Extremal and Probabilistic Graph Theory

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1 Lecture 20.

Conjecture 1.1. *Let H be a $K_{r,r}$-free bipartite graph on (A, B) such that any $a \in A$ has degree at most r. Then there are constants $c, C > 0$ such that* $\mathrm{ex}(n, H) \le Cn^{2-1/r-c}$.

Theorem 1.2. *Let H be a $K_{2,2}$-free bipartite graph on bipartition (A, B) such that each vertex in A has degree at most 2. Then there are constant $c, C > 0$ such that* $\mathrm{ex}(n, H) \le Cn^{3/2-c}$.

Definition 1.3. For a graph H, the k**-subdivision** $H^{(k)}$ of H is a graph obtained from H by replacing each edge ab of H with a internally distinct path P_{ab} of length $k+1$ with endpoints a and b, where all such paths P_{ab} are mutually internally distinct.

Theorem 1.4. *For all $t \ge 3$, there exists $c_t > 0$ such that* $\mathrm{ex}(n, K_t^{(1)}) = O(n^{3/2-c_t})$.

We observe that Theorem 1.4 can imply Theorem 1.2.

Definition 1.5. A graph G is called K**-regular** if $\Delta(G) \le K\delta(G)$.

Lemma 1.6 (Erdös-Simonovits; Jiang, Bukh-Jiang, Colon-Lee)**.** *For all $0 < \alpha < 1$, there exists constants $\beta, K > 0$ such that for all $C > 0$ and sufficently large n, every n-vertex graph G with at least $Cn^{1+\alpha}$ edges has a subgraph G' satisfying:*

- *G' is K-regular and bipartite with two parts of size differencing by a factor at most 2;*
- *$v(G') \ge n^{\beta}$;*
- $\frac{e(G')}{v(G')^{1+\alpha}} \ge \frac{1}{10}\frac{e(G)}{v(G)^{1+\alpha}}$.

We will not give a detailed proof for this lemma.

Definition 1.7. Fix t and for $u, v \in V(G)$, we say the pair (u, v) is **light**, if $1 \le |N(u) \cap N(v)| < \binom{t}{2}$, and is **heavy** if $|N(u) \cap N(v)| \ge \binom{t}{2}$.

Lemma 1.8. *Let G be a $K_t^{(1)}$-free bipartite graph with bipartition $X \cup B$, where $d(x) \ge \delta$ for all $x \in X$ and $|X| \ge \frac{4|B|t}{\delta}$. Then there exists $u \in X$ in $\Omega(\delta^2|X|/|B|)$ light pairs in X.*

Proof. Let $S = \{(\{u, v\}, b) : b \in B, u, v \in X \cap N(b)\}$. We see that

$$|S| = \sum_{b \in B} \binom{d(b)}{2} \ge |B|\binom{\frac{e(G)}{|B|}}{2} \ge \frac{|B|}{4}(\frac{\delta|X|}{B})^2 = \frac{\delta^2|X|^2}{4|B|}.$$

Let $B^* = \{b \in B : d(b) \ge 2t\}$. Since

$$\sum_{b \in B \setminus B^*} \binom{d(b)}{2} \le 2t^2|B| \le \frac{\delta^2|X|^2}{8|B|},$$

we have

$$\sum_{b\in B^*}\binom{d(b)}{2}\geq \frac{\delta^2|X|^2}{8|B|}.$$

Next, we claim that there are no t vertices in X such that any pair of which is heavy. If not, suppose there exists t vertices, say $x_1,x_2,...,x_t$, such that $\{x_i,x_j\}$ is heavy, then it is easy to check that there exists a $K_t^{(1)}$ in G, a contradiction!

Consider $b\in B^*$. Any pair in $N(b)$ is eigher light or heavy. By Turán Theorem and the above claim, the number of heavy pairs in $N(b)$ is at most $e(T_{t-1}(d(b)))$. Then for $b\in B^*$, there are at least

$$\binom{d(b)}{2}-e(T_{t-1}(d(b)))\geq \binom{d(b)}{2}-\binom{t-1}{2}\left(\frac{d(b)}{t-1}\right)^2\geq \frac{d(b)^2}{2(t-1)}-\frac{1}{2}d(b)\geq \Omega(d(b)^2)$$

light pairs in $N(b)$. Sum over all $b\in B^*$, then

$$\#\triangleq \mathop{\#}_{\substack{b\in B^* \text{ and}\\ \{u,v\}\text{ is light in } N(b)}}(\{u,v\},b)\geq \sum_{b\in B^*}\Omega(d(b)^2)\geq \Omega(\frac{\delta^2|X|^2}{|B|}).$$

Since $\{u,v\}$ is light, we get

$$\#\text{ light pairs in } X\geq \frac{\#}{\binom{t}{2}}\geq \Omega(\frac{\delta^2|X|^2}{|B|}).$$

Thus there exists a vertex $u\in X$ which is in at least $\Omega(\frac{\delta^2|X|}{|B|})$ light pairs in X. ∎

In the following, we will give a proof of Theorem 1.4 due to Janzer, who proved that $c_t=\frac{1}{4t-6}$ for $t\geq 3$. Since we know $K_3^{(1)}=C_6$ and $\mathrm{ex}(n,C_6)=\Theta(n^{4/3})$, one may ask that whether $c_t=\frac{1}{4t-6}$ is tight for all $t\geq 3$.

Proof of Theorem 1.4. Let G be a $K_t^{(1)}$-free graph on n vertices and with at least $Dn^{3/2-c_t}=Dn^{1+\alpha}$ ($\alpha=\frac{t-2}{2t-3}$) edges. By lemma 1.6, there exists a $G'\subset G$ which is K-regular and bipartite on parts $A\cup B$, such that $e(G')/v(G')^{1+\alpha}\geq e(G)/(10v(G)^{1+\alpha})$ and $v(G')$ is large, $|B|/2\leq |A|\leq 2|B|$. If $\delta(G')\leq C(v(G'))^{\alpha}$, we have

$$\Delta(G')\leq KC(v(G'))^{\alpha}\Rightarrow e(G')\leq KC(v(G'))^{1+\alpha}\Rightarrow e(G)\leq 10KCn^{1+\alpha},$$

then we are done. Therefore, we may assume that $\delta\triangleq\delta(G')>C(v(G'))^{\alpha}=C(v(G'))^{\frac{t-2}{2t-3}}$.

Our plan is to find t vertices $u_1,...,u_t\in A$ such that $\{u_i,u_j\}$ is light for all $1\leq i<j\leq t$ and u_i,u_j,u_k has no commom neighbors for all distinct i,j,k. If so, then we can find a $K_t^{(1)}$ in G easily.

We will find these t vertices by repeatedly using lemma 1.8 on a stronger hypothesis: for each $1\leq i\leq t$, there exists $A=X_1\supset X_2\supset ...\supset X_i$ and $u_1\in X_1,...,u_i\in X_i$ such that:

(1) u_j is in at least $\Theta(\delta^2|X_j|/v(G'))$ light pairs in X_j, for $1\leq j\leq i-1$.

(2) u_j is light to every vertex w in X_{j+1}, for $1\leq j\leq i-1$.

(3) No 3 vertices of $v_1, ..., v_i$ has commom neighbors.

(4) $|X_{j+1}| = \Omega(\delta^2 |X_j|/v(G'))$, for $1 \leq j \leq i$.

This is holds clearly for $i = 1$ by choices u_1 to be the vertex founded by lemma 1.8 when applied to $A \cup B$. Now suppose we have obtained this for $i-1$: $A = X_1 \supset X_2 \supset ... \supset X_{i-1}$ with $u_j \in X_j, j \leq i-1$. Let $Y_i = \{y \in X_{i-1} : \{y, u_{i-1}\}$ is light$\}$. By (1), $|Y_i| \geq \delta^2 |X_j|/v(G')$. Consider any $u_j u_l$ with $j, l \leq i-1$, take any common neighbor u of them and delete $N(u)$ from Y_i. We know there are $\binom{i-1}{2}$ pairs $u_i u_l$ and $|N(u)| < K\delta$. For each $u_i u_l$ there are at most $\binom{t}{2}$ many choices of u as otherwise we can get a $K_t^{(1)}$. So the number of deleted vertices is at most

$$\binom{i-1}{2}\binom{t}{2} K\delta = O(\delta).$$

As long as $|Y_i| \geq \delta^2 |X_{i-1}|/v(G') \geq \Omega(\delta)$, we can get a $X_i \subseteq Y_i$ of size at least $\Omega(\delta^2/v(G'))|X_{i-1}|$, which satisfies (3). This is true, because $i \leq t$ and

$$\left(\frac{\delta^2}{v(G')}\right)^{i-1} |A| \geq \left(\frac{\delta^2}{n}\right)^{i-1} \cdot n \geq \Omega(\delta),$$

which implies $\delta^{2t-3} \geq n^{t-2}$. This shows that the algorithm can keep going until we have $X_1 \supset ... \supset X_t$ and $u_j \in X_j$ for $1 \leq j \leq t$. It is clear from this to see $\{u_i u_j\}$ is light and any 3 of $u_1, ..., u_t$ has no common neighbors. ∎