Extremal and Probabilistic Graph Theory

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1 Applications of Pósa's Rotation

Theorem 1.1 (Bondy-Simonovits). *There exists a constant $c > 0$ such that for any $k \geq 2$, $ex(n, C_{2k}) \leq ckn^{1+1/k}$.*

Lemma 1.2 (Pósa's lemma). *Let P be a longest path (from u to v) in a graph G. Let S be the set of all endpoints of paths obtained by repeatedly applying Pósa's rotations from P, while preserving u as an endpoint. Clearly, $S \subset V(P)$ and $N(S) \subset V(P)$. Then $N(S) \subset S^+ \cup S^-$*

1.1 Finding cycles of consecutive even lengths

Theorem 1.3 (Sudakov-Verstraëte). *Let G be a graph with average degree $d(G)$ and with girth $g(G) \geq 2g + 1$. Then G has cycles of $\Omega(d^g)$ consecutive even lengths.*

Lemma 1.4 (Exercise in Hw4). *Let G be a graph. If $|N(X)| > 2|X|$ for all subsets $X \subset V(G)$ with $|X| \leq k$, then G contains a cycle C of length at least $\min\{3k, |V(G)|\}$, which contains a vertex x and all its neighbors in G.*

Definition 1.5. Let

$$n_0(d, g) = \begin{cases} 1 + d\sum\limits_{i=0}^{r-1}(d-1)^i & if\ g = 2r+1 \\ 2\sum\limits_{i=0}^{r-1}(d-1)^i & if\ g = 2r. \end{cases}$$

This is called Moore Bound.

Lemma 1.6 (Moore Bound). *Let G be an n-vertex graph with girth g and with min-degree at least d. Then $n \geq n_0(g, d)$.*

The proof is assigned as a homework.(Consider a BFS-tree)

Lemma 1.7. *Let G be a graph with $\delta(G) \geq 6(d+1)$ and with girth at least $2g+1$. Then for any subset $X \subseteq V(G)$ with $|X| \leq d^g/3$, we have $|N(X)| > 2|X|$.*

Proof. Suppose for a contradiction that there is a set X with $|X| \leq \frac{d^g}{3}$ but $|N(X)| \leq 2|X|$. Let $H = G[X \cup N(X)]$. So $|V(H)| \leq 3|X|$, and

$$e(H) \geq \frac{1}{2}\sum_{v \in X} d_G(v) \geq 3(d+1)|X| \geq (d+1)|V(H)|.$$

So H has a subgraph H' with $\delta(H') \geq d+1$, and its girth is no less than the girth of G, that is at least $\geq 2g+1$. By Lemma 1.6,

$$3|X| \geq |V(H)| \geq |V(H')| \geq n_0(d+1, 2g+1) \geq 1 + (d+1)\sum_{i<g} d^i > d^g.$$

This implies $|X| > d^g/3$, a contradiction. ∎

Now we choose $r \in V(G)$ and apply breadth-first search to G. We get a BFS-tree and let $L_i = \{x : d_G(x, r) = i\}$.

Lemma 1.8. *If $G(L_i, L_{i+1})$ has a cycle C of length $2l$ with a chord, then for some $m \in [i]$, G contains cycles $C_{2m+2}, C_{2m+4} ... C_{2m+2l-2}$.*

The proof is assigned as an exercise(Using A-B path lemma).

Proof of Theorem 1.3. Let G be a graph with $d(G) \geq 48(d+1)$ and girth at least $2g+1$. Then G has a connected bipartite subgraph H with $d(H) \geq 24(d+1)$. Consider BFS-tree T of H. Then $e(H) = \sum_{i \geq 0} e(L_i, L_{i+1}) \geq 12n(d+1)$, where $n = |V(H)|$.

We claim that there exists an i such that $e(L_i, L_{i+1}) \geq 6(d+1)(|L_i| + |L_{i+1}|)$. Otherwise, $e(H) < 6(d+1)[(|L_0| + |L_1|) + (|L_1| + |L_2|) + ... + (|L_{r-1}| + |L_r|)] < 6(d+1)2n = 12(d+1)n$, a contradiction. Then $H(L_i, L_{i+1})$ has a subgraph H' with $\delta(H') \geq 6(d+1)$ with girth at least $2g+1$. By lemma 1.7, any $X \subset V(H')$ with $|X| \leq d^g/3$ has $|N(X)| > 2|X|$. By Lemma 1.4, H' has a cycle of length $\geq d^g$ with a chord. By Lemma 1.8, G has $\Omega(d^g)$ cycles of consecutive even length. □ ∎

1.2 Finding Hamilton cycles in $G(n, p)$

Lemma 1.9. *Let $G = G(n, p)$, where $p = 9 \log n/n$. Let S be the set defined in Pósa's lemma for G. Then $P(|S| < n/4) \leq n^{-1.2}$ for large n.*

Proof. Suppose $|S| = k$. Then $|S^+|, |S^-| \leq k$. By Pósa's rotation, $N(S) \subset S^+ \cup S^-$, then $e(S, V(G) \setminus (S \cup S^+ \cup S^-)) = 0$, where $|S| = k$ and $|V(G) \setminus (S \cup S^+ \cup S^-)| \geq n - 3k$. The probability that G has a set of k vertices all of which are non-adjacent to a set of $n - 3k$ vertices is at most $\binom{n}{k}(1-p)^{k(n-3k)}$. Therefore, for large n, the probability that S has at most $\ell = n/4 - 1$ vertices is at most

$$\sum_{k=1}^{\ell} \binom{n}{k} (1-p)^{k(n-3k)} < \sum_{k=1}^{\ell} n^k e^{-pk(n-3k)} < \sum_{k=1}^{\ell} (ne^{-pn/4})^k \leq \sum_{k=1}^{\ell} (n^{-5/4})^k \leq n^{-1.2},$$

a contradiction. ∎

Lemma 1.10. *Let $G = G(n, p)$, where $p = 9 \log n/n$. Then G has a Hamilton path almost surely. In other words, $\mathbb{P}_r(G$ has a H-path$) \to 1$ as $n \to +\infty$.*

Proof. For $v \in V(G)$, consider $G - v = G(n-1, p)$. Let S_v be the set defined in Pósa's lemma, applied to $G - v$. We consider the following two events:
(1). $A_v : |S_v| < (n-1)/4$,
(2). $B_v : |S_v| \geq (n-1)/4$ and there is a longest path in G which does not contain the vertex v.
By Lemma 1.9, $P(A_v) \leq (n-1)^{-1.2}$. On the other hand, if $|S_v| \geq (n-1)/4$ and there is a longest path in G which does not contain v, then none of the longest paths in $G - v$ which terminate in a vertex of S_v can be extended to contain v. This means v is not adjacent to each vertex in S_v. Thus, $P(B_v) \leq (1-p)^{|S_v|} \leq (1-p)^{(n-1)/4} \leq e^{-p(n-1)/4} \ll n^{-2}$. Thus, $\sum_{v \in V} (P_r(A_v) + P_r(B_v)) \leq n((n-1)^{-1.2} + n^{-2}) \to 0$.

We have that $\cap_{v\in V}(A_v^c \cap B_v^c) \subseteq$ the event that for any $v \in V$, every longest path in G contains $v \subseteq$ the event that every longest path contain every vertex, that means G has a Hamilton path. So as $n \to \infty$,

$$\mathbb{P}_r(G \text{ has a H-path}) \geq P_r(\cap_{v\in V}(A_v^c \cap B_v^c)) \geq 1 - \sum_{v\in V}(P_r(A_v) + P_r(B_v)) \to 1,$$

finishing the proof. ∎

Theorem 1.11. *Let $G = G(n,p)$, where $p = 10\log n/n$. Then G has a Hamilton cycle almost surely .*

Proof. Let $H = G_1 \cup G_2$, where $G_i = G(n, p_i)$. Let $p_1 = 9\log n/n$ and $p_2 = \log n/n$. This means $H = G(n,p)$ for $p = 10\log n/n - 9(\log n/n)^2$. It suffices to show that H almost surely has a Hamilton cycle. By Lemma 1.10, G_1 almost surely has a Hamilton path P. Let S be the set defined in Pósa's lemma(applied to G_1). By Lemma 1.9, almost surely $|S| \geq n/4$. Let u be the fixed end point of the Hamilton path. The probability that u is non-adjacent to S in G_2 is at most $(1-p_2)^{|S|} \leq e^{-p_2 n/4} \to 0$ (as $n \to +\infty$), in other words, almost surely u is adjacent to S in G_2. Therefore, almost surely H has a Hamilton cycle. ∎