Extremal and Probabilistic Graph Theory

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Lecture 21. Bipartite graphs with bounded degree in one side

Theorem 1.1 (Füredi, Alon-Krivelevich-Sudakov)**.** *Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most t. We have*

$$\mathrm{ex}(n, H) = O(n^{2-\frac{1}{t}}).$$

It's tight for $H = K_{t,s}$ when $s \gg t$.

Conjecture 1.2. *Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most t. If H is $K_{t,t}$-free, then there exists a constant c=c(H)>0, such that $\mathrm{ex}(n, H) = O(n^{2-\frac{1}{r}-c})$.*

We confirm the conjecture for $r = 2$ by the following theorem.

Theorem 1.3 (Sudakov-Tomon)**.** *Let H be a $K_{t,t}$-free bipartite graph with bipartition (A, B) such that every vertex in A has degree at most t. Then*

$$\mathrm{ex}(n, H) = o(n^{2-\frac{1}{r}}).$$

Notation 1.4.

- $X^{(t)} = \{$all subset of size t in $X\}$.
- $K_k^{(t)}$=complete t-graph on k vertices.
- $N_G(S) = \{v \notin V\backslash S |\ vs \in E(G)$ for any $s \in S\}$.
- A graph G is K-almost regular if $\Delta(G) \leq K \cdot \delta(G) \qquad (K \geq 1)$.

Lemma 1. *Let $0 < c < 10^{-4}$ and $\frac{1}{2} \leq \alpha < 1$. Let n be a suffciently large integer compared to c and α. Let G be an n vertices graph with $e(G) \geq cn^{1+\alpha}$. Then G contains a bipartite graph G', whose both vertex classes have size $m \geq \frac{1}{2}n^{\frac{\alpha-\alpha^2}{4(1+\alpha)}}$, $e(G') \geq \frac{c}{10}m^{1+\alpha}$ and $\Delta(G') < m^{\alpha}$.*

Proof. This can be derived from a lemma of previous lecture. ∎

Lemma 2. *Let k, t be integers. Then there exists $\Delta = \Delta(k, t)$ such that any 2-edge-colouring of $K_\Delta^{(t)}$ contains a monochromatic copy of $K_k^{(t)}$.*

Proof. Omit (HW). ∎

Lemma 3 (Hypergraph Removal Lemma; Nagle-Rödlt-Schacht & Gowers)**.** *Let $k, t \in \mathbb{Z}^+$. For any $\beta > 0$, there is a $\delta = \delta(k, t, \beta) > 0$ such that the following holds. If H is a t-graph on n vertices such that one needs to delete at least βn^t edges of H to make it $K_k^{(t)}$-free, then H contains at least δn^k copies of $K_k^{(t)}$.*

First, we introduce some ideas about Theorem 1.3.

Ideas. Using lemma 1, we assume G is a balanced complete bipartite graph, the two parts U, V has equal size. $e(G) \geq \epsilon n^{2-\frac{1}{r}}$ and K-almost regular $\implies d(v) \approx n^{1-\frac{1}{r}}$.

1. $W \subseteq U$, $G' = G[W \cup U]$, $|W| \sim n^{1-\frac{1}{r}} \implies$ Typical $S \in V^{(t-1)}$ has $\Omega(1)$ common neighbours in W.
2. Define t-graph H on W. $S \in W^{(t)}$ is an edge of H if and only if $|N(S)| \geq t-1$. If $|N(S)| \gg$constant, we colour it by red, otherwise we colour it by blue. $\implies$ No red $K_k^{(t)}$ on $W \implies$ all $K_k^{(t)}$ are blue.
3. Using Hypergraph Removal Lemma $\implies W$ has $\Omega(|W|^k)$ copies of $K_k^{(t)}$.
4. Show NOT many "bad" $K_k^{(t)} \implies$ Done.

∎

Proposition 1.5 (Chernoff's Inequality)**.** *If $X \sim$ binomial distribution B(n,p) i.e. $X = \sum_{i=1}^{n} x_i$, where for any i with $\begin{cases} Pr[x_i = 1] = p \\ Pr[x_i = 0] = 1-p \end{cases}$, we have $\begin{cases} Pr[X \geq (1+\lambda)pn] \leq e^{\frac{-\lambda^2 pn}{3}} \\ Pr[X \leq (1+\lambda)pn] \leq e^{\frac{-\lambda^2 pn}{3}} \end{cases}$.*

Proof of Theorem 1.3. Let H_k be the bipartite graph on parts X, Y such that $|X| = k, |Y| = (t-1)\binom{k}{t}$, and for every S $\in X^{(t)}$, there are exactly $(t-1)$ vertices in Y where neighbourhood is equal to S.

Note that any H satisfy the condition is contained in some H_k.

- It is enough to only prove that for any $k, \mathrm{ex}(n, H_k) = o(n^{2-\frac{1}{t}})$. $(*)$
- We will assume $t \geq 3$.
- We will show that for $0 < \epsilon < 10^{-4}$, if n is suffciently large, $\mathrm{ex}(n, H_k) \leq \epsilon n^{2-\frac{1}{t}}$. $\implies (*)$

Let G_0 be an n_0 vertices graph with $e(G_0) > 10\epsilon n_0^{2-\frac{1}{t}}$. By lemma 1, G_0 has a H-free bipartite subgraph with parts U, V such that $n = |U| = |V| > \frac{1}{2} n_0^{(1-1/t)/(8t-4)}$, $e(G) \geq \epsilon n^{2-\frac{1}{t}}$, $\Delta(G) \leq n^{1-\frac{1}{t}}$ and G is H_k-free with $n \gg k, t, s$. By lemma 2, there exists a $\Delta = \Delta(k, t)$ such that any red-blue edge-colouring of $K_\Delta^{(t)}$ contains either a red or blue copy of $K_k^{(t)}$.

Claim 1. *Let $p = \alpha n^{-\frac{1}{t}}$, where $\alpha = \Delta(\frac{t-1}{\epsilon})^{t-1} 2^{3t-3}$. Then there exists $W \subset U$ such that*

- $\frac{pn}{2} < |W| < 2pn$
- $G' = G[W \cup V]$ *has* $\geq \frac{p}{4} e(G)$ *edges*

- $d_{G'}(x) < 2pn^{1-\frac{1}{t}}$ *for* $x \in V$.

Proof. Pick each vertex of U with probability p (independant with each other) and let W be the set of selected vertices. Then the statement follow by standard concentration inequalities. For $x \in V$ with $d_G(x) \geq n^{\frac{1}{4}}$, by Chernoff's bounds,

$$Pr[|d_{G'}(x) - pd_G(x)| < \frac{1}{2}pd_G(x)] \geq 1 - 2e^{-\frac{pd_G(x)}{12}} \geq 1 - 2e^{-n^{1/4}/12},$$

where $pd_G(x) = \Omega(n^{1-\frac{2}{t}}) \gg n^{\frac{1}{4}}$. So with high probability $(1-o(1))$, $|d_{G'}(x) - pd_G(x)| < \frac{1}{2}pd_G(x)$ for $x \in V$. Also with high probability $(1 - o(1))$, $||W| - pn| < \frac{1}{2}pn$.

Lastly,

$$e(G') = \sum_{x \in V} d_{G'}(x) \geq \sum_{x \in V, d_G(x) \geq n^{\frac{1}{4}}} \frac{1}{2}pd_G(x) \geq \frac{1}{2}pe(G) - n^{1+\frac{1}{4}} \geq \frac{1}{4}pe(G).$$

∎

We consider

$$L = \sum_{C \in V^{(t-1)}} |N_{G'}(C)| = \#(t-1)\text{-stars} = \sum_{x \in W} \binom{d_{G'}(x)}{t-1} \geq |W| \binom{e(G')/|W|}{t-1}$$
$$> (t-1)^{-(t-1)} e(G')^{t-1} |W|^{-(t-2)} \geq \left(\frac{\epsilon}{t-1}\right)^{(t-1)} \cdot 2^{-3t+4} \cdot p \cdot n^{t-1+\frac{1}{t}} = 2\Delta n^{t-1}.$$

We point out $\frac{\alpha}{2}n^{1-\frac{1}{t}} < |W| < 2\alpha n^{1-\frac{1}{t}}$ and any $x \in V$ has degree at most $2pn^{1-\frac{1}{t}} = 2\alpha n^{1-\frac{2}{t}}$ in G', where α is independent of n. Let H be the t-graph on W such that $S \in W^{(t)}$ is an edge of H if and only if $|N_{G'}(S)| \geq t-1$. Then we colour an edge $S \in E(H)$ by red if $|N_{G'}(S)| \geq (t-1)\binom{k}{t}$ and colour it by blue if $t - 1 \leq |N_{G'}(S)| < (t-1)\binom{k}{t}$.

Claim 2. *H has NO red $K_k^{(t)}$.*

Proof. If so, then we can find a copy of H_k by a greedy algorithm applied to this $K_k^{(t)}$. ∎

Let $C \in V^{(t-1)}$ and consider $T = N_{G'}(C)$. Let $r = \lfloor \frac{|T|}{\Delta} \rfloor \geq \frac{|T|}{\Delta} - 1$ and let $T_1, T_2, \cdots, T_r$ be disjoint sets of size Δ in T. Note that $H[T_i]$ is a clique $K_\Delta^{(t)}$. By lemma 2 and claim 2, each T_i contains a blue $K_k^{(t)}$ in H, called A_i. Set $Z_C = \{A_1, A_2, \cdots, A_r\}$ and $Z = \bigcup_{C \in V^{(t-1)}} Z_C$ is a multiset (different sets C may have the same A_i). Note

$$|Z| = \sum_{C \in V^{(t-1)}} |Z_C| \geq \sum_{C \in V^{(t-1)}} (\frac{|N_{G'}(C)|}{\Delta} - 1) \geq \frac{L}{\Delta} - \binom{n}{t-1} \geq n^{t-1}.$$

Claim 3. *There is a constant $\beta = \beta(k, t, \varepsilon) > 0$ and $Z' \subseteq Z$ such that $|Z'| \geq \beta|w|^t$ and any two clique are edge-disjoint.*

Proof. Let D be the auxiliary graph on vertex set Z,where $A, B \in Z$ are jointed by an edge if and only if $|A \bigcap B| \geq t$, we want to show that $\Delta(D) \leq \binom{k}{t}\binom{n}{t-1}$, where $u = (t-1)\binom{k}{t}$.

Let $A \in Z$ be any (blue) ${K_k}^{(t)}$ and $S \in A^{(t)}$. So S is blue implies $|N_G(S)| \leq (t-1)\binom{k}{t} \triangleq u$. There are at most $\binom{u}{t-1}$ sets $C \in V^{(t-1)}$ such that $S \subseteq N_{G'}(C)$. For each such C, at most one element of Z_C can contain S. In total at most $\binom{u}{t-1}$ elements of Z can contain S. Since A has degree at most $\binom{k}{t}\binom{u}{t-1}$.

Therefore, D contains an independent set Z' of size at least $\frac{|Z|}{\Delta(D)+1} \geq \frac{n^{t-1}}{\Delta(D)+1} \geq \beta|w|^t$. ∎

Claim 4. *Let M denote the number of copies of ${K_k}^{(t)}$ in H. Then there is a $\gamma = \gamma(k, t, \varepsilon)$ such that $|M| \geq \gamma n^{\frac{(t-1)k}{t}}$.*

Proof. By claim 3, we see there are at least $|Z'| \geq \gamma\beta|w|^t$ edge-disjoint ${K_k}^{(t)}$. To destroy all copies of ${K_k}^{(t)}$ in H, one needs to delete one edge in each of these edge-disjoint ${K_k}^{(t)}$, which results in the removal of at least $\beta|w|^t$ edges. By lemma 3(HRL), then $|M| \geq \delta|w|^k \geq \gamma n^{\frac{(t-1)k}{t}}$, where $|w| = \Theta(n^{1-1/t})$. ∎

Definition 1.6. A copy R of ${K_k}^{(t)}$ in H is *bad*, if there are two distinct $S, S' \in E(R)$ with $N(S) \bigcap N(S') \neq \emptyset$. Otherwise, R is *good.*

Claim 5. *A good copy R of ${K_k}^{(t)}$ in H can give a copy of H_k in $W \bigcup V$. Then all copies of ${K_k}^{(t)}$ are blue and bad.*

Claim 6. *There exists a $\gamma' = \gamma'(k, t, \varepsilon)$ such that the number of bad copies of ${K_k}^{(t)}$ is at most $\gamma' n^{\frac{(t-1)k-1}{t}}$.*

Proof. If R is bad, we have $S, S' \in E(R)$ with $N(S) \bigcap N(S') \neq \emptyset$. Let $x \in N(S) \bigcap N(S')$. Then $|N_{G'}(x) \bigcap V(R)| \geq |S \bigcup S'| \geq t+1$. Summing over all vertices $X \in V$, we see that the number of bad ${K_k}^{(t)}$ is at most

$$\sum_{x \in V} \binom{N(x) \bigcap W}{t+1} |w|^{k-t-1} \leq n(2\alpha n^{1-2/t})^{t+1}(2\alpha n^{1-1/t})^{k-t-1} = (2\alpha)^k n^{\frac{(t-1)k-1}{t}}.$$

∎

Claim 4 and Claim 6 contradict to each other. This proves the theorem 1.3.

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