Let H be a bipartite graph with bipartition (A, B) such that every vertex in A has degree at most t. $(*)$

Thm 1 (Füredi, Alon-Krivelevich-Sudakov)
$\forall H$ satisfying $(*)$, $ex(n,H) = O(n^{2-1/t})$.

Rmk. tight for $H = K_{t,s}$ where $s \gg t$.

Conjecture (Conlon-Lee) Let H be a bipartite graph satisfying $(*)$. If H is $K_{t,t}$-free, then there exists a constant $c = c(H) > 0$ such that
$$ex(n,H) = O(n^{2-1/r-c})$$

- We proved $r=2$ ✓
- We prove in this lecture that $ex(n,H) = o(n^{2-1/r})$.

Main Thm (Sudakov-Tomon, '20) Let H be a $K_{t,t}$-free bipartite graph satisfying $(*)$. Then $ex(n,H) = o(n^{2-1/r})$.

Notation
- $X^{(t)} = \{$all subsets of size t in $X\}$
- $K_k^{(t)}$ = complete t-graph on k vertices
- $N_G(S) = \{v \notin V \setminus S : vs \in E(G)$ for $\forall s \in S\}$.
- A graph G is K-almost regular if $\Delta(G) \leq K \cdot \delta(G)$. $(K \geq 1)$

Lemma 1. Let $0 < c < 10^{-4}$ and $\frac{1}{2} \leq \alpha < 1$. Let n be a sufficiently large integer compared to c and α. Let G be

Lemma 1. Let $0 < c < 10^{-7}$ and $\frac{1}{2} \le \alpha < 1$. Let n be a sufficiently large integer compared to c and α. Let G be an n-vtx graph with $e(G) \ge c \cdot n^{1+\alpha}$. Then G contains a bipartite subgraph G', whose both vertex classes have size $m \ge \frac{1}{2} n^{\frac{\alpha-\alpha^2}{4(1+\alpha)}}$, $e(G') \ge \frac{c}{10} m^{1+\alpha}$ and $\Delta(G') < m^{\alpha}$.

pf. This can be derived from a lemma of previous lecture ☒

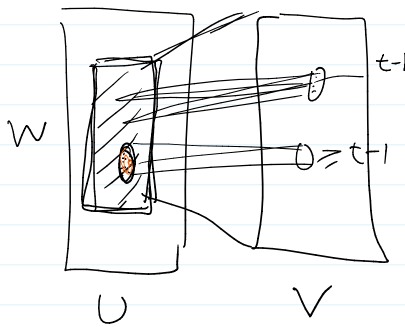
Lemma 2. Let k, t be integers. Then there exists $\Delta = \Delta(k,t)$ such that any 2-edge-coloring of $K_{\Delta}^{(t)}$ contains a monochromatic copy of $K_k^{(t)}$.

pf. omit (HW). ☒

Lemma 3 (Hypergraph Removal Lemma; Nagle-Rödl-Schacht & Gowers)

Let $k, t \in \mathbb{Z}^+$. For $\forall \beta > 0$, $\exists \delta = \delta(k,t,\beta) > 0$ s.t. the following holds. If $\mathcal{H}$ is a t-graph on n vertices s.t. one needs to delete at least βn^t edges of $\mathcal{H}$ to make it $K_k^{(t)}$-free, then $\mathcal{H}$ contains at least δn^k copies of $K_k^{(t)}$.

Ideas



- $e(G) \ge \varepsilon \cdot n^{2-1/r}$
- K-almost regular

$\Rightarrow d(v) \approx n^{1-1/r}$

① $W \subseteq U$, $G' = G[W \cup V]$, $|W| \sim n^{1-1/r}$

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$\Rightarrow$ Typical $S \in V^{(t-1)}$ has $\Omega(1)$ common neighbors in W.

② Define t-graph $\mathcal{H}$ on W

$S \in W^{(t)}$ is an edge of $\mathcal{H}$ iff $|N(S)| \geq t-1$.

$\Rightarrow$ No $\boxed{\text{red}}$ $K_k^{(t)}$ on W $\Rightarrow$ all $K_k^{(t)}$ are $\boxed{\text{blue}}$

③ Using HRL $\Rightarrow$ W has $\Omega(|W|^k)$ copies of $K_k^{(t)}$

④ show NOT many "bad" $K_k^{(t)}$ $\Rightarrow$ Done.

Chernoff's inequality: $X \sim$ binomial distribution $B(n,p)$

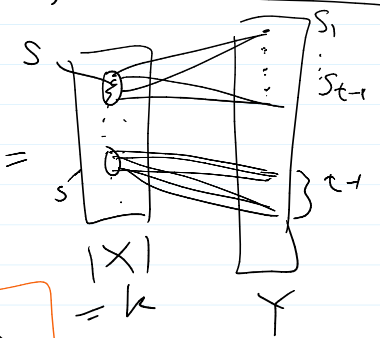
$$\begin{cases} \Pr\left(X \geq (1+\lambda)pn\right) \leq e^{-\lambda^2 pn/3} \\ \Pr\left(X \leq (1-\lambda)pn\right) \leq e^{-\lambda^2 pn/3} \end{cases}$$

i.e. $X = \sum_{i=1}^{n} X_i$, where $\forall i$ $\begin{cases} \Pr[X_i = 1] = p \\ \Pr[X_i = 0] = 1-p \end{cases}$

Pf of Main Thm. Let H_k be the bipartite graph on parts X, Y such that $|X| = k$, $|Y| = (t-1)\binom{k}{t}$, and for every $S \in X^{(t)}$, there are exactly $t-1$ vertices in Y whose neighborhood is equal to S.

$S \subseteq X$, there are exactly $t-1$ vertices in Y whose neighbourhood is equal to S.

Note that any H satisfying the condition is contained in some H_k.

$H_k =$ 

- It is enough to only prove that

$$\forall k, \quad ex(n, H_k) = o\left(n^{2-1/t}\right) \qquad (2)$$

- We will assume $t \geq 3$.
- We will show that $\forall\, 0 < \varepsilon < 10^{-4}$, if n is sufficiently large, then $ex(n, H_k) \leq \varepsilon \cdot n^{2-1/t}$ $\implies$ (2)

Let G_0 be an n_0-vtx graph (H_k-free) with $e(G_0) > 6\varepsilon n_0^{2-1/t}$.

By Lemma 1, G_0 has a bipartite subgraph G with parts U & V such that $n = |U| = |V| > \frac{1}{2}(n_0)^{(1-1/t)/(8t-4)}$

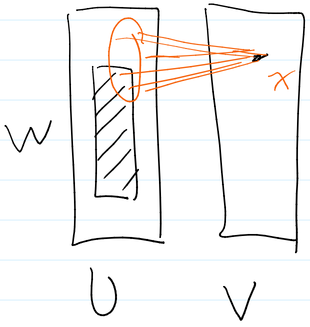
& $e(G) \geq \varepsilon \cdot n^{2-1/t}$, & $\Delta(G) \leq n^{1-1/t}$.

& G is H_k-free. $\boxed{n \gg k, t, \varepsilon}$

By Lem 2, $\exists\, \Delta = \Delta(k, t)$ s.t. any red-blue edge-coloring of $K_\Delta^{(t)}$ contains either a red or blue copy of $K_k^{(t)}$.

Claim 1. Let $p = \alpha n^{-1/t}$ where $\alpha = 2\Delta \binom{t-1}{s}^{t-1} 2^{3t-4}$

Claim 1. Let $\boxed{p = \alpha n^{-1/t}}$, where $\alpha = 2\Delta\left(\frac{t-1}{\varepsilon}\right)^{t-1} 2^{3t-4}$.



Then there exists $W \subseteq U$ such that

- $\frac{pn}{2} < |W| < 2pn$
- $G' = G[W \cup V]$ has $\geq \frac{p}{4} e(G)$ edges
- $\forall x \in V$, $d_{G'}(x) < 2p \cdot n^{1-1/t}$.

Pf. Pick each vertex of U with prob. p (indep. of each other) and let W be the set of selected vertices.

Then the statements follow by standard concentration inequalities.

$\boxed{\text{For } x \in V \text{ with } d_G(x) \geq n^{1/4}}$, by Chernoff's bounds,

the prob. that $|d_{G'}(x) - p \cdot d_G(x)| < \frac{1}{2} p\, d_G(x)$,

$$\geq 1 - 2e^{-p d_G(x)/12} \quad \text{where } p \cdot d_G(x) = \Omega\left(n^{1-2/t}\right) \gg n^{1/4}. \quad \boxed{t \geq 3}$$

$$\geq 1 - 2 \cdot e^{-n^{1/4}/12}.$$

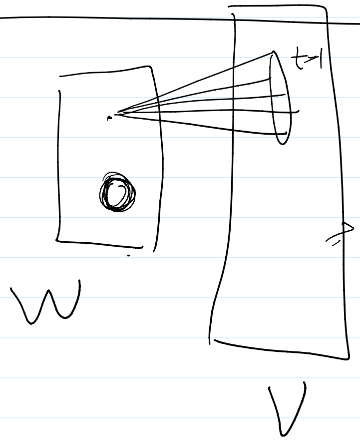
$\Rightarrow$ with high prob. $(1-o(1))$, $|d_{G'}(x) - p \cdot d_G(x)| < \frac{1}{2} p\, d_G(x)$ $\cancel{\forall x \in V}$.

Also with high prob. $(1-o(1))$, $\big||W| - pn\big| < \frac{1}{2} pn$.

Also with high prob. $(1-o(1))$, $\big|\,|W| - pn\,\big| < \frac{1}{2}pn$.

Lastly, $e(G') = \sum_{x\in V} d_{G'}(x) \geq \sum_{x\in V,\ d_G(x)\geq n^{1/4}} \frac{1}{2}p\cdot d_G(x)$

$$\geq \frac{1}{2}p\cdot e(G) - n\cdot n^{1/4} \geq \frac{1}{4}p\cdot e(G) \quad \blacksquare$$



We consider

$$L = \sum_{C\in V^{(t-1)}} |N_{G'}(C)| = \boxed{\#\ (t-1)\text{-stars}}$$

$$= \sum_{x\in W}\binom{d_{G'}(x)}{t-1} \geq |W|\binom{e(G')/|W|}{t-1}$$

$$> (t-1)^{-(t-1)}\, e(G')^{t-1}\, |W|^{-(t-2)}$$

$$\geq \left(\frac{\varepsilon}{t-1}\right)^{t-1} 2^{-3t+4}\, p\cdot n^{t-1+1/t} = 2\Delta n^{t-1}.$$

We point out $\frac{\alpha}{2}n^{1-1/t} < |W| < 2\alpha\cdot n^{1-1/t}$

& $\forall x\in V$ has degree at most $2p\cdot n^{1-1/t} = \boxed{2\alpha\cdot n^{1-2/t}}$ in G'.

where α is independent of n.

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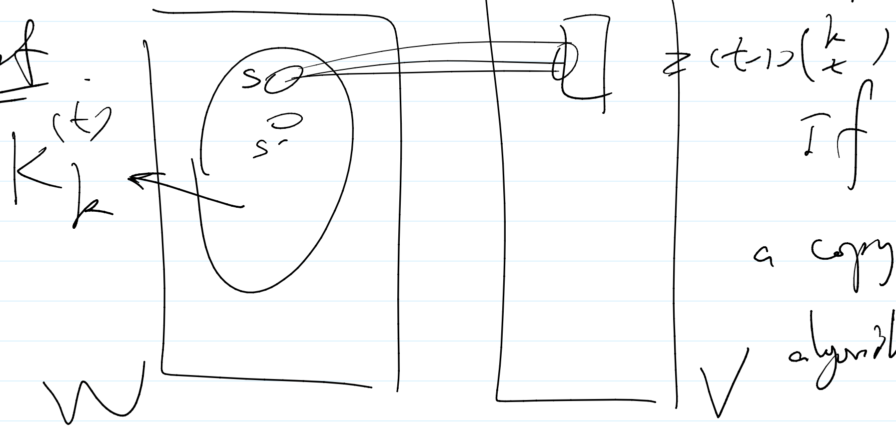
Let $\mathcal{H}$ be the t-graph on W such that $S \in W^{(t)}$ is an edge of $\mathcal{H}$ iff $|N_{G'}(S)| \geq t-1$.

Then we color an edge $S \in E(\mathcal{H})$ by <u>red</u> if $|N_{G'}(S)| \geq (t-1)\binom{k}{t}$ and color it by $\boxed{\text{blue}}$ if $t-1 \leq |N_{G'}(S)| < (t-1)\binom{k}{t}$.

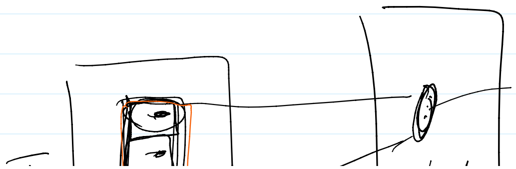
<u>Claim 2</u>. $\mathcal{H}$ has No red $K_k^{(t)}$.

<u>pf</u>.



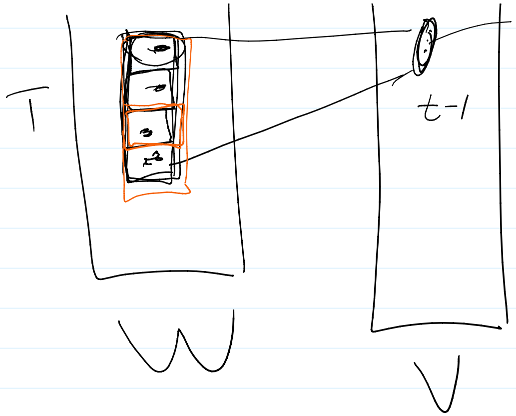
If so, then we can find a copy of H_k by a greedy algorithm applied to this $K_k^{(t)}$. ∎

Let $C \in V^{(t-1)}$ and consider $T = N_{G'}(C) \subseteq W$



$C \in V^{(t-1)}$ Let $r = \left\lfloor \frac{|T|}{\Delta} \right\rfloor \geq \frac{|T|}{\Delta} - 1$

and let $T_1, T_2, \cdots, T_r$ be



$c \in V$ … $\subseteq$ …

and let $T_1, T_2, \ldots, T_r$ be disjoint sets of size Δ in T

Note that $H[T_i]$ is a clique $K_\Delta^{(t)}$

By Lemma 2 & claim 2,

each T_i contains a $\boxed{\text{blue } K_k^{(t)}}$ in H, called A_i

Set $\mathcal{Z}_c = \{A_1, \ldots, A_r\}$

& $\mathcal{Z} = \bigcup_{c \in V^{(t-1)}} \mathcal{Z}_c$ is a multiset

(diff. sets c may have the same A_i)

Note $|\mathcal{Z}| = \sum_{c \in V^{(t-1)}} |\mathcal{Z}_c| \geq \sum_{c \in V^{(t-1)}} \left(\frac{|N_{G'}(c)|}{\Delta} - 1 \right)$

$\geq \frac{L}{\Delta} - \binom{n}{t-1} \geq n^{t-1}$

IV

Claim 3 $\exists\ \beta = \beta(k, t, s) > 0$ and $\mathcal{Z}' \subseteq \mathcal{Z}$ such that

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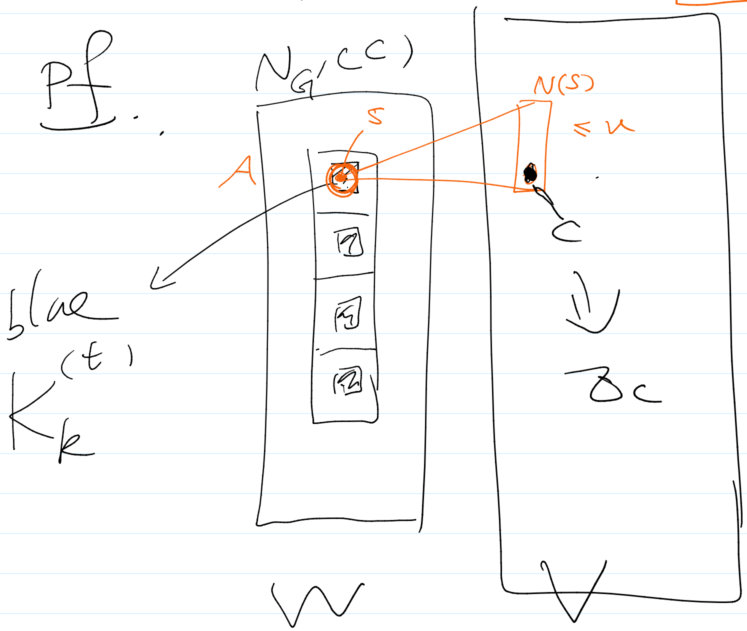
Claim 3. $\exists\ \beta=\beta(k,t,\varepsilon)>0$ and $Z'\subseteq Z$ such that $|Z'|\geq \beta\cdot|W|^t$ and $\boxed{\text{any two cliques in } Z' \text{ are edge-disjoint}}$.

Pf.



Let D be the auxiliary graph on vertex set Z, where $A,B\in Z$ are joined by an edge iff $|A\cap B|\geq t$.

We want to show:

$$\Delta(D)\leq \binom{k}{t}\binom{u}{t-1},\ \text{where } u=(t-1)\binom{k}{t}.$$

Let $A\in Z$ be any (blue) $K_k^{(t)}$ and let $S\in A^{(t)}$.

So S is blue $\Rightarrow |N_G(S)|\leq (t-1)\binom{k}{t}\triangleq u$

There are at most $\binom{u}{t-1}$ sets $C\in V^{(t-1)}$ such that $\boxed{S\subseteq N_{G'}(C)}$.

For each such C, at most one element of Z_C can contain S.

For each such C, at most one element of $\mathcal{Z}_C$ can contain S.

$\Rightarrow$ In total at most $\binom{v}{t-1}$ elements of $\mathcal{Z}$ can contain $\underline{\underline{S}}$.

Since A has $\binom{k}{t}$ edges S, A has degree at most $\binom{k}{t}\binom{v}{t-1}$.

Therefore, D contains an independent set $\mathcal{Z}'$ of size at least $\frac{|\mathcal{Z}|}{\Delta(D)+1} \geq \frac{n^{t-1}}{\Delta(D)+1} \geq \beta |W|^t$. $\boxed{\Xi}$

<u>Claim 4</u>. Let M denote the number of copies of $K_k^{(t)}$ in H.

Then $\exists\, \gamma = \gamma(k, t, \varepsilon)$ such that $\boxed{|M| \geq \gamma\, n^{(t-1)k/t}}$.

<u>pf</u>: By claim 3, we see there are at least $|\mathcal{Z}'| \geq \beta |W|^t$ edge-disjoint $K_k^{(t)}$. To destroy all copies of $K_k^{(t)}$ in H, one needs to delete one edge in each of these edge-disjoint $K_k^{(t)}$, which results in the removal of at least $\beta |W|^t$ edges.

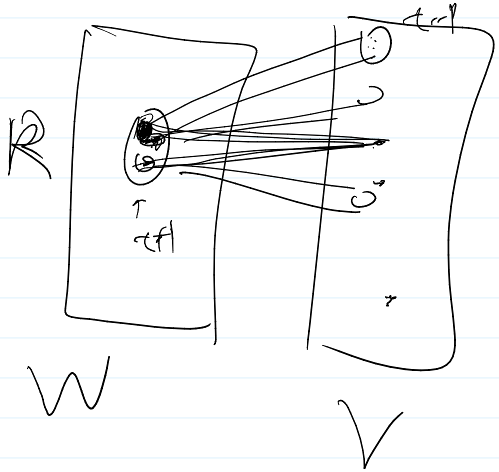
which results in the removal of at least $\beta |W|$ edges.

By Lemma 3 (HRL). Then $|M| \geq \delta |W|^k \geq \gamma \cdot n^{(t-1)k/t}$. □

$|W| = \Theta(n^{1-1/t})$

Def: A copy R of $K_k^{(t)}$ in H is **bad**, if $\exists$ distinct $S, S' \in E(R)$ with $N(S) \cap N(S') \neq \phi$.

Otherwise, R is **good**



Claim 5. A good copy R of $K_k^{(t)}$ in H can give a copy of H_k in $W \cup V$.

$\Longrightarrow$ All copies of $K_k^{(t)}$ are blue & bad.

Claim 6 $\exists\, \gamma' = \gamma'(k, t, \varepsilon)$ s.t. the number of bad copies of $K_k^{(t)}$ is at most $\boxed{\gamma' \, n^{(k(t-1)-1)/t}}$.

of $K_k^{(t)}$ is at most $\boxed{\gamma' \cdot n^{(k(t-1)-1)/t}.}$

Pf: If R is bad, then $\exists\ s, s' \in E(R)$ s.t.

$N(s) \cap N(s') \neq \phi$. Let $x \in N(s) \cap N(s')$.

$\Rightarrow \boxed{|N_G(x) \cap V(R)| \geq |s \cup s'| \geq t+1.}$

Summing over all vertices $x \in V$, we see that the number of bad $K_k^{(t)}$ is at most

$$\sum_{x \in V} \binom{|N(x) \cap W|}{t+1} |W|^{k-t-1} \leq n \cdot \left(2\alpha n^{1-\frac{2}{t}}\right)^{t+1} \left(2\alpha n^{1-\frac{1}{t}}\right)^{k-t-1}$$

$$= (2\alpha)^k \cdot n^{(k(t-1)-1)/t}$$

∎.

Claims 4 and 6 contradict to each other.

This proves the Main Thm. (Sudakov-Tomon) ∎