Extremal and Probabilistic Graph Theory

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1 Lecture 16. Randomized Constructions and Erdős-Renyi Poloairty

Kövari-Sós-Turán Theorem tells us that for any bipartite H, $\mathrm{ex}(n, H) = O(n^{2-c})$. Now we will use Randomized construction, Algebraic construction, and Randomized Algebraic construction to find the lower bound of Turán numbers for bipartite graphs.

1.1 Randomized construction

Theorem 1.1. *For any graph H with at least 2 edges, there exists a constant $c > 0$ such that*

$$\mathrm{ex}(n, H) \geq cn^{2-\frac{v(H)-2}{e(H)-1}}.$$

Proof. The idea is to use random graphs and the deletion/alternation method. Consider a random graph $G = G(n, p)$ where $p = \frac{1}{2}n^{\frac{v(H)-2}{e(H)-1}}$. Let $\#H$ be the number of H-copies in G. Then

$$\mathbb{E}[\#H] = \frac{n(n-1)\cdots(n-v(H)+1)}{|Aut(H)|}p^{e(H)} \leq n^{v(H)}p^{e(H)}.$$

Since $p = \frac{1}{2}n^{-\frac{v(H)-2}{e(H)-1}}$ and $\mathbb{E}[e(G)] = p\binom{n}{2}$, we get $\mathbb{E}[e(G)] \geq 2\mathbb{E}[\#H]$, which implies that

$$\mathbb{E}[e(G) - \#H] \geq \frac{1}{2}\mathbb{E}[e(G)] \geq \frac{1}{2}p\binom{n}{2} \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

Thus there exists an n-vertex graph G with $e(G) - \#H \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}$.

Let G' be obtained from G by deleting one edge for each copy of H in G. Then G' is H-free and

$$e(G') \geq e(G) - \#H \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

So

$$ex(n, H) \geq e(G') \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

∎

Definition 1.2. The *2-density* of H is

$$m_2(H) = \max_{\substack{H' \subset H \\ v(H') \geq 3}} \frac{e(H') - 1}{v(H') - 2}.$$

Theorem 1.3. *For any H with at least 2 edges,*

$$\mathrm{ex}(n, H) = \Omega(n^{2-\frac{1}{m_2(H)}}).$$

Proof. The proof is similar to Theorem 1.1. ∎

Let us look at some examples: if $H = K_{s,t}$ with $2 \leq s \leq t$, then

$$n^{2-\frac{s+t-2}{st-1}} \lesssim \mathrm{ex}(n, K_{s,t}) \lesssim n^{2-1/s}.$$

In particular, when $s = t$,

$$n^{2-1/(s+1)} \lesssim \mathrm{ex}(n, K_{s,t}) \lesssim n^{2-1/s}.$$

1.2 Algebraic construction

For C_4, we have Reiman's bound:

$$\mathrm{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = (\frac{1}{2} + o(1))n^{\frac{3}{2}}.$$

By using algebraic construction we can prove the following theorem.

Theorem 1.4. $\mathrm{ex}(n, C_4) = (\frac{1}{2} + o(1))n^{3/2}$.

Proof. For a prime q, we first define the Erdös-Rényi polarity graph ER_q as following:

- Its vertex set is $\{U : U$ is 1-dimension subspace in a 3-dimension space $\mathbb{F}_q^3\}$.
- U, W are adjacent in ER_q if and only if U and W $(U \neq W)$ are perpendicular as 1-dimension subspace.

Obviously $|V(ER_q)| = \frac{q^3-1}{q-1} = q^2 + q + 1$.

We see each vertex U has degree q or $q+1$, since there are exactly $\frac{q^2-1}{q-1} = q+1$ 1-dimension subspaces W perpendicular to U and it is possible $U \perp U$. Also ER_q is C_4-free, because given any two vertices U, W, there is exactly one line L perpendicular to both U and W. Then we have

$$e(ER_q) \geq \frac{1}{2}q(q^2 + q + 1) = (\frac{1}{2} + o(1))|V(ER_q)|^{3/2},$$

where $|V(ER_q)| = q^2 + q + 1$ for primes q. By the number theory we know that for any large integer n there exists a prime in the interval $[n - n^{0.525}, n]$. Thus there exists an n-vertex C_4-free graph with at least $\frac{1}{2} + o(1))n^{3/2}$ edges for any large n.
∎

Remark 1.5. In ER_q there are exactly $q+1$ vertices of degree q which implies $e(ER_q) = \frac{1}{2}q(q+1)^2$.

Remark 1.6. We call vertex U is *isotropic* if $U \perp U$. Any isotropic vertex is not contained in any triangle of ER_q. No edges of ER_q can join two isotropic vertices. And unless adjacent to the isotropic vertex each edge is in exactly one triangle.

Theorem 1.7 (Füredi)**.** *For large prime power q.*

$$\mathrm{ex}(q^2+q+1, C_4) = \frac{1}{2}q(q+1)^2,$$

where the ER_q provides an extremal graph.

We also know the following bounds:

- (Füredi-Naor-Verstraete) $0.538n^{\frac{4}{3}} \le \mathrm{ex}(n, C_6) \le 0.627n^{\frac{4}{3}}$.
- (Lazebnik etal) $0.58n^{6/5} \approx \frac{4}{5^{6/5}}n^{6/5} \le \mathrm{ex}(n, C_{10}) \le O(n^{6/5})$.
- (Lazebnik etal) $((\frac{1}{2})^{3/2} + o(1))n^{3/2} \le \mathrm{ex}(n, \{C_3, C_4\}) \le (\frac{1}{2} + o(1))n^{3/2}$.

Conjecture 1.8 (Erdős-Simonovits)**.** $\mathrm{ex}(n, \{C_3, C_4\}) \le (\frac{n}{2})^{3/2} + o(n^{3/2}))$

Definition 1.9. A *Berge ℓ-cycle BC_ℓ* in k-graphs is a k-graph consisting of ℓ distinct hyperedges $e_1, e_2, \dots, e_\ell$ such that there are ℓ distinct vertices $v_1, v_2, \dots, v_\ell$ satisfying $v_i \in e_i \bigcap e_{i+1}$ for $1 \le i \le l-1$ and $v_l \in e_1 \bigcap e_\ell$.

Let $\mathrm{ex}_k(n, \mathcal{F})$ be the Turán number of $\mathcal{F}$ in k-graph. For example $\mathrm{ex}_k(n, BC_2) = \binom{n}{2}/\binom{k}{2} + o(n^2)$.

Theorem 1.10. $\Omega(n^{2-\epsilon}) = 2^{-c\sqrt{logn}}n^2 \le ex_3(n, BC_2, BC_3) = o(n^2)$

Theorem 1.11 (Lazebnik-Verstraete)**.** *(i) $ex_3(n.BC_2, BC_3, BC_4) \le \frac{n}{6}\sqrt{n - \frac{4}{3}} + \frac{n}{12}$.*

(ii) There exists a 3-graph H on q^2 vertices with $\binom{q+1}{3}$ edges which is $\{BC_2, BC_3, BC_4\}$-free where q is a prime. Thus $ex_3(n, \{BC_2, BC_3, BC_4\}) = (\frac{1}{6} + o(1))n^{\frac{3}{2}}$.

Proof. First, let H be a $\{BC_2, BC_3, BC_4\}$-free n-vertex 3-graph. For any $v \in V(H)$ and edges A, B which contain v, let $v(A, B)$ be the set of $\{a, b\}$ where $a \in A - v$ and $b \in B - v$. Let $D_v = \bigcup_{\{A,B\}: v \in A \bigcap B} v(A, B)$. Then $|D_v| = 4\binom{d_v}{2}$. Obviously we can get the following two claims.

- For $u \ne v$ we have $D_u \bigcap D_v = \emptyset$.
- No pair in D_v is contained in an edge.

Let $m = e(H)$, we have

$$\binom{n}{2} - 3m \ge \sum_v |D_v| = 4\sum_v \binom{d_v}{2} \ge 4n\binom{\sum_v d_v/n}{2} = 2(\frac{9m^2}{n} - 3m).$$

Thus

$$e(H) = m \le \frac{n}{6}\sqrt{n - \frac{4}{3}} + \frac{n}{12}.$$

Next, for a prime q, consider ER_q. Let H be a 3-graph obtained from ER_q such that $V(H)$ is the set of all non-isotropic vertices and any triangle in ER_q forms an hyperedge. By Remark 1.6, it is easy to see that H is $\{BC_2, BC_3, BC_4\}$-free with $V(H) = q^2 + q + 1 - (q+1) = q^2$ and

$$e(H) = \frac{e(ER_q) - q(q+1)}{3} = \frac{(q+1)q(q-1)}{6} = \binom{q+1}{3}.$$

∎