Extremal and Probabilistic Graph Theory
2020 Spring, USTC
Homework 1

- The due is on Mar. 8, 2020.

1. Prove that any n-vertex graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges has at least $\left\lfloor \frac{n}{2} \right\rfloor$ triangles.

2. For any $k \geq 3$, formalize the Turán number for any family $\mathcal{F}$ of k-graphs. Then prove that the Turán density $\pi(\mathcal{F})$ always exists.

3. For any $k \geq 3$, characterize all families $\mathcal{F}$ of k-graphs with $\pi(\mathcal{F}) = 0$.

4. Let G be a K_{r+1}-free graph with $V(G) = [n]$.[1] Prove that when the function

$$P = \sum_{ij \in E(G)} p_i p_j$$

achieves its maximum over all choices of $p_i \in [0,1]$ with $\sum_{i \in [n]} p_i = 1$ and subject to it, the number of vertices i with $p_i > 0$ is minimum, then these vertices i with $p_i > 0$ form a clique in G. Then use this to show that $ex(n, K_{r+1}) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$.

5. Let $\alpha(G)$ be the maximum size of an independent set in G. Prove that if a graph G has n vertices and m edges, then $\alpha(G) \geq \frac{n^2}{2m+n}$.

6. Let $k \geq 1$ and $n \geq k+2$ be integers. Prove that any n-vertex graph G with at least $kn - \frac{k^2+k-2}{2}$ edges contains a subgraph of minimum degree at least $k+1$.

[1] Throughout this course, $[n]$ denotes the set $\{1, 2, ..., n\}$.