Extremal and Probabilistic Graph Theory

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1 Lecture 17. Algebraic Constructions

1.1 New constructions of $\mathrm{ex}(n, C_4)$

Theorem 1.1 (Erdős-Renyi-Sos)**.** $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$

Proof. We have seen that the Erdös-Rényi polarity graphs ER_q can give this lower bound. Now we give a different construction for C_4-free graphs which also yield the same lower bound.

Suppose $n = q^2 - 1$ for a prime q. Consider the following graph $G = (V, E)$, where $V = F_q^2 \backslash \{0, 0\}$, $E = \{(x, y) \sim (a, b) | ax + by = 1 \text{ in } F_q\}$. First, we see G is C_4-free: for any distinct vertices $(a, b) \neq (a', b')$, there is at most one solution(common neighbor) satisfying both $ax + by = 1$ and $a'x + b'y = 1$. It is easy to see that the degree of each vertex is q or $q - 1$. So $e(G) \geq \frac{1}{2}(q^2 - 1)(q - 1) \approx (\frac{1}{2} - o(1))n^{3/2}$ (where $n = q^2 - 1$). Since primes are dense in integer, we can get $\mathrm{ex}(n, C_4) \geq (\frac{1}{2} - o(1))n^{3/2}$. ∎

Remark 1.2. One can generalize this this to $K_{2,t}$-free graphs.

1.2 Constructions of $\mathrm{ex}(n, K_{3,3})$

Theorem 1.3 (Brown)**.**

$$\mathrm{ex}(n, K_{3,3}) \geq \frac{1}{2}n^{5/3} + O(n^{4/3}).$$

Proof. Let $n = q^3$ for some odd prime q. Consider the following G: $V(G) = F_q^3$ and $E(G) = \{(x, y, z) \sim (a, b, c) | (x - a)^2 + (y - b)^2 + (z - c)^2 = d \text{ in } F_q\}$, where $d \neq 0$ is a quadratic residue[1] if $q = 4k - 1$ and d is a quadratic non-residue if $q = 4k - 3$.

One can check that G is $K_{3,3}$-free. We should omit the detailed proof, instead we give the following intuition : The $K_{3,3}$-freeness is equivalent to the statement that any 3 unit spheres have at most two common points. It is not hard to see that most vertices (x, y, z) have around q^2 neighbors. Thus we have $e(G) \geq \frac{1}{2}q^3q^2 \approx (\frac{1}{2} - o(1))n^{5/3}$ when $n = q^3$. ∎

Note that the best upper bound is due to Füredi: $\mathrm{ex}(n, K_{3,3}) \leq \frac{1}{2}n^{5/3} + n^{4/3} + 3n$.

Lemma 1.4. *Let K be a field and $a_{ij}, b_i \in K$ for $1 \leq i, j \leq 2$ such that $a_{1j} \neq a_{2j}$. Then the system of equations*

$$\begin{cases} (x_1 - a_{11})(x_2 - a_{12}) = b_1 \\ (x_1 - a_{21})(x_2 - a_{22}) = b_2 \end{cases}$$

has at most two solutions $(x_1, x_2) \in K \times K$.

[1] there is an integer x such that $x^2 \equiv d \pmod{q}$.

Proof. Considering the difference of two equations, we get $(a_{11}-a_{21})x_2+(a_{12}-a_{22})x_1+a_{21}a_{22}-a_{11}a_{12}=b_2-b_1$. Since $a_{11}-a_{21}\neq 0$, we can express x_1 by an expression of x_2. Substituting this expression to any one of the equation, we get a quadratic equation in the variable x_2. It has at most two solutions for x_2, each of which determines the valus of x_1. So we have at most two solutions $(x_1,x_2)\in K\times K$. ∎

Lemma 1.5. *Let K be a field with characteristic q. Then any $x,y\in K$ satisfy $(x+y)^q=x^q+y^q$*

Definition 1.6. Let q be a prime. The norm map N is: $F_{q^s}\to F_q$ by $N(x)=xx^qx^{q^2}\dots x^{q^s-1}$, for $x\in F_{q^s}$.

Note that since $x^{q^s}=x$ for any $x\in F_{q^s}$, we have $(N(x))^q=x^qx^{q^2}\dots x^{q^s}=N(x)$, then $N(x)\in F_q$.

Theorem 1.7 (Alon-Rónyai-Szabó)**.** *For every $n=q^3-q^2$ with prime power q,*

$$\mathrm{ex}(n,K_{3,3})\geq \frac{1}{2}n^{5/3}+\frac{1}{3}n^{4/3}+C.$$

Proof. Let $N:F_{q^s}\to F_q$ be the norm map. The graph $H=H(q,3)$ is as follows. The vertex set of H is $F_{q^2}\times F_q^*$. Two vertices (A,a) and (B,b) in $V(H)$ are adjacent if and only if $N(A+B)=a\cdot b$. The degree of each vertex $(A,a)\in V(H)$ is the number of pairs (B,b) with $N(A+B)=ab$. For any $B\neq -A$, we can have a unique b. So the degree of (A,a) is q^2-1 or q^2-2, as $N(A+A)=a^2$ may happen.

Now it suffices to show H is $K_{3,3}$-free, which is enough to show that for any three distinct vertices (D_i,d_i) with $i\in[3]$, they have at most 2 common neighbors. That is, the system of equations:

$$\begin{cases} N(X+D_1)=xd_1 & (1.1)\\ N(X+D_2)=xd_2 & (1.2)\\ N(X+D_3)=xd_3 & (1.3)\end{cases}$$

has at most 2 solutions $(X,x)\in F_{q^2}\times F_q^*$. Observe that if (X,x) is a solution, then:

1) $X\neq -D_i$, for $i\in[3]$, and

2) $D_i\neq D_j$, for $i\neq j$.

Divide equations 1.1 and 1.2 by equation 1.3, we can get

$$\frac{d_i}{d_3}=\frac{N(X+D_i)}{N(X+D_3)}=N\left(\frac{X+D_i}{X+D_3}\right)=N\left(1+\frac{D_i-D_3}{X+D_3}\right),\ \text{for } i=1,2.$$

Let $Y=\frac{1}{X+D_3}, A_i=\frac{1}{D_i-D_3}$, and $b_i=\frac{d_i}{d_3N(D_i-D_3)}, i\in[2]$. Then,

$$\begin{cases}(Y+A_1)(Y^q+A_1^q)=N(Y+A_1)=b_1\\ (Y+A_2)(Y^q+A_2^q)=N(Y+A_2)=b_2\end{cases}$$

It is clear that $A_1\neq A_2$ and $A_1^q\neq A_2^q$. Then by lemma 1.4, this system has at most 2 solutions (Y,Y^q). Therefore, we have at most two pairs of (X,x). ∎

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1.3 Construction of $\mathrm{ex}(n, K_{s,t})$

Lemma 1.8. *Let K be a filed and $a_{ij}, b_i \in K$ for $1 \le i, j \le s$ such that $a_{i_1,j} \neq a_{i_2,j}$ for $i_1 \neq i_2$. Then the system*

$$\begin{cases} (X_1 - a_{11})(X_2 - a_{12}) \dots (X_s - a_{1s}) = b_1 \\ (X_1 - a_{21})(X_2 - a_{22}) \dots (X_s - a_{2s}) = b_2 \\ \quad \cdots \\ (X_1 - a_{t1})(X_2 - a_{s2}) \dots (X_s - a_{ss}) = b_s \end{cases}$$

has at most $s!$ solutions $(x_1, x_2, \dots, x_s) \in K^s$.

Proof. Omit ∎

Definition 1.9. Let q be a prime. The *norm-graph* $G = G(q, s)$ is as follows: $V(H) = F_{q^s}$ and two vertices A, B are adjacent if and only if $N(A+B) = 1$, where $N : F_{q^s} \to F_q$ is the norm map.

Theorem 1.10 (Kollár-Rónyai-Szabó)**.** *Graph $G(q, s)$ is $K_{s,s!+1}$-free. Therefore, for $t \ge s! + 1$,*

$$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s}).$$

Proof. It suffices to show that any s vertices $d_1, d_2 \dots d_s$ have at most $s!$ common neighbors. That is, the system

$$\begin{cases} N(X + d_1) = 1 \\ N(X + d_2) = 1 \\ \quad \cdots \\ N(X + d_s) = 1 \end{cases}$$

has at most $s!$ solutions. Then

$$\begin{cases} 1 = (x + d_1)(x + d_1)^q \dots (x + d_1)^{q_{s-1}} \\ 1 = (x + d_2)(x + d_2)^q \dots (x + d_2)^{q_{s-1}} \\ \quad \cdots \\ 1 = (x + d_s)(x + d_s)^q \dots (x + d_s)^{q_{s-1}}. \end{cases}$$

Let $x_j = x^{q^{j-1}}$ for $j \in [s]$ and $a_{ij} = -d_j^{q^{i-1}}$. Then we have the system in Lemma 1.8. So G is $K_{s,s+1}$-free. Also, since the number of solutions in F_{q^s} for the equation $N(x) = 1$ is $\frac{q^s-1}{q-1}$, the minimum degree $\delta(G)$ is at least $\frac{q^s-1}{q-1} - 1 \ge q^{s-1}$. Thus, when $n = q^s$, $e(G) \ge \frac{1}{2} q^s \cdot q^{s-1} \ge \frac{1}{2} n^{2-1/s}$. ∎

Definition 1.11. For a prime q, the *projective norm graph* $H = H(q, s)$ is as follows: $V(H) = F_{q^{s-1}} \times F_q^*$ and two vertices $(A, a), (B, b) \in V(H)$ are adjacent if and only if $N(A + B) = a \cdot b$, where $N : F_{q^{s-1}} \to F_q$ is the norm map.

Theorem 1.12 (Alon-Rónyai-Szabó)**.** *$H(q, s)$ is $K_{s,(s-1)!+1}$-free. Therefore, for $t \ge (s-1)! + 1$,*

$$\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s}).$$

Proof. Exercise (similar to the proof of Theorem 1.7 for $H(q, 3)$). ∎