Extremal and Probabilistic Graph Theory
2020 Spring, USTC
Homework 2

- The due is on Mar. 22, 2020.

1. Prove that any k-graph H contains a k-partite k-uniform subhypergraph H' such that

$$\frac{e(H')}{e(H)} \geq \frac{k!}{k^k}.$$

2. Let $k \geq 2$ and $t_1, t_2 ..., t_k \geq 1$ be integers. Then for $K = K^{(k)}_{(t_1,t_2,...,t_k)}$,

$$ex(n, K) = O\left(n^{k - \frac{1}{t_1 t_2 \cdots t_{k-1}}}\right).$$

Prove it in two different ways.

(i) Using supersaturation method we gave in the class.

(ii) Double counting on the number of $K_{1,...,1,t_k}$ in G.

3. Prove the following version of Erdős-Stone-Simonovits Theorem that for any family $\mathcal{F}$ of graphs, we have

$$\pi(\mathcal{F}) = 1 - \frac{1}{\chi(\mathcal{F}) - 1}.$$

4. Let $P = \{V_1, ..., V_k, U_1, ..., U_t\}$ be a partition of an n-vertex graph G, where $|V_1| = \cdots = |V_k|$ and $\sum_{j=1}^{t} |U_j| \leq \varepsilon n$. Prove that there exists an equipartition P^* of order k in G such that

$$q(P^*) \geq q(P) - 10\varepsilon.$$

5. Let $d(G)$ denote the edge-density of a graph G. Let R be a regularity graph of a graph G with parameters ϵ, ℓ and d. Prove that

$$d(G) \leq d(R) + o(1)$$

where $o(1)$ goes to 0, as ϵ, d go to 0.

6. Assume that A, B, C are disjoint vertex sets such that $d(A, B) = c$, $d(A, C) = b$, $d(B, C) = a$, where $a, b, c \geq 2\varepsilon$, and all these biaprtite graphs are ε-regular. Then the tripartite graph (A, B, C) has at least $(1 - 2\varepsilon)(a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|A||B||C|$ triangles.

7. For graphs H_1, H_2, let the *Ramsey number* $R(H_1, H_2)$ denote the minimum integer r such that for any coloring of K_r which assigns red or blue to each edge of K_r, there is either a copy of H_1 with all red edges or a copy of H_2 with all blue edge. Prove that $R(K_s, K_t) \leq \binom{s+t-2}{s-1}$.

8. Prove that for every Δ there is $c = c(\Delta)$ such that the following holds. If K, H are any graphs on n vertices with maximum degree at most Δ then $R(H, K) \leq cn$.