Day 21

Wednesday, May 20, 2020 1:51 PM

Let H be a bipartite graph with bipartition (A,B) such that every vertex in A has degree at most t. $(*)$

Thm 1 (Füredi, Alon-Krivelevich-Sudakov)

$\forall H$ satisfying $(*)$, $ex(n,H) = O(n^{2-1/t})$.

Rmk. tight for $H = K_{t,s}$ where $s \gg t$.

Conjecture (Conlon-Lee) Let H be a bipartite graph satisfying $(*)$. If H is $K_{t,t}$-free, then there exists a constant $c = c(H) > 0$ such that

$$ex(n,H) = O(n^{2-1/r-c})$$

- We proved $r=2$ ✓
- We prove in this lecture that $ex(n,H) = o(n^{2-1/r})$.

Main Thm (Sudakov-Tomon, '20) Let H be a $K_{t,t}$-free bipartite graph satisfying $(*)$. Then $ex(n,H) = o(n^{2-1/r})$.

Notation.
- $X^{(t)} = \{$all subsets of size t in $X\}$
- $K_k^{(t)}$ = complete t-graph on k vertices
- $N_G(S) = \{v \notin V \setminus S : vs \in E(G)$ for $\forall s \in S\}$.
- A graph G is K-almost regular, if $\Delta(G) \le K \cdot \delta(G)$ $(K \ge 1)$.

Lemma 1. Let $0 < c < 10^{-4}$ and $\frac{1}{2} \le \alpha < 1$. Let n be a sufficiently large integer compared to c and α. Let G be an n-vtx graph with $e(G) \ge c \cdot n^{1+\alpha}$. Then G contains a bipartite subgraph G', where both vertex classes have size $m \ge \frac{1}{2} n^{\frac{\alpha-\alpha^2}{4(1+\alpha)}}$, $e(G') \ge \frac{c}{10} m^{1+\alpha}$ and $\Delta(G') < m^{\alpha}$.

pf. This can be derived from a lemma of previous lecture ☒

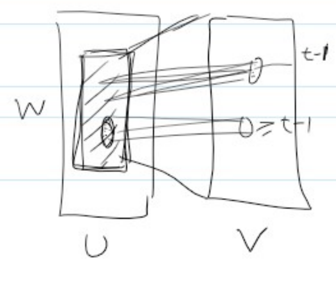
Lemma 2. Let k, t be integers. Then there exists $\Delta = \Delta(k,t)$ such that any 2-edge-coloring of $K_{\Delta}^{(t)}$ contains a monochromatic copy of $K_k^{(t)}$.

pf. omit (HW). ☒

Lemma 3 (Hypergraph Removal Lemma; Nagle-Rödl-Schacht & Gowers)

Let $k, t \in \mathbb{Z}^+$. For $\forall \beta > 0$, $\exists \delta = \delta(k,t,\beta) > 0$ s.t. the following holds. If $\mathcal{H}$ is a t-graph on n vertices s.t. one needs to delete at least βn^t edges of $\mathcal{H}$ to make it $K_k^{(t)}$-free, then $\mathcal{H}$ contains at least δn^k copies of $K_k^{(t)}$.

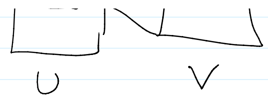
Ideas



$e(G) \ge \varepsilon \cdot n^{2-1/r}$

K-almost regular

$\Rightarrow d(v) \approx n^{1-1/r}$



① $W \subseteq U$, $G' = G[W \cup V]$, $|W| \sim n^{1-1/t}$

$\Rightarrow$ Typical $S \in V^{(t-1)}$ has $\Omega(1)$ common neighbors in W.

② Define t-graph $\mathcal{H}$ on W

$S \in W^{(t)}$ is an edge of $\mathcal{H}$ iff $|N(S)| \geq t-1$.

$\Rightarrow$ No $\boxed{\text{red}}$ $K_k^{(t)}$ on W $\Rightarrow$ all $K_k^{(t)}$ are $\boxed{\text{blue}}$

③ Using HRL $\Rightarrow$ W has $\Omega(|W|^k)$ copies of $K_k^{(t)}$

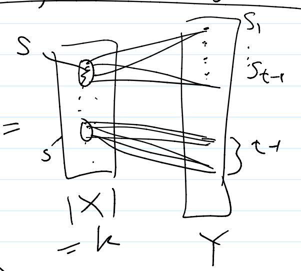
④ Show NOT many "bad" $K_k^{(t)}$ $\Rightarrow$ Done.

Chernoff's inequality: $X \sim$ binomial distribution $B(n,p)$

$$\Pr(X \geq (1+\lambda)pn) \leq e^{-\lambda^2 pn/3}$$
$$\Pr(X \leq (1-\lambda)pn) \leq e^{-\lambda^2 pn/3}$$

i.e. $X = \sum_{i=1}^{n} X_i$, where $\forall i$ $\Pr[X_i = 1] = p$, $\Pr[X_i = 0] = 1-p$

Pf of Main Thm. Let H_k be the bipartite graph on parts X, Y such that $|X| = k$, $|Y| = (t-1)\binom{k}{t}$, and for every $S \in X^{(t)}$, there are exactly $t-1$ vertices in Y whose neighborhood is equal to S.

$H_k =$ 

Note that any H satisfying the condition is contained in some H_k.

- It is enough to only prove that $\forall k$, $\mathrm{ex}(n, H_k) = o(n^{2-1/t})$ ②
- We will assume $t \geq 3$.
- We will show that $\forall\, 0 < \varepsilon < 10^{-4}$, if n is sufficiently large, then $\mathrm{ex}(n, H_k) \leq \varepsilon \cdot n^{2-1/t}$ $\Rightarrow$ ②

Let G_0 be an n_0-vtx H_k-free graph with $e(G_0) > 6\varepsilon n_0^{2-1/t}$.

By Lemma 1, G_0 has a bipartite subgraph G with parts U & V such that $n = |U| = |V| > \frac{1}{2}(n_0)^{(1-1/t)/(8t-4)}$

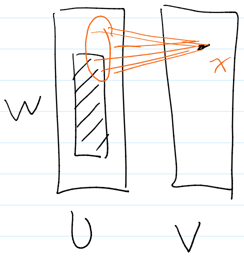
& $e(G) \geq \varepsilon \cdot n^{2-1/t}$ & $\Delta(G) \leq n^{1-1/t}$

& G is H_k-free. $\boxed{n \gg k, t, \varepsilon}$

By Lemma 2, $\exists\, \Delta = \Delta(k,t)$ s.t. any red&blue edge-coloring

By Lem 2, $\exists\ \Delta = \Delta(k,t)$ s.t. any red/blue edge-coloring of $K_\Delta^{(t)}$ contains either a red or blue copy of $K_k^{(t)}$.

Claim 1. Let $\boxed{p = \alpha n^{-1/t}}$, where $\alpha = 2\Delta\left(\frac{t-1}{\varepsilon}\right)^{t-1} 2^{3t-4}$.



Then there exists $W \subseteq U$ such that

- $\frac{pn}{2} < |W| < 2pn$
- $G' = G[W \cup V]$ has $\geq \frac{p}{4} e(G)$ edges
- $\forall x \in V$, $d_{G'}(x) < 2p \cdot n^{1-1/t}$.

pf. Pick each vertex of U with prob. p (indep. of each other) and let W be the set of selected vertices.

Then the statements follow by standard concentration inequalities.

$\boxed{\text{For } x \in V \text{ with } d_G(x) \geq n^{1/4}}$, by chernoff's bounds,

the prob. that $|d_{G'}(x) - p \cdot d_G(x)| < \frac{1}{2} p\, d_G(x)$ $\boxed{t \geq 3}$

$$\geq 1 - 2e^{-p d_G(x)/12} \qquad \text{where } p \cdot d_G(x) = \Omega\left(n^{1-2/t}\right) \gg n^{1/4}.$$

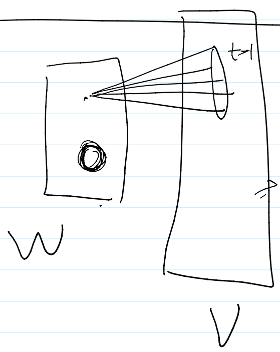
$$\geq 1 - 2 \cdot e^{-n^{1/4}/12}$$

$\Rightarrow$ with high prob. $(1-o(1))$, $|d_{G'}(x) - p \cdot d_G(x)| < \frac{1}{2} p\, d_G(x)$ $\forall x \in V$.

Also with high prob. $(1-o(1))$, $\big|\,|W| - pn\,\big| < \frac{1}{2} pn$.

Lastly, $e(G') = \sum_{x \in V} d_{G'}(x) \geq \sum_{x \in V,\ d_G(x) \geq n^{1/4}} \frac{1}{2} p \cdot d_G(x)$

$$\geq \frac{1}{2} p \cdot e(G) - n \cdot n^{1/4} \geq \frac{1}{4} p \cdot e(G) \quad \blacksquare$$



We consider

$$L = \sum_{C \in V^{(t-1)}} |N_{G'}(C)| = \boxed{\#\ (t-1)\text{-stars}}$$

$$= \sum_{x \in W} \binom{d_{G'}(x)}{t-1} \geq |W| \binom{e(G')/|W|}{t-1}$$

$$\geq (t-1)^{-(t-1)} e(G')^{t-1} |W|^{-(t-2)}$$

$$\geq \left(\frac{\varepsilon}{t-1}\right)^{t-1} 2^{-3t+4} \cdot p \cdot n^{t-1+1/t} = 2\Delta n^{t-1}.$$

We point out $\alpha n^{1-1/t}$, $|W| < 2\alpha n^{1-1/t}$

$= (t-1) \quad \cdot \quad p \cdot n$

We point out $\frac{\alpha}{2} n^{1-1/t} < |W| < 2\alpha \cdot n^{1-1/t}$

& $\forall x \in V$ has degree at most $2p \cdot n^{1-1/t} = 2\alpha \cdot n^{1-2/t}$ in G'.

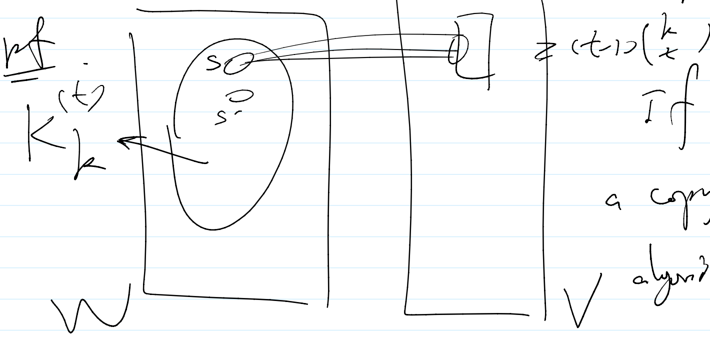
where α is independent of n.

Let $\mathcal{H}$ be the t-graph on W such that $S \in W^{(t)}$ is an edge of $\mathcal{H}$ iff $|N_{G'}(S)| \geq t-1$.

Then we color an edge $S \in E(\mathcal{H})$ by red if $|N_{G'}(S)| \geq (t-1)\binom{k}{t}$ and color it by blue if $t-1 \leq |N_{G'}(S)| < (t-1)\binom{k}{t}$.

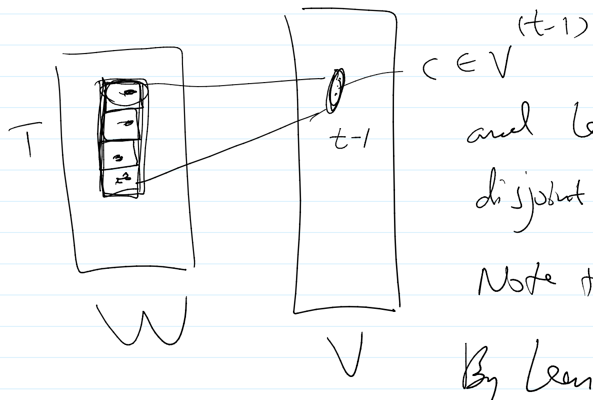
Claim 2. $\mathcal{H}$ has no red $K_k^{(t)}$.

pf.



If so, then we can find a copy of H_k by a greedy algorithm applied to this $K_k^{(t)}$ ∎

Let $C \in V^{(t-1)}$ and consider $T = N_{G'}(C)$



Let $r = \lfloor \frac{|T|}{\Delta} \rfloor \geq \frac{|T|}{\Delta} - 1$ and let $T_1, T_2, \ldots, T_r$ be disjoint sets of size Δ in T.

Note that $\mathcal{H}[T_i]$ is a clique $K_\Delta^{(t)}$.

By Lemma 2 & claim 2,

each T_i contains a blue $K_k^{(t)}$ in $\mathcal{H}$, called A_i.

Set $\mathcal{Z}_C = \{A_1, \ldots, A_r\}$.

& $\mathcal{Z} = \bigcup_{C \in V^{(t-1)}} \mathcal{Z}_C$ is a multiset

(diff. sets C may have the same A_i)

Note $|\mathcal{Z}| = \sum |\mathcal{Z}_C| \geq \sum \left(\frac{|N_{G'}(C)|}{\quad} - 1 \right)$

Note $|\mathcal{Z}| = \sum_{C \in V^{(t-1)}} |\mathcal{Z}_C| \geq \sum_{C \in V^{(t-1)}} \left(\frac{|N_{G'}(C)|}{\Delta} - 1 \right)$

$$\geq \frac{L}{\Delta} - \binom{n}{t-1} \geq n^{t-1}.$$

<u>claim 3</u>. $\exists\ \beta = \beta(k, t, \varepsilon) > 0$ and $\mathcal{Z}' \subseteq \mathcal{Z}$ such that $|\mathcal{Z}'| \geq \beta \cdot |W|^{t}$ and any two cliques in $\mathcal{Z}'$ are <u>edge-disjoint</u>