Extremal and Probabilistic Graph Theory
2020 Spring, USTC
Homework 4

- The due is on April 20, 2020.

1. Let G be an n-vertex graph and A be its adjacency matrix. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of A. Prove that

(1). $\lambda_1 = \max_{\vec{x} \neq \vec{0}} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$.

(2). $|\lambda_n| \leq \lambda_1$.

(3). If G is bipartite, then $\lambda_n = -\lambda_1$.

(4). If G is connected and $\lambda_n = -\lambda_1$, then G is bipartite.

2. Let G be an n-vertex graph and $u, v \in V(G)$ with $d(u) + d(v) \geq n$. Prove that G is Hamiltonian if and only if $G + \{uv\}$ is Hamiltonian.

3. Let G be a graph with degree sequence $(a_1, a_2, ..., a_n)$ where $a_1 \leq a_2 \leq ... \leq a_n$. Suppose that there is no integer $1 \leq i < n/2$ such that $a_i \leq i$ and $a_{n-i} < n - i$. Then G is Hamiltonian.

4. Let $\mathcal{C}_t$ be the set of all cycles of length at least $t + 1$. Prove that $ex(n, \mathcal{C}_t) \leq t(n-1)/2$ and show that it is tight when $(t-1)|(n-1)$.

5. Let G be a graph. Prove that if $|N(X)| > 2|X|$ holds for all subsets $X \subseteq V(G)$ with $|X| \leq k$, then G contains a cycle C of length at least $3k$, which contains a vertex x and all its neighbors in G.

6. Show that every $2k$-regular simple graph on $4k + 1$ vertices is Hamiltonian.

7. Let $k \geq 2\ell \geq 0$. If G is a graph with $\delta(G) \geq 1000 \cdot \ell \cdot k^{1/2}$, then G contains a cycle of length 2ℓ modulus k.