Extremal and Probabilistic Graph Theory

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1 Lecture 13. Extremal Bounds on Cycles

Theorem 1.1. *If G is a 2-connected graph, then G has a cycle of length at least* $\min\{|V(G)|, 2\delta(G)\}$.

Proof. Let $P = x_0x_1 \cdots x_m$ be a longest path in G. By an earlier theorem, $m \geq \min\{n-1, 2\delta(G)\}$.
Case 1. Suppose there exist two vertices $x_i \in N(x_0)$ and $x_j \in N(x_m)$ with $i > j$. If $i = j+1$, obviously we are done. So we assume $j \leq i-2$. Pick a pair $\{i, j\}$ with $|i-j|$ minimum. Let C be the cycle $x_0x_1 \cdots x_jx_mx_{m-1} \cdots x_ix_0$, then we have $N(x_0) \subseteq V(C)$, $(N(x_m) \setminus \{x_j\})^+ \subseteq V(C)$. Since $N(x_0) \cap N(x_m)^+ = \varnothing$, we see that

$$|C| \geq |N(x_0) \cup \{x_0\}| + |(N(x_m) \setminus \{x_j\})^+| \geq d(x_0) + d(x_m) \geq 2\delta(G).$$

Case 2. Suppose for any $x_j \in N(x_m), x_i \in N(x_0)$, we have $i \leq j$. Let i be the maximum index such that $x_i \in N(x_0)$, and let j be the minimum index such that $x_j \in N(x_j)$. Let $G_1 = G[V(x_0Px_i)], G_2 = G[V(x_jPx_m)]$. By Menger's Theorem, there are 2 internally disjoint paths P_1, P_2 from $V(G_1)$ to $V(G_2)$. If neither P_1 nor P_2 starts at x_i, we begin traveling from x_i along x_iPx_j, until we encounter some vertex of P_1 or P_2, then we would define two new paths one of which starts at x_i and do the same for x_j. Hence we can choose such P_1, P_2 such that each of x_i and x_j is an endpoint of P_1 or P_2.

Then there are two cases to consider: **(a)** $x_i, x_j \in V(P_1)$; **(b)** $x_i \in V(P_1), x_j \in V(P_2)$.
For (a), we can construct a cycle using P_1, P_2 and all vertices in $\{x_0, x_m\} \cup N(x_0) \cup N(x_m)$, that is $C = x_0x_1 \cdots x_sP_2x_t \cdots x_mx_l \cdots x_jP_1x_i \cdots x_kx_0$, where x_s, x_t are two endpoints of P_2, x_k is the first neighbor of x_0 after x_s and x_l is the first neighbor of x_m before x_t. Since it is possible that $x_i = x_j$,

$$|C| \geq d(x_0) + d(x_m) + 2 - 1 \geq 2\delta(G) + 1.$$

For (b), we can find a cycle of length at least $2\delta(G) + 1$ in a similar way. ∎

Remark: Consider a union of cliques K_{k+1}, where any pair of the cliques share 2 common vetices. The largest cycle in this graph has length $2k = 2\delta(G)$.

Theorem 1.2 (Erdös-Gallai)**.**

$$ex(n, \{C_{t+1}, C_{t+2}, \cdots\}) \leq t(n-1)/2.$$

This is tight for all n with $(t-1)|(n-1)$.

Proof. This can be derived from Theorem 1.1. (exercise) ∎

Theorem 1.3. *If G is a graph with $\delta(G) \geq d \geq 5$, then G has a cycle C and a subgraph H with $V(H) \subseteq V(C)$, $E(H) \subseteq E(G) - E(C)$, and $\delta(H) \geq d/6 + 1$.*

Proof. Let $P = x_0x_1 \cdots x_p$ be a longest path in G, and S be the set from Posá's Lemma. For each path $P' = v_0 \cdots v_p$ obtained from Posá's rotation, denote by $l(P')$ the minimum index i such that $v_i \in N(v_p)$.

For a new path Q from Posá's rotation with $l(Q)$ minimum among all new paths. Let $C = v_iQv_pv_i$ be the cycle. Then $S \subseteq V(C)$. By Posá's lemma, $|N(S)| \leq 2|S|$. Let H^* be a subgraph of G defined on $S \cap N(S)$, when $xy \in E(H^*)$ if and only if at least one of x and y is in S. Let H be obtained from H^* by deleting all edges in C and then deleting those isolated vertices which may result after the deletion of $E(C)$. Clearly, H is edge-disjoint from C and $V(H) \subseteq V(C)$. For any vertex $a \in S$, $d_H(x) \geq d_G(x) - d_C(x) \geq d - 2 \geq 3$, thus $S \subseteq V(H)$. Let $S^* = V(H) \setminus S$, so $S^* \subseteq N(S)$ and $|S^*| \leq 2|S|$.

Now it suffices to show that H has a subgraph F with $\delta(F) \geq d/6 + 1$. Suppose not, we have

$$e(H) < \frac{d}{6}|V(H)| = \frac{d}{6}(|S| + |S^*|). \tag{1.1}$$

On the other hand, $e(H) = \frac{1}{2}\sum_{x \in S} |N_H(x) \cap S| + \sum_{x \in S} |N_H(x) \cap S^*|$, where for any $x \in S$, $d_H(x) = |N_H(x) \cap S| + |N_H(x) \cap S^*| \geq d - 2 \geq 3$. Since there is no isolated vertices in $H \cap S^*$, $\sum_{x \in S} |N_H(x) \cap S^*| = e(S, S^*) \geq |S^*|$, implies that

$$e(H) \geq \frac{1}{2}\sum_{x \in S} d_H(x) + \frac{1}{2}\sum_{x \in S} |N_H(x) \cap S^*| \geq \frac{1}{2}(d-2)|S| + \frac{1}{2}|S^*|. \tag{1.2}$$

By inequalities 1.1 and 1.2, we get $(\frac{d}{6} - \frac{1}{2})|S^*| > (\frac{d}{3} - 1)|S|$, that is, $|S^*| > 2|S|$, a contradiction. ∎

Corollary 1.4. *Let $f(k) = \frac{6}{5}(4 \cdot 6^{k-2} + 1)$. Let G be any graph with $\delta(G) \geq f(k)$. Then G has k edge-disjoint cycles $C_1, C_2, \cdots, C_k$ with $V(C_1) \supseteq V(C_2) \supseteq \cdots \supseteq V(C_k)$. (nested)*

Corollary 1.5. *Let G be a graph on $n \geq f(k) + 1$ vertices. If G has at least $[(f(k) - 1)n - f(k)] \cdot (f(k) - 1)/2 + 1$ edges, then G has k edge-disjoint cycles $C_1, C_2, \cdots, C_k$ with $V(C_1) \supseteq V(C_2) \supseteq \cdots \supseteq V(C_k)$.*

Definition 1.6. Let $g(n)$ be the smallest integer such that any n-vertex graph with at least $g(n)$ edges contains two edge-disjoint cycles C_1 and C_2 with $V(C_1) \subseteq V(C_2)$.

Question 1.7 (Chen-Erdös-Staton, 1994, open)**.** *$g(n) = 3n - 6$, for any $n \geq 6$.*

Consider the graph $P_2 \bigoplus I_{n-3}$, obtained by adding two edges to the 3-vertex part of $K_{3,n-3}$. It has $3(n-3) + 2 = 3n - 7$ edges and only contains cycles of length 4, so $g(n) \geq 3n - 6$.

Question 1.8 (Erdös)**.** *How many edges or what min-degree will force the existence of a cycle with as many chords as its vertices?*

Show that $\delta(G) \geq 2\sqrt{n}$ will do in your exercise.

Question 1.9. *How many edges are necessary to force the existence of 2 edge-disjoint cycles with the same vertex set? (The real question is: if $\delta(G) \geq C$, where C is constant, there exist such cycles.)*