Extremal and Probabilistic Graph Theory
2020 Spring, USTC
Homework 3

- The due is on April 6, 2020.

1. Prove the following version of Graph Counting Lemma, using induction on $|V(H)|$: Let H be a graph with $V(H) = [h]$. Given $\epsilon > 0$, let G be a graph with disjoint vertex subsets $V_1, ..., V_h \subseteq V(G)$ such that (V_i, V_j) is ϵ-regular whenever $ij \in E(H)$. Then the number of tuples $(v_1, v_2, ..., v_h) \in V_1 \times V_2 \times ... \times V_h$ such that $v_i v_j \in E(G)$ whenever $ij \in E(H)$ is

$$\left(\prod_{ij \in E(H)} d(V_i, V_j) \pm c_H \epsilon \right) \prod_{i \in [h]} |V_i|,$$

where c_H is a constant only depending on H.

2. Give a full proof of Graph Removal Lemma: For any graph H and any $\epsilon > 0$, there exists some $\delta = \delta(H, \epsilon) > 0$ such that any n-vertex graph with less $\delta n^{|V(H)|}$ copies of H can be made H-free by deleting at most ϵn^2 edges.

3. Give a full proof of Erdős-Simonovits Stability Theorem: For any $\epsilon > 0$ and any graph F with $\chi(F) = r + 1$, there exist some $\delta > 0$ and n_0 such that if G is an n-vertex F-free graph with $e(G) \geq (1 - \frac{1}{r})\binom{n}{2} - \delta n^2$ where $n \geq n_0$, then the edit distance $d(G, T_r(n)) \leq \epsilon n^2$.

4. Let K_4^- be obtained from K_4 by deleting one edge. Prove that for sufficiently large n, the unique extremal graph achieving $ex(n, K_4^-)$ is $T_2(n)$.

Hint: use the above stability. (Ask Yang Tianchi for more hints if needed.)

5. Prove Induced Embedding Lemma: For any $c \in (0, 1/2]$ and h, there exists some $\epsilon = \epsilon(c, h)$ such that the following holds for any h-vertex graph H. If $V_1, V_2, ..., V_h$ are disjoint sets of size at least $1/\epsilon$ and each (V_i, V_j) is ϵ-regular with density in $[c, 1-c]$ for all $1 \leq i < j \leq h$, then one can find an induced copy of H in $(V_1, V_2, ..., V_h)$.

Hint: If some (V_i, V_j) is ϵ-regular with density in $[c, 1-c]$, so is the complement of (V_i, V_j).

6. Let $t(n)$ be the maximum number of edges in an n-vertex graph where every edge lies in a unique triangle. Use triangle removal lemma to show that $t(n) = o(n^2)$.

7. Let $f(n)$ be the maximum number of edges in an n-vertex 3-uniform hypergraph which does not contain 6 vertices that span at least 3 edges. Prove that $f(n) \leq t(n) + n$.

Hint: This is the famous $(6, 3)$-problem. Let H be a 3-graph achieving $f(n)$. Let us assume that every vertex in H has degree at least two. Construct a graph G from the 3-graph H such that G satisfies the condition in Problem 6.

8. Let $g(n)$ be the maximum number of edges in an n-vertex bipartite graph whose edge set can be partitioned into at most n induced matchings. Prove that $g(n) \le f(2n)$.

Hint: Let G be a bipartite graph achieving $g(n)$. Construct a 3-graph H obtained from G by adding n new vertices, and show that H satisfies the condition in Problem 7.

Remark 1. Combining Problems 6, 7 and 8, these functions $t(n), f(n), g(n)$ are all $o(n^2)$.

Remark 2. It is also true that $\frac{t(n)}{2} \le g(n)$. Can you see it? Putting all together, it holds

$$\frac{1}{2}t(n) \le g(n) \le f(2n) \le t(2n) + 2n.$$

Therefore, all these functions $t(n), f(n), g(n)$ are equivalent (up to constant factors)!!