Extremal and Probabilistic Graph Theory

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1 Lecture 14, Bondy-Simonovits Theorem on even cycles

We consider the upper bound of $\text{ex}(n, C_{2k})$ for $k \geq 2$ in this lecture.

Theorem 1.1 (Bondy-Simonovits)**.** *There is a constant $c > 0$ such that for any $k \geq 2$,*

$$\text{ex}(n, C_{2k}) \leq ckn^{1+1/k}.$$

Remark: The orignal proof gives $c = 100$.

In the following lecture, we will give two different proofs of Theorem 1.1. Let us get into the first one by introducing the A-B path lemma.

Theorem 1.2 (A-B path Lemma)**.** *Let H be a graph consisting of a cycle with a chord, and let (A, B) be a non-trivial partition of $V(H)$. Then for any $\ell < |V(H)|$, there is an (A, B)-path of length ℓ in H, unless ℓ is even and H is bipartite with the partition (A, B).*

The first proof of Theorem 1.1. Let the cycle $C = (0, 1, \ldots, n-1, 0)$ with chord $(0, r)$. We take indices under modulos n. Denote $\chi : V(H) \to \{0, 1\}$ by $\chi(i) = 1$ for $i \in A$ and $\chi(i) = 0$ for $i \in B$. Let $P = \{p \in Z_n^+ : \chi(i) = \chi(i+p)$ holds for any $i\}$. So if $\ell \notin P$, we can find an (A, B)-path of length ℓ using only the edges of C.

It suffices for us to consider $\ell \in P$. Let $m \in P$ be the smallest positive integer in P. Then $m|n$ (exercise). For all ℓ with $m \nmid \ell$, there exists some (A, B)-path of length ℓ.(By the definition of m.) So we only need to consider $\ell = km$.

Case 1: Suppose the chord $(0, r)$ satisfies that $1 < r \leq m$. Since $m \nmid (m + r - 1)$, there is some $-m < j \leq 0$ such that $\chi(j) \neq \chi(j + m + r - 1) = \chi(j + km + r - 1)$. Consider the path $(j, j+1, \ldots, 0, r, r+1, \ldots, j+m+r-1, \ldots, j+km+r-1)$. This is an (A, B)-path of length $km = \ell$.

Case 2: Suppose $m < r < n - m$. For $-m \leq j \leq 0$, we define 2 paths: $P_j = (j, j+1, \ldots, 0, r, r-1, \ldots, r-j-m+1)$ and $Q_j = (m+j, m+j-1, \ldots, 0, r, r+1 \ldots, r-j-1)$. We see both paths have length m.

(i) Suppose there is a j with $-m \leq j \leq 0$ such that P_j or Q_j is an (A, B)-path. Then we can extend it to an (A, B)-path of length $km = \ell$ by adding a subpath of length m at a time.

(ii) We may assume that P_j and Q_j are not (A, B)-paths for all $-m \leq j \leq 0$. Then we have $\chi(j) = \chi(r-j-m+1)$, $\chi(m+j) = \chi(r-j-1)$ for any $-m \leq j \leq 0$. So $\chi(r-j+1) = \chi(r-j-1)$, for any $-m \leq j \leq 0$. That is $\chi(i) = \chi(i+2)$ for any i. Then for $m = 2$, we have $2|n$ and the vertices of C alternate between A and B. If the chord $(0, r)$ is in the same part, we can check that H contains A-B paths of all possible lengths. Otherwise, the chord $(0, r)$ is between A and B, then H is bipartite with the partition (A, B).

Case 3: $n - m \leq r < n - 1$. This case is the same as **Case 1**. ∎

Proof of Theorem 1.1. We will show

$$\mathrm{ex}(n, C_{2k}) \le 2kn^{1+1/k} + 6(k-1)n.$$

Let G be an n-vertex C_{2k}-free graph with more than $2kn^{1+1/k} + 6(k-1)n$ edges. Then G has a bipartite subgraph H' with $e(H') > kn^{1+1/k} + 3(k-1)n$. Further, H' contains a bipartite subgraph H with $\delta(H) > kn^{1/k} + 3(k-1)$. Let T be a breadth-first search tree (BFS tree) with root x in H. Let $L_i = \{u \in V(H) : d_H(x,u) = i\}$ for $i \ge 1$. Since H is bipartite, each L_i is stable.

First we claim that $e(L_{i-1}, L_i) \le (k-1)(|L_{i-1}| + |L_i|)$ for each $1 \le i \le k$. Suppose not, $e(L_{i-1}, L_i) > (k-1)(|L_{i-1}| + |L_i|)$ for some $i \ge 2$. Then $H(L_{i-1}, L_i)$ contains a subgraph H_1 with $\delta(H_1) \ge k$. Then H_1 has an even cycle C of length at least $2k$ with a chord. Let $A = V(C) \cap L_{i-1}$ and $B = V(C) \cap L_i$. Let T' be a subtree of T such that $A \subseteq V(T')$ and subject to this, T' is minimal. Let y be the root of T'. As T' is minimal, y has at least 2 branches. Let A' be the subset of A formed by all vertices from one branch of T'. Then $A \setminus A' \neq \emptyset$. Let $B' = B \cup (A \setminus A')$. Then (A', B') is not a bipartition of H_1. Let ℓ be the distance between x and y. Then $\ell < i - 1$ and $2k - 2i + 2\ell + 2 < 2k \le |V(C)|$. By A-B path Lemma, we can find an (A', B')-path P of length $2k - 2i + 2\ell + 2$ in H_1 between $a \in A'$ and $b \in B'$. As $|P|$ is even, $b \in A \setminus A'$. Let P_a, P_b be the unique paths in T' that connect y to a and b respectively. Then $P \cup P_a \cup P_b$ is a cycle of length $2k$ in H, a contradiction.

Next we show that $|L_i| \ge n^{1/k}|L_{i-1}|$ for any $i \in [k]$. We prove this by induction on i. Base case $i = 1$ is trivial since $\delta(H) > kn^{1/k} + 3(k-1)$. For $i \ge 2$, we have

$$\begin{aligned}(kn^{1/k} + 3(k-1))|L_{i-1}| \le \sum_{v \in L_{i-1}} d_H(v) &= e(L_{i-2}, L_{i-1}) + e(L_{i-1}, L_i) \\ &\le (k-1)(|L_{i-2}| + 2|L_{i-1}| + |L_i|) \le (k-1)(3|L_{i-1}| + |L_i|).\end{aligned}$$

So $|L_i| \ge \frac{kn^{1/k}}{k-1}|L_{i-1}| \ge n^{1/k}|L_{i-1}|$, as desired. Now we see $|L_k| \ge n$, a contradiction. ∎

Next, we move into the second proof of Theorem 1.1.

Lemma 1.3 (Lemma 2.6 in [1])**.** *Let H be a connected graph where each edge is colored by color 1 or color 2. Suppose that there is at least one edge of each color. If the number of edges of color 1 is at least $(p+1)|V(H)|$, then there exists a path of length p in H, whose first edge is colored by color 2 and all other edges are colored by color 1.*

Proof. Exercise. ∎

The second proof of Theorem 1.1. This is gave by Jiang-Ma in [1]. We aim to show

$$\mathrm{ex}(n, C_{2k}) \le 8kn^{1+1/k} + 24kn.$$

Let G be a n-vertex C_{2k}-free graph with more than $8kn^{1+1/k} + 24kn$ edges. Then G has a bipartite subgraph H' with $e(H') > 4kn^{1+1/k} + 12kn$. Further, H' contains a bipartite subgraph H with $\delta(H) > 4kn^{1/k} + 12k$. Similarly, let T be a breadth-first search tree (BFS tree) with root x in H. Let $L_i = \{u \in V(H) | d_H(x,u) = i\}$ for $i \ge 1$. Since H is bipartite, each L_i is stable.

First we claim that $e(L_{i-1}, L_i) \le 4k(|L_{i-1}| + |L_i|)$ for each $1 \le i \le k$. Suppose not, $e(L_{i-1}, L_i) > 4k(|L_{i-1}| + |L_i|)$ for some $i \ge 2$. Take a connected component H^* with $d(H^*) \ge 8k$ in $H(L_{i-1}, L_i)$. Let T' be a subtree of T with $V(H^*) \cap L_{i-1} \subseteq V(T')$, and subject to this, T' is minimal. Let X be the subset of $V(H^*) \cap L_{i-1}$ which formed by all vertices from one branch of T'. Let $Y = (V(H^*) \cap L_{i-1}) \setminus X$. Color all edges in H^* by color 1 if it has an end in X and by color 2 if it has an end in Y. Then we can assume that the number of edges with color 1 is at least $2k|V(H^*)|$. By Lemma 1.3, there is a path P of length at least $2k-1$ whose first edge is colored by color 2 and all other edges are colored by color 1. So we can find consecutive even cycles of length $2t+2, 2t+4, \ldots, 2t+2k-2$ where t is the distance between L_{i-1} and the root of T'. Since $t < i \leq k$, there is a cycle of length $2k$, a contradiction.

Next, we claim that $|L_i| \geq n^{1/k}|L_{i-1}|$ for any $i \in [k]$. We prove this by induction on i. Base case $i = 1$ holds as $\delta(H) > 4kn^{1/k} + 12k$. For $i \geq 2$, we have

$$\begin{aligned}(4kn^{1/k} + 12k)|L_{i-1}| \leq \sum_{v \in L_{i-1}} d_H(v) &= e(L_{i-2}, L_{i-1}) + e(L_{i-1}, L_i) \\ &\leq 4k(|L_{i-2}| + 2|L_{i-1}| + |L_i|) \leq 4k(3|L_{i-1}| + |L_i|),\end{aligned}$$

then $|L_i| \geq n^{1/k}|L_{i-1}|$. Finally, we get $|L_k| \geq n$, a contradiction.

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In the end, let us give some remarks. The current best bound on $\mathrm{ex}(n, C_{2k})$ is as follows.

Theorem 1.4 (Bukh-Jiang, 2016)**.**

$$\mathrm{ex}(n, C_{2k}) \leq 80\sqrt{k}\log k \cdot n^{1+1/k} + 10k^2 n.$$

Their proof heavily replies on A-B path Lemma.

Conjecture 1.5 (Erdős-Simonovits)**.** *For $k \geq 2$,*

$$\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+1/k}).$$

This conjecture is known for $k = 2, 3, 5$ only.

References

[1] T. Jiang and J. Ma, Cycles of given lengths in hypergraphs, *J. Combin. Theory Ser. B* **133** (2018), 54–77.