Extremal and Probabilistic Graph Theory
2020 Spring, USTC
Homework 5

- The due is on May 4, 2020.

1. Prove that there exists an absolute constant $c > 0$ such that any graph with average degree at least ck contains k cycles of consecutive even lengths.

2. Let $k \geq 2\ell \geq 0$. If G is a graph with $\delta(G) \geq 1000 \cdot \ell \cdot k^{1/2}$, then G contains a cycle of length 2ℓ modulus k.

3. Let d, g be positive integers and let $n_0(d,g)$ be a function as following:

$$n_0(d,g) = 1 + d\sum_{i=0}^{r-1}(d-1)^i \text{ if } g = 2r+1 \quad \text{and} \quad n_0(d,g) = 2\sum_{i=0}^{r-1}(d-1)^i \text{ if } g = 2r.$$

Prove that if G is a graph with min-degree d and girth at least g, then $|V(G)| \geq n_0(d,g)$.

4. Let H be a connected graph where each edge is colored by color 1 or color 2. Suppose that there is at least one edge of each color. Prove that if the number of edges of color 1 is at least $(p+1)|V(H)|$, then there exists a path of length p in H whose first edge is colored by color 2 and all other edges are colored by color 1.

5. Prove that the Erdős-Renyi polarity graph ER_q has exactly $q+1$ vertices of degree q.

6. Let Zarankiewicz number $z(n,m,s,t)$ be the maximum number of 1 entries in an $n \times m$ matrix M (whose entries are either 0 or 1) such that M does not contain an $s \times t$ submatrix consisting entirely of 1 entries. Prove that $z(n,m,s,t) \leq (s-1)^{1/t}mn^{1-1/t} + (t-1)n$.

7. Let $t \geq 2$ and $s > t!$ be fixed. Prove that if $n^{1/t} \leq m \leq n^{1+1/t}$, then

$$z(n,m,s,t) = \Theta\left(mn^{1-1/t}\right).$$

8. The k-color Ramsey number $R_k(F)$ is the maximum integer m such that one can color the edges of the complete graph K_m using k colors with no monochromatic copy of F.

Prove that $R_k(K_{3,3}) \geq (1+o(1))k^3$. (Remark: in fact we have $R_k(K_{3,3}) = (1+o(1))k^3$.)