Extremal and Probabilistic Graph Theory

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1 Lecture 19. Dependent Random Choice

Theorem 1.1. *Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. Then there exists a constant $C = C_H$ such that*

$$ex(n, H) \leq Cn^{2-1/r}$$

Remark 1.2. This theorem was first proved by Füredi(1991) and then was reproved by Alon-Krivelevich-Sudakov(2002).

We will give the proof of Alon-Krivelevich-Sudakov, which has been extended to a powerful probabilistic tool called "dependent random choice". The main idea of this is the following lemma: If G has many many edges, then one can find a large subset A in G such that all small subsets of A have many common neighbors.

Definition 1.3. For $S \subseteq V(G)$, $N(S) = \{w \in V(G) : ws \in E(G) \text{ for every } s \in S\}$.

Lemma 1.4 (Dependent random choice)**.** *Let $u, n, r, m, t \in \mathbb{N}$ and a real number $\alpha \in (0, 1)$ be such that*

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u$$

Then every n-vertex graph G with at least $\frac{\alpha}{2}n^2$ edges contains a subset U of at least u vertices such that every r-element subset S of U has at least m common neighbors.

Proof. Let T be a list of t vertices chosen uniformly at random from $V(G)$ (allowing repetition). Let $A = N(T)$. Then

$$\mathbb{E}[|A|] = \sum_{v \in V} \mathbb{P}(v \in A) = \sum_{v \in V} \mathbb{P}(T \subseteq N(v)) = \sum_{v \in V} \left(\frac{d(v)}{n}\right)^t \geq n\left(\frac{1}{n}\sum_{v \in V}\frac{d(v)}{n}\right)^t \geq n\alpha^t.$$

Call an r-element subset $S \subseteq V(G)$ bad if S has less than m common neighbors ($|N(S)| < m$). Given an r-element subset $S \subseteq V(G)$, we have

$$\mathbb{P}(S \subseteq A) = \mathbb{P}(T \subseteq N(S)) = \left(\frac{|N(S)|}{n}\right)^t.$$

So

$$\mathbb{E}[\#\text{ bad } r\text{-element subsets in } A] < \binom{n}{r}\left(\frac{m}{n}\right)^t.$$

Combining, there exists a choice of T such that $A = N(T)$ satisfies that

$$|A| - \#\text{ bad } r\text{-element subsets in } A \geq n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

Let U be obtained from A by deleting one vertex from each bad r-element subset in A. Then we have that $|U| \geq u$ and U satisfies the condition. ∎

Now we can prove the Theorem 1.1.

Proof. (Theorem 1.1) Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most r. We want to show $\mathrm{ex}(n, H) \leq Cn^{2-1/r}$, where $C = C_H$ is a constant. Let G be any n-vertex graph with at least $Cn^{2-1/r}$ edges, where C satisfies

$$n(2Cn^{-1/r})^r - \binom{n}{r}\left(\frac{|A| + |B|}{n}\right)^r \geq |B|.$$

By dependent random choice lemma, taking $u = |B|$, $m = |A| + |B|$, $t = r$, $\alpha = 2Cn^{-1/r}$, we see

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

So there exists a subset U with $|U| \geq u$ such that any r-element subsets of U has at least $m = |A| + |B|$ common neighbors.

We label $A = \{v_1, v_2, ..., v_a\}$ and $B = \{u_1, u_2, ..., u_b\}$. We find any one-to-one mapping $\phi : B \to U$, $u_i \mapsto \phi(u_i)$. Next, we want to extend this ϕ from B to $A \cup B$ and then we can find a copy of H in G. Suppose for $A' = \{v_1, v_2, ..., v_s\}$, we have $\phi : A' \cup B \to V(G)$ such that $H[A' \cup B] \subseteq G[\phi(A') \cup \phi(B')]$. Consider v_{s+1} and $N_H(v_{s+1}) \subseteq B$, we have that $N_H(v_{s+1}) \leq r$. We consider $\phi(N_H(v_{s+1})) \subseteq U$ of size at most r. By the property of U, $\phi(N_H(v_{s+1}))$ has at least $|A| + |B|$ common neighbors in G. Then we can get a vertex $\phi(v_{s+1})$ which is a common neighbor of $\phi(N_H(v_{s+1}))$ but is not in $\phi(A' \cup B)$. Repeatedly, we can extend ϕ to be $\phi : A \cup B \to V(G)$ such that $\phi(A \cup B)$ is a copy of H, a contradiction.

∎

The result $\mathrm{ex}(n, H) = O(n^{2-1/r})$ is tight for $H = K_{r,s}$ if $s \gg r$.

Conjecture 1.5. *Let H be a bipartite graph with bipartition (A, B) such that each vertex in A has degree at most r and H is $K_{r,r}$-free. Then there exist $C, c > 0$ depending on H such that*

$$ex(n, H) \leq Cn^{2-1/r-c}$$

Remark 1.6. The conjecture is only known for $r = 2$. For $r \geq 3$, $\mathrm{ex}(n, H) = o(n^{2-1/r})$ is proved by Sudakov and Tomon recently.