Day 7

Monday, March 16, 2020 5:17 PM

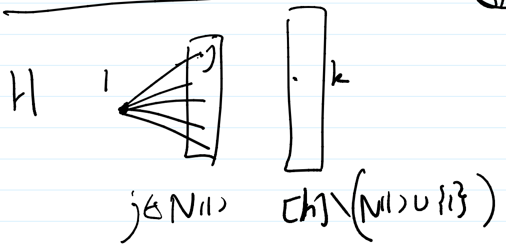
Thm 1 (Graph Counting Lemma) Let H be a graph with $V(H) = [h]$. For $\varepsilon > 0$, let G be a graph with vertex subsets $V_1, \ldots, V_h \subseteq V(G)$ such that (V_i, V_j) is ε-regular whenever $ij \in E(H)$.
Then, the number of tuples $(v_1, v_2, \ldots, v_h) \in V_1 \times V_2 \times \cdots \times V_h$ such that $v_i v_j \in E(G)$ whenever $ij \in E(H)$ is

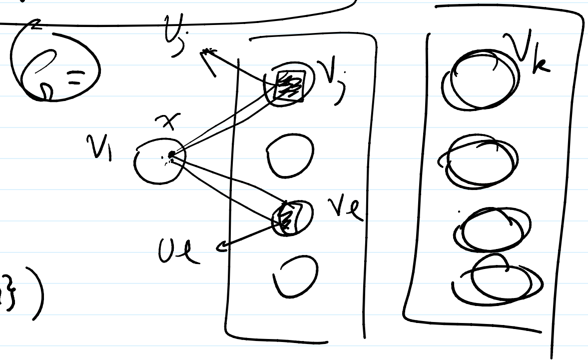
$$\left(\prod_{ij \in E(H)} d(V_i, V_j) \pm C_H \varepsilon \right) \cdot \prod_{i \in [h]} |V_i| ,$$

where C_H is a constant depending on H.

$$t = (a \pm \varepsilon) b \iff \left| \frac{t}{b} - a \right| \le \varepsilon$$

Pf 1 (Sketch)





Fix $x \in V_1$. Define the following subsets $U_j \subseteq V_j$ for $j \neq 1$.

If $j \in N(1)$, then $U_j = V_j \cap N(x)$;

o.w. $j \notin N(1)$, then $U_j = V_j$

$\Rightarrow$ $(U_2, U_3, \ldots, U_h)$ and we count the number of tuples $(u_2, u_3, \ldots, u_h) \in U_2 \times \cdots \times U_h$ s.t. $u_i u_j \in E(G)$ whenever $ij \in E(H')$ where $H' = H \setminus \{1\}$

HW EX (to extend this to a full proof). $\boxtimes$

Thm 1 (Graph Counting Lemma) Let H be a graph with $V(H) = [h]$. For $\varepsilon > 0$, let G be a graph with vertex subsets $V_1, \ldots, V_h \subseteq V(G)$ such that (V_i, V_j) is ε-regular whenever $ij \in E(H)$. Then the number of tuples $(v_1, \ldots, v_h) \in V_1 \times V_2 \times \cdots \times V_h$ such that $v_i v_j \in E(G)$ whenever $ij \in E(H)$ is

$$\left(\prod_{ij \in E(H)} d(V_i, V_j) \pm e(H) \cdot \varepsilon \right) \cdot \prod_{i \in [h]} |V_i| .$$

$(*)$

$$\left(\prod_{ij\in E(H)} d(V_i,V_j) \pm e(H)\cdot\varepsilon\right)\cdot\prod_{i\in[h]}|V_i| .$$

$\boxed{\text{pf 2}}$ This can be rephrased into the following probabilistic form:

Choose $v_1\in V_1, \dots, v_h\in V_h$, uniformly and independently at random;

If we let $p=\mathbb{P}\left(v_iv_j\in E(G) \text{ for all } ij\in E(H)\right)$,

then $(*)$ is the same as $\boxed{\left|p-\prod_{ij\in E(H)} d(V_i,V_j)\right|\le e(H)\cdot\varepsilon}$ $(*)$

Assume $12\in E(H)$.

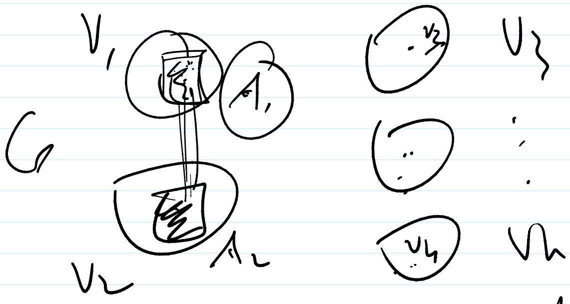
Let $p'=\mathbb{P}\left(v_iv_j\in E(G) \text{ for all } ij\in E(H)\setminus\{12\}\right)$.

$\boxed{\text{We claim: } \left|p-d(V_1,V_2)\,p'\right|\le\varepsilon \quad (1)}$

$\boxed{\text{pf of claim}}$ It suffices to show (1) holds whenever $v_3,\dots,v_h$ are fixed arbitrarily and only v_1,v_2 are random.

Define $A_1=\{v_1\in V_1 : v_1v_j\in E(G) \text{ whenever } j\in N_H(1)\setminus\{2\}\}$

$A_2=\{v_2\in V_2 : v_2v_j\in E(G) \text{ whenever } j\in N_H(2)\setminus\{1\}\}$



$$\begin{cases} p=\dfrac{e(A_1,A_2)}{|V_1|\cdot|V_2|} \\[2ex] p'=\dfrac{|A_1|\cdot|A_2|}{|V_1|\cdot|V_2|}\end{cases}$$

Therefore, for us it suffices to show $\left|\dfrac{e(A_1,A_2)}{|V_1||V_2|}-d(V_1,V_2)\cdot\dfrac{|A_1||A_2|}{|V_1||V_2|}\right|\le\varepsilon$. (2)

If $|A_1|\le\varepsilon|V_1|$ or $|A_2|\le\varepsilon|V_2|$,

then $\dfrac{e(A_1,A_2)}{|V_1||V_2|}\le\dfrac{|A_1|\,|A_2|}{|V_1|\cdot|V_2|}\le\varepsilon$ & $d(V_1,V_2)\cdot\dfrac{|A_1|\,|A_2|}{|V_1||V_2|}\le\varepsilon$

thus the inequality (2) holds. ✓

Otherwise, we have $|A_1|>\varepsilon|V_1|$ and $|A_2|>\varepsilon|V_2|$.

By ε-regularity of (V_1, V_2),

$$\left| \frac{e(A_1,A_2)}{|V_1||V_2|} - d(V_1,V_2)\cdot \frac{|A_1||A_2|}{|V_1||V_2|} \right| = \left| \frac{e(A_1,A_2)}{|A_1|\cdot|A_2|} - d(V_1,V_2) \right| \frac{|A_1||A_2|}{|V_1||V_2|} \leq \varepsilon \quad \checkmark$$

This proves the claim $\blacksquare$

To complete the proof, we do induction on $e(H)$.

Let $H' = H \setminus \{12\}$. We have

$$\left| p - \prod_{ij \in E(H)} d(V_i, V_j) \right| \leq \left| p - d(V_1,V_2)\cdot p' \right| + d(V_1,V_2)\cdot \left| p' - \prod_{ij \in E(H')} d(V_i, V_j) \right|$$

$$\overset{\text{by claim}}{\leq} \varepsilon + d(V_1,V_2)\left(e(H')\cdot \varepsilon\right) \leq e(H)\cdot \varepsilon. \quad \blacksquare$$

Thm 2 (Graph removal Lemma) For $\forall$ graph H and $\forall \varepsilon > 0$, $\exists\ \delta = \delta(H, \varepsilon) > 0$ s.t. any n-vertex graph with less than $\delta n^{|V(H)|}$ copies of H can be made H-free by deleting at most εn^2 edges.

pf: HW EX. (Similar to triangle removal Lemma; use Graph counting Lemma) $\blacksquare$

Def. For two graphs G and H with the same number of vertices, the edit-distance $d(G,H)$ is the minimum integer k such that G can be obtained from H by adding or deleting k edges.

$$\Leftrightarrow \quad d(G,H) = |E(G) \triangle E(H)|$$

symmetric difference

Next lecture, we will show Erdős–Simonovits Stability Theorem!