to show that for any $x \in S$, we have $N(x) \subseteq S^+ \cup S \cup S^-$. Suppose not, there is a vertex $y \in N(x)$ but $y \notin S^+ \cup S \cup S^-$. Since $y \notin S^+ \cup S \cup S^-$, then y^-yy^+ is always a subpath of any new path obtained by Pósa's rotation. Since $xy \in E(G)$, we can use a Pósa's rotation to find a new longest P', which ends at y^+ or y^-. So y^+ or $y^- \in S$ implies $y \in S^+ \cup S^-$, a contradiction. ∎

Theorem 1.19. *Suppose any* $S \subseteq V(G)$ *satisfies* $N(S) \geq \min\{n - |S|, 2|S| + 1\}$. *Then* G *has a Hamiltonian path.*

Proof. Let P be a longest path in G. Define S as in the Pósa's lemma. So $2|S| \geq |N(S)| \geq \min\{n - |S|, 2|S| + 1\}$, which means $\min\{n - |S|, 2|S| + 1\} = n - |S|$ and $|N(S)| \geq n - |S|$. As $N(S) \cap S = \emptyset$, $|V(P)| \geq |N(S)| + |S| \geq n$, as desired. ∎