Day 12

Wednesday, April 1, 2020 1:55 PM

We now turn to extremal problems (mostly on Turán numbers) for bipartite graphs. We start by considering trees, paths and cycles.

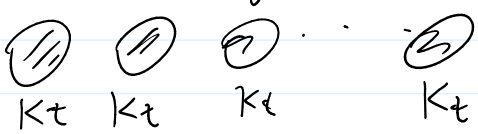
Prop 1. For $n \geq k+2$, any n-vertex graph G with at least $kn - \frac{k^2+k-2}{2}$ edges contains a subgraph of minimum degree at least $k+1$. (If $e(G) \geq kn$, then $\exists$ such a subgraph)

Prop 2. For any tree T on $t+1$ vertices, $ex(n,T) < (t-1)n$.

Pf. Let G be an n-vtx graph with at least $(t-1)n$ edges.
By Prop 1, $\exists$ subgraph G' of G with $\delta(G') \geq t$.
Using a greedy algorithm, one can find any tree on $t+1$ vertices. $\boxed{\not{p}}$

Conjecture (Erdős–Sós) Let T be any tree on $t+1$ vertices.
Then $ex(n,T) \leq \frac{(t-1)n}{2}$.

- Note that a vertex-disjoint union of cliques K_t. $\Rightarrow$ tight.



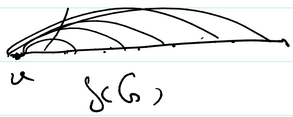
- An approximate version of this conjecture was confirmed by Ajtai–Komlós–Simonovits–Szemerédi.

Def. The length $|P|$ of a path P denotes the number of edges in P.

Let P_t be the path of length t.

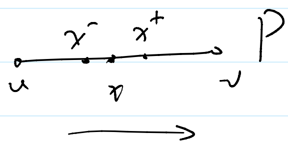
Prop 3. Any graph G has a path of length at least $\delta(G)$.

longest path



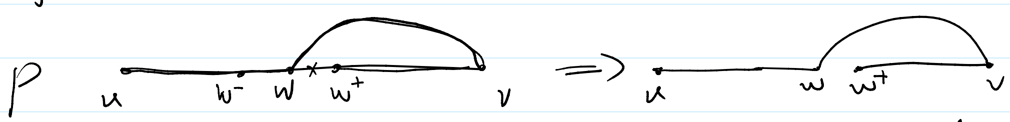
$|P| \geq d(u) \geq \delta(G)$.

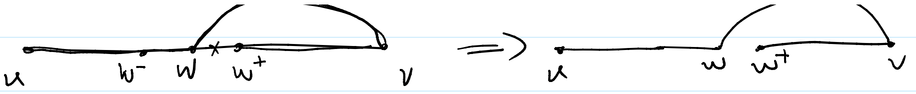
Def. Let P be a path in G from u to v.



For $x \in V(P)$, denote x^- and x^+ to be the immediate predecessor and immediate successor of x on P.
For $S \subseteq V(P)$, let $S^+ = \{x^+ : x \in S\}$ and $S^- = \{x^- : x \in S\}$.

Def. Let P be a longest path in G from u to v.



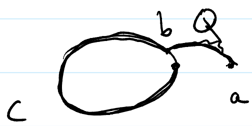
P 

For $w \in N(v)$, the path $P' = P - \{ww^+\} + \{wv\}$ is also a longest path in G. This transformation from P to P' is called a <u>Pósa rotation</u>.

Def. A path or a cycle is called <u>Hamiltonian</u>, if it contains all vertices of the graph G. And G is Hamiltonian if G has such a cycle.

Prop 4. Let G be connected and P be a longest path in G. If there exists a cycle C with $V(C) = V(P)$, then G is Hamiltonian.

Pf. Suppose $V(C) \neq V(G)$. As G is connected, there exist a vertex $a \notin V(C)$ and a path Q from a to $b \in V(C)$ internally disjoint from $V(C)$.



Then we can find a longer path than P, a contradiction. $\boxtimes$

Thm 1. If G is connected, then G has a path with at least $\min\{n, 2\delta(G)+1\}$ vertices.

Pf. Let $P = x_0 x_1 \cdots x_m$ be a longest path in G.

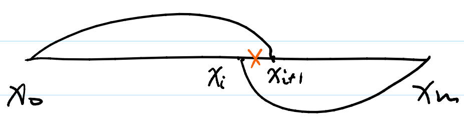
So $N(x_0), N(x_m) \subseteq V(P)$. Suppose $|V(P)| < \min\{n, 2\delta(G)+1\}$.

We claim that $\exists\, i \in \{0, 1, \cdots, m-1\}$ s.t. $x_0 x_{i+1}, x_m x_i \in E(G)$

[Suppose not. Then $N(x_0) \cap N(x_m)^+ = \phi$.

Also $x_0 \notin N(x_0) \cup N(x_0^+)$ $\Rightarrow$ $|V(P)| \geq 1 + |N(x_0) \cup N(x_m)^+|$

$\geq 1 + d(x_0) + d(x_m) \geq 1 + 2\delta(G)$, a contradiction.]

$\Rightarrow$ 

We can find a cycle $C = P - \{x_i x_{i+1}\} + \{x_0 x_{i+1}, x_m x_i\}$ with $V(C) = V(P)$.

By Prop 4, G is Hamiltonian. $\boxtimes$

Thm 2. (Erdős–Gallai) For $\forall\, n \geq t$, $ex(n, P_t) \leq \frac{(t-1)n}{2}$

Rmk: a special case of Erdős–Sós Conjecture.

Pf. By induction on n. It is trivial for $n \leq t$.

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For $\boxed{n \geq t+1}$, let G be a P_t-free graph on n vertices.

We want to show $e(G) \le \frac{(t-1)n}{2}$

We may assume that G is connected. (O.W. by induction)

If $\delta(G) \geq \frac{t}{2}$, by Thm 1, G has a path with at least $\min\{n, 2\delta(G)+1\} \geq \min\{n, t+1\} \geq t+1$, a contradiction.

So there is a vertex v of degree at most $\frac{t-1}{2}$.

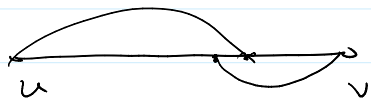
Let $G' = G - \{v\}$. By induction, $e(G') \le \frac{(t-1)(n-1)}{2}$

$$\Longrightarrow \quad e(G) = e(G') + d(v) \le \frac{(t-1)n}{2} \qquad \boxed{\text{QED}}$$

Rmk. The unique extremal graph is a disjoint union of K_t's (EX)

Thm 3 (Ore's Thm) Let G be an n-vtx graph such that for any non-adjacent vertices u and v, $d(u) + d(v) \geq n$. Then G is Hamiltonian.

Pf. First, such G is connected. Let P be a longest path in G, say from u to v.

P 

By Prop 4, we may assume $uv \notin E(G)$.

So $d(u) + d(v) \geq n \Longrightarrow N(u) \cap N(v)^+ \neq \emptyset \Longrightarrow \exists$ a cycle C with $V(C) = V(P) \Longrightarrow$ By Prop 4, G is Hamiltonian. $\boxed{\text{QED}}$

Corollary (Dirac's Thm) If $\delta(G) \geq \frac{|V(G)|}{2}$, then G is Hamiltonian.

Def. The closure of a graph G is the graph obtained from G by recursively joining pairs of non-adjacent vertices u, v whose degree sum is at least n until no such pairs exist.

Prop 5. The closure of G is well-defined ($\Leftrightarrow$ it is unique $\Leftrightarrow$ the ordering of adding edges will not affect the final graph)

Thm 4 (Bondy-Chvátal) A graph is Hamiltonian iff its closure (will not appear in the final [illegible])

Thm 4 (Bondy–Chvátal) A graph is Hamiltonian iff its closure is Hamiltonian.

pf. It suffices to show:

EX $\Big[$ For any non-adjacent $\{u,v\}$ with $d_G(u)+d_G(v) \geq n$, G is Ham. iff $G+\{uv\}$ is Ham. $\Big]$ ☒

Rmk: Thm 4 $\Rightarrow$ Thm 3. ✓

Def. For $S \subseteq V(G)$, define $N(S) = \{v \notin S : v \in N(w) \text{ for some } w \in S\}$ to be the neighborhood of S in G.

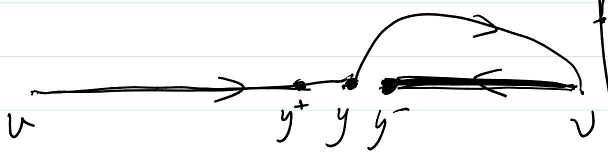
Pósa's lemma Let P be a longest path (say from u to v) in a graph G. Let S be the set of all endpoints of paths obtained by repeatedly applying Pósa's rotations from P, while preserving u as an endpoint. Clearly $S \subseteq V(P)$ & $N(S) \subseteq V(P)$. Then $|N(S)| \leq 2|S|$.

pf: As before, let S^+ & S^- be subsets of $V(P)$.

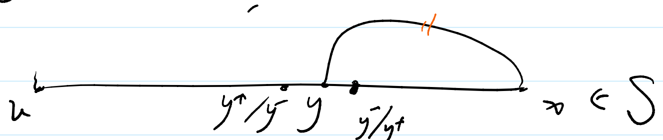
claim: $N(S) \subseteq S^+ \cup S^- \Leftrightarrow \forall x \in S,\ N(x) \subseteq S^+ \cup S \cup S^-$.

pf. Suppose $\exists\, y \in N(x)$ but $y \notin S^+ \cup S \cup S^-$

If $y \notin S^+ \cup S \cup S^-$, then $y^- y y^+$ is always a subpath of any new path obtained by Pósa's rotation. ✓



Using $xy \in E(G)$,



we can have a Pósa's rotation to get a new longest P', which ends at y^+ or y^-. $\Rightarrow y^+$ or $y^- \in S \Rightarrow y \in S^+ \cup S^-$, a contradiction. ☒

a contradiction. $\boxtimes$

Using this claim, we see $|N(S)| \le 2|S|$. $\boxtimes$

Thm 5. Suppose any $S \subseteq V(G)$ satisfying
$$|N(S)| \ge \min\{n-|S|,\ 2|S|+1\}. \Rightarrow G \text{ has a Hamiltonian path.}$$

Pf: Let P be a longest path in G. Define S as in the page's Lemma. $\Rightarrow 2|S| \ge |N(S)| \ge \min\{n-|S|,\ 2|S|+1\}$.

$\Rightarrow \min\{n-|S|,\ 2|S|+1\} = n-|S|$ & $|N(S)| \ge n-|S|$.

As $N(S) \cap S = \emptyset$, $|V(P)| \ge |N(S)| + |S| \ge n$. $\boxtimes$

Thm 6. If G is 2-connected, then G has a cycle of length at least $\min\{n, 2\delta(G)\}$.