Day 18

Wednesday, April 29, 2020 1:54 PM

Monday: $ex(n, K_{s,t}) = \Theta(n^{2-\frac{1}{s}})$ if $t \geq (s-1)!+1$

using algebraic construction $\boxed{K_{3,3}, K_{4,7}}$

Today, we consider a new construction

§ Randomized algebraic constructions

Let q be a prime power and let $\mathbb{F}_q$ be a finite field of order q.

Let $s \geq 4$ be fixed and $q \gg s$. Let $d = s^2 - s + 2$, and $n = q^s$.

Def. Let $\vec{X} = (x_1, \ldots, x_s) \in \mathbb{F}_q^s$ & $\vec{Y} = (y_1, \ldots, y_s) \in \mathbb{F}_q^s$.

Let $\mathcal{P}$ be the family of all polynomials $f(\vec{X}, \vec{Y})$ of degree at most d in each of $\vec{X}$ and $\vec{Y}$, that is,

$$\boxed{f(\vec{X}, \vec{Y}) = \sum_{(\vec{a}, \vec{b})} \alpha_{\vec{a}, \vec{b}}\, x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} \cdot y_1^{b_1} \cdots y_s^{b_s}},$$

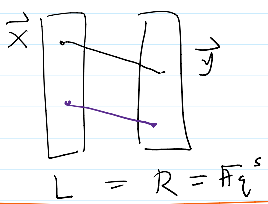
over all possible choices that $\sum_{i \in [s]} a_i \leq d$ and $\sum_{j \in [s]} b_j \leq d$

where $\alpha_{\vec{a}, \vec{b}} \in \mathbb{F}_q$.

Def. For any $f(\vec{X}, \vec{Y}) \in \mathcal{P}$, we can define a bipartite graph G_f on bipartition (L, R) as follows.

$L = R = \mathbb{F}_q^s$, and $\vec{X} \in L \sim \vec{Y} \in R$ iff $f(\vec{X}, \vec{Y}) = 0$.

$n = q^s$



$f(\vec{X}, \vec{Y}) = 0$

$\Rightarrow \mathbb{E}[e(G)] = n^2/q$

(#) The key idea is to choose a polynomial $f \in \mathcal{P}$ randomly at uniform and use it to define a bipartite G_f.

Lemma 1. $\forall \vec{u}, \vec{v} \in \mathbb{F}_q^s$, $\mathbb{P}_r[f(\vec{u}, \vec{v}) = 0] = 1/q$

Pf: Note that if c is a uniformly random constant in $\mathbb{F}_q$, then $f(\vec{u}, \vec{v})$ and $f(\vec{u}, \vec{v}) + c$ are identically distributed. Hence, each of the q possibilities are equally likely.

So $\mathbb{P}_r[f(\vec{u}, \vec{v}) = 0] = 1/q$. ∎

Fact: Let A be an event in a probability space:

Sampling / conditional probability

$$\mathbb{P}(A) = \sum_{\text{events } B} \mathbb{P}[A \mid B] \cdot \mathbb{P}[B]$$

If $\mathbb{P}[A \mid B] = a$ for $\forall$ event B, then $\mathbb{P}[A] = a$. (#)

Lemma 2. Suppose $r, s \leq \min\{\sqrt{q}, d\}$. Let $U \subseteq \mathbb{F}_q^s$ and $V \subseteq \mathbb{F}_q^s$ be sets with $|U| = s$ and $|V| = r$.

Then $\mathbb{P}_r[f(\vec{u}, \vec{v}) = 0 \text{ for all } \vec{u} \in U \text{ \& } \vec{v} \in V] = \left(\frac{1}{q}\right)^{sr}$

Pf. Call ... $\mathbb{F}_q^s$ ... first coordinates of

Then $\Pr[f(\vec{u},\vec{v})=0$ for all ... $] \leq (q)$

Pf. Call a set of points in $\mathbb{F}_q^s$ *simple* if the first coordinates of the points are distinct.

① First, we give the proof when both U and V are simple.

In this case, we decompose $f = g + h$, where h contains the sr monomials $x_1^i y_1^j$ for $i=0,1,\dots,s-1$ and $j=0,1,\dots,r-1$, and g is the linear combination of other monomials.

To prove $\Pr[f(\vec{u},\vec{v})=0$ for all $\vec{u}\in U$ & $\vec{v}\in V] = q^{-sr}$, it suffices to prove that the system of sr equations

$$h(\vec{u},\vec{v}) = -g(\vec{u},\vec{v}) \quad \text{for all } \vec{u}\in U,\ \vec{v}\in V \qquad (*)$$

has a unique solution when all $-g(\vec{u},\vec{v})$ are given

KEY.

Note that $h(\vec{x},\vec{y}) = \sum_{\substack{i<s\\ j<r}} \alpha_{ij}\, x_1^i y_1^j$ has sr terms

& the system $(*)$ consists of sr equations with sr unknown variables α_{ij}, $0\le i\le s-1$ & $0\le j\le r-1$.

This is a consequence of the Lagrange interpolation theorem twice.

- The 1^{st} application gives for all fixed $\vec{u}\in U$, the single-variable polynomials $h_{\vec{u}}(\vec{Y})$ of degree $r-1$ such that $h_{\vec{u}}(\vec{v}) = -g(\vec{u},\vec{v})$ for all $\vec{v}\in V$.
- The second application gives a polynomial $h(\vec{x},\vec{Y}) = \sum_{0\le j\le r-1} a_j(x_1)\cdot y_1^j$ such that each of the coefficients of $h(\vec{u},\vec{Y})$ is equal to the respective coefficient of $h_{\vec{u}}(\vec{Y})$ for all $\vec{u}\in U$.

Using this twice, we show the solution is unique. ∎

② Now we consider the general U and V. It suffices to find invertible linear transformations T and $S: \mathbb{F}_q^s \to \mathbb{F}_q^s$ such that TU and SV are simple. Indeed, $\Pr$ is invariant under the actions of linear transformations on the first s variables $(\vec{x})$ and then on the latter s variables $(\vec{Y})$.

Hence if we arrange for TU and SV to be simple, we reduce to ①.

first s variables (x) and then on the latter s variables (y).

Hence, if we arrange for TU and SV to be simple, we reduce to ①.

To find such $T : \mathbb{F}_q^s \to \mathbb{F}_q^s$, it suffices to find a linear map $T_1 : \mathbb{F}_q^s \to \mathbb{F}_q$ that injective on U ($\forall \vec{u}_1, \vec{u}_2 \in U$, $T_1(\vec{u}_1) \neq T_1(\vec{u}_2)$).

We can then find an invertible map $T : \mathbb{F}_q^s \to \mathbb{F}_q^s$ whose first coordinate is T_1. To find such a T_1, we pick T_1 uniformly at random among all linear maps $\mathbb{F}_q^s \to \mathbb{F}_q$.

Then for all pairs $(\vec{u}_1, \vec{u}_2)$ in U, $\Pr(T_1(\vec{u}_1) = T_1(\vec{u}_2)) = \frac{1}{q}$.

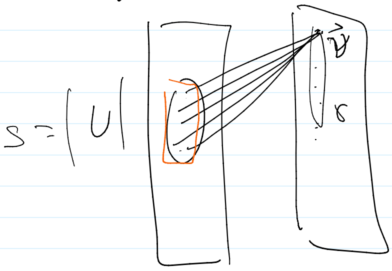
So by union bound,

$$\Pr[\exists\, \vec{u}_1, \vec{u}_2 \in U \text{ with } T_1(\vec{u}_1) = T_1(\vec{u}_2)] \le \binom{|U|}{2} \frac{1}{q} < 1,$$

implying the existence of the desired $T_1 : \mathbb{F}_q^s \to \mathbb{F}_q$.

And the construction for S is similar. $\square$

Fix $U \subseteq \mathbb{F}_q^s$ with $|U| = s$. We want to count the common neighbor of the vertices in U. We will use the moments method.



Let $I(\vec{v}) = \begin{cases} 1, & \vec{v} \text{ is adj. to any } \vec{u} \in U \\ 0 & \text{o.w.} \end{cases}$

Let $X = |N(U)|$. Then $X = \sum_{\vec{v}} I(\vec{v})$.

Then $\mathbb{E}[X^d] = \mathbb{E}\left[\left(\sum_{\vec{v} \in \mathbb{F}_q^s} I(\vec{v})\right)^d\right]$

(may repeated) $= \sum_{\vec{v}_1, \dots, \vec{v}_d \in \mathbb{F}_q^s} \mathbb{E}[\underbrace{I(\vec{v}_1) I(\vec{v}_2) \cdots I(\vec{v}_d)}_{r}]$

$r = \#$ distinct vectors in $\vec{v}_1 .. \vec{v}_d$

$$= \sum_{1 \le r \le d} \binom{q^s}{r} q^{-rs} M_r \le \sum_{r \le d} M_r \overset{\Delta}{=} M,$$

where M_r is defined to the number of surjections from $[d]$ to $[r]$.

By Markov's inequality, $\Pr(X \ge \lambda) = \Pr(X^d \ge \lambda^d)$

$$\le \frac{\mathbb{E}[X^d]}{\lambda^d} \le \frac{M}{\lambda^d}$$

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Lemma 3. For all s, d, there exists a constant C such that if $f_1(\vec{Y}), f_2(\vec{Y}), \dots, f_s(\vec{Y})$ are polynomials over $\vec{Y} \in \mathbb{F}_q^s$ of degree at most d, then

$\{\vec{y} \in \mathbb{F}_q^s : f_1(\vec{y}) = f_2(\vec{y}) = \dots = f_s(\vec{y}) = 0\}$ has size either at most C or at least $q - C\sqrt{q} \ge \frac{q}{2}$.

Rmk: This lemma can be reduced from an important result in algebraic geometry, known as the Lang-Weil Bound (1954). It says that roughly # points in an r-dimensional algebraic variety in $\mathbb{F}_q^s$ is around q^r (assuming some irreducibility conditions)

As
$$\begin{aligned}
X &= \#\text{ common neighbors of vertices } \vec{u}_1, \vec{u}_2, \dots, \vec{u}_s \in U \\
&= \left| \{ \vec{v} \in \mathbb{F}_q^s : \vec{v} \sim \vec{u}_i \ \forall i \in [s] \text{ in } G \} \right| \\
&= \left| \{ \vec{v} \in \mathbb{F}_q^s : f(\vec{u}_i, \vec{v}) = 0 \ \forall i \in [s] \} \right| \\
&= \left| \{ \vec{y} \in \mathbb{F}_q^s : f_{\vec{u}_1}(\vec{y}) = f_{\vec{u}_2}(\vec{y}) = \dots = f_{\vec{u}_s}(\vec{y}) = 0 \} \right|,
\end{aligned}$$

by Lemma 3, if $X > C$, then $X > \frac{q}{2}$

$$\Rightarrow \Pr(X > C) = \Pr\left(X \ge \frac{q}{2}\right) \le \frac{E[X^d]}{(q/2)^d} \le \frac{M}{(q/2)^d}$$

So the number of s-subsets in L or in R with more than C common neighbors is at most

$$2\binom{n}{s} \frac{M}{(q/2)^d} = O(q^{s-2}) \text{ in expectation.}$$

Take such a G and remove a vertex from every such s-subset

Take such a G and remove a vertex from every such s-subset to create a new graph G'. First, we see G' is $K_{s,C+1}$-free

$$\mathbb{E}[e(G')] \geq \mathbb{E}[e(G)] - \mathbb{E}[\#\text{ such } s\text{-subsets}]\cdot n$$

$$= n^2/q - \underline{O(q^{s-2})\cdot n} = (1-o(1))\, n^{2-1/s},$$

and $v(G') \leq 2n$. ∎

$$\Longrightarrow \boxed{\begin{array}{l}\underline{\text{Thm}}\ (\text{Bukh, 2015})\quad \forall s,\ \exists\, C = C(s) \\ \text{s.t.}\quad ex(n, K_{s,C+1}) = \Theta(n^{2-1/s})\end{array}}$$