Some sharp results on the generalized Turán numbers

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Abstract

For graphs T, H, let $\text{ex}(n, T, H)$ denote the maximum number of copies of T in an n-vertex H-free graph. In this paper we prove some sharp results on this generalization of Turán numbers, where our focus is for the graphs T, H satisfying $\chi(T) < \chi(H)$. This can be dated back to Erdős [8], where he generalized the celebrated Turán's theorem by showing that for any $r \geq m$, the Turán graph $T_r(n)$ uniquely attains $\text{ex}(n, K_m, K_{r+1})$. For general graphs H with $\chi(H) = r + 1 > m$, Alon and Shikhelman [3] showed that $\text{ex}(n, K_m, H) = \binom{r}{m}(\frac{n}{r})^m + o(n^m)$. Here we determine this error term $o(n^m)$ up to a constant factor. We prove that $\text{ex}(n, K_m, H) = \binom{r}{m}(\frac{n}{r})^m + \text{biex}(n, H) \cdot \Theta(n^{m-2})$, where $\text{biex}(n, H)$ is the Turán number of the decomposition family of H. As a special case, we extend Erdős' result, by showing that $T_r(n)$ uniquely attains $\text{ex}(n, K_m, H)$ for any edge-critical graph H. We also consider T being non-clique, where even the simplest case seems to be intricate. Following from a more general result, we show that for all $s \leq t$, $T_2(n)$ maximizes the number of $K_{s,t}$ in n-vertex triangle-free graphs if and only if $t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$.

1 Introduction

Let T and H be two fixed graphs. Throughout the paper we denote by $\mathcal{N}(G, T)$ the number of copies of T in a graph G, and let $\text{ex}(n, T, H)$ be the maximum number of copies of T in an n-vertex H-free graph.

The well-known Turán's theorem [28] states that the maximum number of edges in an n-vertex K_{r+1}-free graph is uniquely attained by the *Turán graph* $T_r(n)$, i.e., the complete balanced r-partite graph on n vertices. This was generalized by Erdős [8] as following.

Theorem 1.1 ([8]). *For all $n \geq r \geq m \geq 2$, the Turán graph $T_r(n)$ uniquely attains the maximum number of cliques K_m in an n-vertex K_{r+1}-free graph.*

Since then the function $\text{ex}(n, T, H)$ for $T \neq K_2$ was studied for certain pairs $\{T, H\}$ (such as [5, 19, 20, 21]; see [3] for an elaborated discussion). This was culminated in [3] by Alon and Shikhelman, where they systematically studied the function $\text{ex}(n, T, H)$. Among other results, they [3] proved that for any graph H with chromatic number $\chi(H) = r + 1 > m$,

$$\text{ex}(n, K_m, H) = \mathcal{N}(T_r(n), K_m) + o(n^m). \tag{1}$$

Recently this function has been the subject of extensive research, including [2, 14, 15, 16, 17, 18, 23, 24, 25] (by no means a comprehensive list).

In this paper we determine the error term $o(n^m)$ in (1) up to a constant factor. Given a graph H with $\chi(H) = r + 1$, the *decomposition family* of H, denoted by $\mathcal{F}_H$, is the

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family of all bipartite graphs that are obtained from H by deleting $r-1$ color classes in some $(r+1)$-coloring of H. By $\mathrm{biex}(n, H)$ we denote the maximum number of edges in an n-vertex graph which does not contain any graph in $\mathcal{F}_H$ as a subgraph. Our main result is as following.

Theorem 1.2. *For any integer m and any graph H with $\chi(H) = r+1 > m \geq 2$,*

$$\mathrm{ex}(n, K_m, H) = \mathcal{N}(T_r(n), K_m) + \mathrm{biex}(n, H) \cdot \Theta(n^{m-2}).$$

Since $\mathrm{biex}(n, H) = O(n^{2-\alpha_H})$ for some $\alpha_H > 0$ by the classic result of Kövári, Turán and Sós [22], this improves the error term to $O(n^{m-\alpha_H})$. A graph is *edge-critical* if there exists some edge whose deletion reduces its chromatic number. Simonovits [26] proved that for any edge-critical graph H with $\chi(H) = r+1 \geq 3$ and for sufficiently large n, the Turán graph $T_r(n)$ is the unique graph which attains the maximum number of edges in an n-vertex H-free graph. It is clear that if H is edge-critical, then $\mathrm{biex}(n, H) = 0$. This enables us to obtain the following

Corollary 1.3. *Let H be an edge-critical graph with $\chi(H) = r+1 > m \geq 2$ and n be sufficiently large. Then the Turán graph $T_r(n)$ is the unique graph attaining the maximum number of K_m's in an n-vertex H-free graph.*

This can be viewed as a common generalization of the result of Erdős [8] and the result of Simonovits [26]. To prove the upper bound of Theorem 1.2, we establish a stability result.

Theorem 1.4. *Let H be a graph with $\chi(H) = r+1 > m \geq 2$. If G is an n-vertex H-free graph with $\mathcal{N}(G, K_m) \geq \mathcal{N}(T_r(n), K_m) - o(n^m)$, then G can be obtained from $T_r(n)$ by adding and deleting a set of $o(n^2)$ edges.*

Let us prove the lower bound of Theorem 1.2 here. The construction we use can be found in [1]. Let F'' be an n-vertex $\mathcal{F}_H$-free graph with $\mathrm{biex}(n, H)$ edges. Then F'' contains an $\lceil \frac{n}{r} \rceil$-vertex subgraph F' with at least $\mathrm{biex}(n, H)/2r^2$ edges. One can further find a bipartite subgraph F of F' with at least $\mathrm{biex}(n, H)/4r^2$ edges. Consider the graph G obtained by inserting F into the largest part of $T_r(n)$. Since $\chi(H) = r+1$ and F is bipartite, it's easy to see G is H-free. As each edge in F is contained in $\Omega(n^{m-2})$ copies of K_m in G, we see that $\mathrm{ex}(n, K_m, H) \geq \mathcal{N}(G, K_m) \geq \mathcal{N}(T_r(n), K_m) + \mathrm{biex}(n, H) \cdot \Omega(n^{m-2})$.

We also consider the function $\mathrm{ex}(n, T, H)$ for some T not being a clique. In the next result we maximize the number of some complete r-partite graphs T in K_{r+1}-free graphs. It reveals that the relatively sizes of the parts in T will play an important role.

Theorem 1.5. *(i) Let n be sufficiently large and T be any complete balanced r-partite graph. Then the Turán graph $T_r(n)$ is the unique n-vertex K_{r+1}-free graph which maximizes the number of T-copies.*
(ii) Let n be sufficiently large and $t \geq s$. Then $T_2(n)$ maximizes the number of copies of $K_{s,t}$ in n-vertex triangle-free graphs if and only if $t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$.

This will follow from Theorem 5.1 in Section 5 in a more general setting.

The remaining of this paper is organized as follows. In Section 2 we give out some preliminaries. In Section 3 we prove Theorem 1.4. The proofs of Theorem 1.2 and Corollary 1.3 will be completed in Section 4. In Sections 5 and 6, we show Theorem 5.1, which would imply Theorem 1.5. Section 7 contains some concluding remarks. Throughout the paper, let $[k] = \{1, \cdots, k\}$ for a positive integer k.

2 Preliminaries

In this section we will present some definitions and results needed in the subsequent sections.

Let $\sigma(H)$ be the smallest size of a color class in a proper $\chi(H)$-coloring of a graph H. So if H is edge-critical, then $\sigma(H) = 1$. The next proposition can be found in [1]; we include its short proof for the completeness.

Proposition 2.1 ([1]). *If H is a graph with $\chi(H) \geq 3$ and $\sigma(H) \geq 2$, then* $\text{biex}(n, H) \geq n - 1$.

Proof. We have $\sigma(H) \geq 2$. Then any $F \in \mathcal{F}_H$ contains a matching of size 2. So $K_{1,n-1}$ must be $\mathcal{F}_H$-free, implying that $\text{biex}(n, H) \geq e(K_{1,n-1}) = n - 1$. □

Next we collect some properties on the counts of cliques in Turán graphs $T_r(n)$.

Proposition 2.2. *For any integers $n \geq r \geq s \geq m \geq 2$, it holds that*

$$\mathcal{N}(T_r(n), K_m) \geq \mathcal{N}(T_s(n), K_m) \quad \textit{and} \quad \mathcal{N}(T_r(n), K_m) = \binom{r}{m}\left(\frac{n}{r}\right)^m + O(n^{m-1}).$$

Fix a graph H and consider a graph G. For each $v \in V(G)$, let $d_G(v, H)$ denote the number of copies of H in G containing the vertex v, and let $\delta(G, H) = \min_{x \in V} d_G(x, H)$. If $H = K_m$, then we write $d_G(v, K_m)$ and $\delta(G, K_m)$ as $d_G^{(m)}(v)$ and $\delta^{(m)}(G)$, respectively.

Proposition 2.3. *For any integers $n - 1 \geq r \geq m \geq 2$, it holds that*

$$\delta^{(m)}(T_r(n)) = \mathcal{N}(T_r(n), K_m) - \mathcal{N}(T_r(n-1), K_m).$$

Proof. Let $V_1, V_2, \cdots, V_r$ be the partition classes of $T_r(n)$. Then for any $v \in V_i$ with $|V_i| = \lceil \frac{n}{r} \rceil$, we have $T_r(n-1) = T_r(n) - \{v\}$ and thus

$$d^{(m)}(v) = \mathcal{N}(T_r(n), K_m) - \mathcal{N}(T_r(n-1), K_m).$$

It then suffices to show that $d^{(m)}(v) = \delta^{(m)}(T_r(n))$. Suppose to the contrary that $d^{(m)}(v) > \delta^{(m)}(T_r(n))$ for some $v \in V_i$. Then there exists a vertex $u \in V_j$ with $d^{(m)}(u) = \delta^{(m)}(T_r(n)) < d^{(m)}(v)$. Then we must have $|V_j| = \lfloor \frac{n}{r} \rfloor < \lceil \frac{n}{r} \rceil$. Thus the graph G' obtained from $T_r(n)$ by deleting the vertex u is not $T_r(n-1)$. Since $\mathcal{N}(G', K_m) = \mathcal{N}(T_r(n), K_m) - d^{(m)}(u)$, it follows that

$$\mathcal{N}(G', K_m) > \mathcal{N}(T_r(n), K_m) - d^{(m)}(v) = \mathcal{N}(T_r(n-1), K_m).$$

This contradicts Theorem 1.1, completing the proof. □

The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a clique in G. We will use a result due to Eckhoff [7].

Theorem 2.4 ([7]). *Let G be an n-vertex graph with the clique number $\omega := \omega(G) \geq m \geq 2$. Let n_1 and n_2 be the unique integers satisfying that $e(G) = e(T_\omega(n_1)) + n_2$ and $0 \leq n_2 < \frac{\omega-1}{\omega} n_1$. Then, $\mathcal{N}(G, K_m) \leq \mathcal{N}(T_\omega(n_1), K_m) + \mathcal{N}(T_{\omega-1}(n_2), K_{m-1})$.*

Note that in the setting we have $n_1 \leq n$. To see this, we notice that as G is $K_{\omega+1}$-free, it follows by $e(T_\omega(n_1)) \leq e(G) \leq e(T_\omega(n))$.

The following structural stability theorem was originally proved by Erdős and Simonovits [9, 10, 11, 26] (also see Füredi [13] for a new proof in the case of H being cliques).

Theorem 2.5 (Erdős-Simonovits Stability Theorem). *Let H be a graph with $\chi(H) = r+1 \geq 3$. Then, for every $\varepsilon > 0$, there exist $\delta = \delta(H, \varepsilon) > 0$ and $n_0 = n_0(H, \varepsilon) \in \mathbb{N}$ such that the following holds. If G is an H-free graph on $n \geq n_0$ vertices with $e(G) \geq e(T_r(n)) - \delta n^2$, then there exists a partition of $V(G) = V_1 \dot\cup \cdots \dot\cup V_r$ such that $\sum_{i=1}^{r} e(V_i) < \varepsilon n^2/2$. Therefore, G can be obtained from $T_r(n)$ by adding and deleting a set of at most εn^2 edges.*

A classical result of Andrásfai, Erdős and Sós [4] asserts that a K_{r+1}-free graph with large minimum degree must be r-partite.

Theorem 2.6 ([4]). *Let $n > r \geq 2$. If G is a K_{r+1}-free graph on n vertices with $\delta(G) > \frac{3r-4}{3r-1}n$, then G is r-partite.*

We need the celebrated Szemerédi's regularity lemma [27]. Let X, Y be disjoint subsets in a graph G. By $G[X, Y]$ we denote the bipartite subgraph of G consisting of all edges that has one endpoint in X and another in Y; let $e_G(X, Y)$ (respectively, $e_G(X)$) be the number of edges in $G[X, Y]$ (respectively, in $G[X]$). For mutually disjoint $V_1, \cdots, V_r \subseteq V(G)$, similarly we define $G[V_1, \cdots, V_r]$ to be the r-partite subgraph of G consisting of all edges in $\cup_{1 \leq i < j \leq r} E(G[V_i, V_j])$. The subscripts will be dropped if there is no confusion. The *density* of the pair (X, Y) is defined by $d(X, Y) := e_G(X, Y)/|X||Y|$. The pair (X, Y) is called *ε-regular* if $|d(X, Y) - d(A, B)| < \varepsilon$ for all $A \subseteq X$ and $B \subseteq Y$ with $|A| \geq \varepsilon|X|$ and $|B| \geq \varepsilon|Y|$. A partition $V_0, \cdots, V_k$ of V is *ε-regular*, if $|V_0| \leq \varepsilon|V|$, $|V_1| = \cdots = |V_k|$, and all but at most εk^2 of pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ε-regular.

Theorem 2.7 (Regularity Lemma). *For every $\varepsilon > 0$, there exists $M = M(\varepsilon)$ such that every graph of order at least ε^{-1} admits an ε-regular partition $\{V_0, \cdots, V_k\}$ with $\varepsilon^{-1} \leq k \leq M$.*

For a real $d \in (0, 1]$, an ε-regular pair (X, Y) is called *(ε, d)-regular* if the density $d(X, Y) \geq d$. Given an ε-regular partition $\{V_0, \cdots, V_k\}$ of a graph G, the *(ε, d)-cluster graph* is a graph R with the vertex set $V(R) = [k]$ and with edges $ij \in E(R)$ if and only if (V_i, V_j) is an (ε, d)-regular pair. For an integer $s \geq 1$, the *s-blowup* of G, denoted by $G(s)$, is the graph obtained from G by replacing every vertex $v \in V(G)$ with an independent set I_v of size s and replacing every edge $uv \in E(G)$ with the complete bipartite graph between I_u and I_v. Let $\Delta(G)$ be the maximum degree of G.

Theorem 2.8 (Embedding Lemma; see [6]). *For all $d \in (0, 1]$ and $\Delta \geq 1$ there exists a $\gamma_0 > 0$ with the following property. If a graph G has a γ-regular partition $\{V_0, \cdots, V_k\}$ with $|V_1| = \cdots = |V_k| = \ell$ and the (γ, d)-cluster graph R, where $\gamma \leq \gamma_0$ and $\ell d^{\Delta} \geq 2s$ for some integer $s \geq 1$, then any subgraph H of the s-blowup of R with $\Delta(H) \leq \Delta$ is also a subgraph of G.*

3 A stability result on the number of cliques

In this section we prove Theorem 1.4, which is restated as the following.

Theorem 3.1. *For any $\varepsilon > 0$, integers $r \geq m \geq 2$ and a fixed graph H with $\chi(H) = r+1$, there exist $\delta = \delta(H, \varepsilon) > 0$ and $n_0 = n_0(H, \varepsilon) \in \mathbb{N}$ such that the following holds. Let G be an H-free graph on $n \geq n_0$ vertices with $\mathcal{N}(G, K_m) \geq \mathcal{N}(T_r(n), K_m) - \delta n^m$. Then G can be obtained from $T_r(n)$ by adding and deleting a set of at most εn^2 edges.*

We first establish a lemma, which says that it will be enough to find a partition of $V(G)$ into r parts such that the number of edges contained in a part is at most $o(n^2)$.

Lemma 3.2. *Let H, G be from Theorem 3.1 and $\varepsilon \gg \eta \gg \delta \gg 1/n_0$.*[1] *If $V_1, \cdots, V_r$ is a partition of $V(G)$ with $\sum_{i=1}^r e(V_i) < \eta n^2$, then $e(G[V_1, \cdots, V_r]) > e(T_r(n)) - \varepsilon n^2$.*

Proof. Let $G' = G[V_1, \cdots, V_r]$. So $\omega := \omega(G') \leq r$. Every K_m-copy in G either contains some edge in $\cup_{i=1}^r E(G[V_i])$ or is contained in G'. Since $\sum_{i=1}^r e(V_i) < \eta n^2$, the number of K_m-copies of the former type is at most ηn^m. So we have $\mathcal{N}(G, K_m) \leq \mathcal{N}(G', K_m) + \eta n^m$.

Let n_1, n_2 be the unique integers satisfying that $e(G') = e(T_\omega(n_1)) + n_2$ and $0 \leq n_2 < \frac{\omega-1}{\omega} n_1$. If $\omega < m$, then $\mathcal{N}(G', K_m) = 0$ and thus $\mathcal{N}(G, K_m) \leq \varepsilon n^m$, a contradiction. So $\omega \geq m$. Then by Theorem 2.4,

$$\mathcal{N}(G', K_m) \leq \mathcal{N}(T_\omega(n_1), K_m) + \mathcal{N}(T_{\omega-1}(n_2), K_{m-1}).$$

We also have $n_2 < n_1 \leq n$ and thus $\mathcal{N}(T_{\omega-1}(n_2), K_{m-1}) \leq n^{m-1} \leq \eta n^m$. Now combining the above inequalities, we have

$$\mathcal{N}(T_r(n), K_m) - \delta n^m \leq \mathcal{N}(G, K_m) \leq \mathcal{N}(T_\omega(n_1), K_m) + 2\eta n^m,$$

where the first inequality is given by the conditions. Since $\omega \leq r$, $n_1 \leq n$ and $\varepsilon \gg \eta \gg \delta \gg 1/n$, it yields $\omega = r$ and $n_1 > (1-\varepsilon)n$. By the definition of n_1, we can conclude that

$$e(G') \geq e(T_r(n_1)) > e(T_r(n)) - \varepsilon n^2.$$

This completes the proof of the lemma. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We are given $\varepsilon > 0$ and a fixed graph H with $\chi(H) = r+1 > m \geq 2$. We will choose the constants appeared in this proof satisfying the following hierarchy:

$$\varepsilon \gg \eta \gg \delta \gg 1/k_0 \gg \gamma_0 \gg 1/n_0, \tag{2}$$

where η is from Lemma 3.2 and each of $\delta, k_0, \gamma_0, n_0$ can be expressed as functions of H, ε, η and the previous constants in this order. Let G be an H-free graph on $n \geq n_0$ vertices with

$$\mathcal{N}(G, K_m) \geq \mathcal{N}(T_r(n), K_m) - \delta n^m \geq \binom{r}{m}\left(\frac{n}{r}\right)^m - 2\delta n^m, \tag{3}$$

where the last inequality follows by Proposition 2.2. We will show that

$$\text{there exists a partition of } V(G) = V_1 \dot\cup \cdots \dot\cup V_r \text{ such that } \sum_{i=1}^r e(V_i) < \eta n^2. \tag{4}$$

Note that by Lemma 3.2 and Theorem 2.5, this would imply that G can be obtained from $T_r(n)$ by adding and deleting a set of at most εn^2 edges.

Let $d := \frac{\delta}{2}$ and $\Delta := \Delta(H)$. Then there exists a real $\gamma_0 > 0$ such that the conclusion of Lemma 2.8 holds for d and Δ, and in addition, γ_0 satisfies the hierarchy (2). By Theorem 2.7, there exists a γ_0-regular partition $\mathcal{A} := \{A_0, \cdots, A_k\}$ of G with $\gamma_0^{-1} \leq k \leq M(\gamma_0)$. Let $\ell = |A_1| = \cdots = |A_k|$. As $|A_0| < \gamma_0 n$, we have $\ell \geq \frac{1-\gamma_0}{k} n \geq \frac{1-\gamma_0}{M(\gamma_0)} n_0$ and thus we can choose n_0 so that $\ell d^\Delta \geq 2|V(H)|$. Let R be the (γ_0, d)-cluster graph of $\mathcal{A}$.

We first show that the clique number $\omega := \omega(R)$ is at most r. Suppose for a contradiction that $K_{r+1} \subseteq R$. Then $H \subseteq K_{r+1}(|V(H)|) \subseteq R(|V(H)|)$, which, together with Lemma 2.8, implies that $H \subseteq G$, a contradiction. Thus R is a K_{r+1}-free graph on $k \geq \gamma_0^{-1}$ vertices.

[1]Throughout this paper, the notation $\varepsilon_1 \gg \varepsilon_2$ simply means that ε_2 is a sufficiently small function of ε_1 which is needed to satisfy some inequalities in the proof.

The following claim gives an estimation on the number of edges in R.

Claim. $e(R) \geq e(T_r(k)) - c_H\delta^{1/m}k^2$, where $c_H > 0$ is a constant only depending on H.

Proof of the claim. Let n_1 and n_2 be the unique integers satisfying $e(R) = e(T_\omega(n_1)) + n_2$ and $0 \leq n_2 < \frac{\omega-1}{\omega}n_1$. By Theorem 2.4 and its remark, $n_2 < n_1 \leq |V(R)| = k$ and

$$\mathcal{N}(R, K_m) \leq \mathcal{N}(T_\omega(n_1), K_m) + \mathcal{N}(T_{\omega-1}(n_2), K_{m-1}).$$

Since $\omega \leq r$ and $\mathcal{N}(T_{\omega-1}(n_2), K_{m-1}) \leq k^{m-1}$, by Proposition 2.2, we have

$$\mathcal{N}(R, K_m) \leq \binom{r}{m}\left(\frac{n_1}{r}\right)^m + O(k^{m-1}).$$

By the choices of γ_0 and k, we have $\frac{1}{k} \leq \frac{1}{k_0} \ll \delta$, implying that

$$\mathcal{N}(R, K_m) \leq \binom{r}{m}\left(\frac{n_1}{r}\right)^m + \delta \cdot k^m. \tag{5}$$

We then estimate the number $\mathcal{N}(G, K_m)$ of the copies of K_m in G, which must belong to one of the following five types. For those copies of K_m containing some vertex in A_0, since $|A_0| < \gamma_0 n$, these copies will contribute no more than $\gamma_0 n^m$ to $\mathcal{N}(G, K_m)$. For those copies of K_m containing at least two vertices in A_i for some $i \in [k]$, since $\gamma_0^{-1} \leq k$ and $k\ell \leq n$, they will contribute at most $k\binom{\ell}{2}n^{m-2} \leq \gamma_0 n^m$. For those copies of K_m containing some edge in non-γ_0-regular pairs of $\mathcal{A}$, since there are at most $\ell^2 \cdot \gamma_0 k^2$ such edges, there are at most $\ell^2\gamma_0 k^2 n^{m-2} \leq \gamma_0 n^m$ such copies. For those copies of K_m containing some edge in γ_0-regular pairs of $\mathcal{A}$ with density $< d$, since there are at most $d\ell^2\binom{k}{2}$ such edges, there are at most $d\ell^2\binom{k}{2}n^{m-2} \leq dn^m$ such copies. For those copies of K_m not belonging to the above types, all of their edges must be in (γ_0, d)-regular pairs of $\mathcal{A}$, and thus there are at most $\mathcal{N}(R, K_m) \cdot \ell^m \leq \mathcal{N}(R, K_m) \cdot \left(\frac{n}{k}\right)^m$ such copies of K_m. Summing up these five types, we have, as $3\gamma_0 + d \leq \delta$, that

$$\mathcal{N}(G, K_m) \leq 3\gamma_0 n^m + dn^m + \frac{\mathcal{N}(R, K_m)}{k^m}n^m \leq \delta n^m + \frac{\mathcal{N}(R, K_m)}{k^m}n^m.$$

Together with (3), this implies that

$$\frac{\mathcal{N}(R, K_m)}{k^m} \geq \binom{r}{m}\left(\frac{1}{r}\right)^m - 3\delta. \tag{6}$$

Combining with (5) and (6), we have

$$\frac{n_1}{k} \geq 1 - c \cdot \delta^{1/m},$$

where the constant $c > 0$ depends on r and m (and thus only depends on H). By the definition of n_1, we have

$$e(R) \geq e(T_r(n_1)) \geq e(T_r(k)) - c_H\delta^{1/m} \cdot k^2,$$

completing the proof of the claim. □

We now choose $\delta = \delta(\eta, K_{r+1})$ and $k_0 = k_0(\eta, K_{r+1})$ according to Theorem 2.5 such that for any K_{r+1}-free graph $\mathcal{G}$ on $k \geq k_0$ vertices with $e(\mathcal{G}) \geq e(T_r(k)) - c_H\delta^{1/m} \cdot k^2$, there exists a partition of $V(\mathcal{G}) = W_1\dot\cup \cdots \dot\cup W_r$ such that $\sum_{i=1}^r e_{\mathcal{G}}(W_i) < \eta k^2/2$.

We have seen that the cluster graph R is a K_{r+1}-free graph on $k \geq \gamma_0^{-1} \geq k_0$ vertices. Therefore, by the claim, there is a partition of $V(R) = [k] = W_1\dot\cup \cdots \dot\cup W_r$ such that

$$\sum_{i=1}^r e_R(W_i) < \eta k^2/2.$$

Then one can partition $V(G)$ into the following r parts: $V_1 = (\cup_{j\in W_1} A_j) \cup A_0$ and $V_i = \cup_{j\in W_i} A_j$ for $i \in \{2, \cdots, r\}$. It remains to estimate the number of edges in $\cup_{i=1}^r G[V_i]$, each of which belongs to one of following five types: (Note that $d = \delta/2$, $k\ell \le n$ and $\frac{1}{k} \le \gamma_0$.)

- edges incident to some vertex in A_0, the number of which is at most $\gamma_0 n^2$,
- edges in $G[A_i]$ for some $i \in [k]$, the number of which is at most $k\binom{\ell}{2} \le \gamma_0 n^2$,
- edges in non-γ_0-regular pairs of $\mathcal{A}$, the number of which is at most $\ell^2\gamma_0 k^2 \le \gamma_0 n^2$,
- edges in γ_0-regular pairs $\mathcal{A}$ with density $< d$, the number of which is at most $d\ell^2 k^2 \le \delta n^2/2$, and
- edges in some (γ_0, d)-regular pair (A_{j_1}, A_{j_2}) for some $j_1, j_2 \in W_i$ and $i \in [r]$, the number of which is at most $\ell^2 \sum_{i=1}^r e_R(W_i) < \ell^2\eta k^2/2 \le \eta n^2/2$.

Combining, as $3\gamma_0 + \delta/2 + \eta/2 < \eta$, we have that $\sum_{i=1}^r e(V_i) < \eta n^2$. This proves (4) and thus completes the proof of Theorem 3.1. □

4 Counting cliques

This section will be devoted to the proof of Theorem 1.2 (from which Corollary 1.3 will also follow). We have established the lower bound. So it suffices to show for sufficiently large n, if G is an n-vertex H-free graph with

$$\mathcal{N}(G, K_m) \ge \mathcal{N}(T_r(n), K_m), \tag{7}$$

then

$$\mathcal{N}(G, K_m) \le \mathcal{N}(T_r(n), K_m) + \text{biex}(n, H) \cdot O(n^{m-2}). \tag{8}$$

We will proceed with a sequence of claims.

Claim 4.1. *We may assume an additional condition for G that $\delta^{(m)}(G) \ge \delta^{(m)}(T_r(n))$.*

Proof. Assume $n \ge n_0 + \binom{n_0}{m}$ for some sufficiently large n_0. Let $G_n := G$. If $\delta^{(m)}(G_n) \ge \delta^{(m)}(T_r(n))$, then there is nothing to show. So we may assume there exists some vertex $v_n \in V(G_n)$ with $d^{(m)}_{G_n}(v_n) \le \delta^{(m)}(T_r(n)) - 1$. Let $G_{n-1} := G_n - \{v_n\}$. Then, by Proposition 2.3, we have $\mathcal{N}(G_{n-1}, K_m) = \mathcal{N}(G_n, K_m) - d^{(m)}_{G_n}(v_n) \ge \mathcal{N}(T_r(n), K_m) - \delta^{(m)}(T_r(n)) + 1 = \mathcal{N}(T_r(n-1), K_m) + 1$.

We then iteratively define graphs G_j satisfying $\mathcal{N}(G_j, K_m) \ge \mathcal{N}(T_r(j), K_m) + (n-j)$ as following. Assume that G_j is defined. If there exists some $v_j \in G_j$ with $d^{(m)}_{G_j}(v_j) \le \delta^{(m)}(T_r(j)) - 1$, then let $G_{j-1} := G_j - \{v_j\}$ and it also follows that $\mathcal{N}(G_{j-1}, K_m) = \mathcal{N}(G_j, K_m) - d^{(m)}_{G_j}(v_j) \ge \mathcal{N}(T_r(j-1), K_m) + (n-j+1)$; otherwise, terminate.

Let G_t be the graph for which the above iteration terminates. So G_t has exactly t vertices and $\delta^m(G_t) \ge \delta^m(T_r(t))$. Suppose that $t < n_0$. Then we have

$$\binom{n_0}{m} > \binom{t}{m} \ge \mathcal{N}(G_t, K_m) \ge \mathcal{N}(T_r(t), K_m) + (n-t) \ge n - n_0 \ge \binom{n_0}{m},$$

a contradiction. So we have $n \ge |V(G_t)| = t \ge n_0$.

Now suppose that under the additional condition $\delta^m(G_t) \ge \delta^m(T_r(t))$, one can derive from the inequality $\mathcal{N}(G_t, K_m) \ge \mathcal{N}(T_r(t), K_m)$ (for $t \ge n_0$) that (8) holds for G_t, i.e.,

$\mathcal{N}(G_t, K_m) \le \mathcal{N}(T_r(t), K_m) + \text{biex}(t, H) \cdot O(t^{m-2})$. Then we would infer that (8) also holds for G, by the following

$$\begin{aligned}\mathcal{N}(G, K_m) &= \mathcal{N}(G_t, K_m) + \sum_{j=t+1}^{n} d^{(m)}_{G_j}(v_j) \\ &\le \mathcal{N}(T_r(t), K_m) + \text{biex}(t, H) \cdot O(t^{m-2}) + \sum_{j=t+1}^{n} \delta^{(m)}(T_r(j)) \\ &= \mathcal{N}(T_r(n), K_m) + \text{biex}(n, H) \cdot O(n^{m-2}),\end{aligned}$$

where the last equality follows from Proposition 2.3. This proves Claim 4.1. □

Choose $\varepsilon > 0$ to be sufficiently small. Let $V_1, \cdots, V_r$ be a partition of $V(G)$ such that $\sum_{i=1}^{r} e(V_i)$ is minimized. In view of (7), by Theorem 1.4, we have that

$$\sum_{i=1}^{r} e(V_i) < \varepsilon n^2. \tag{9}$$

By Lemma 3.2, there exists some $\gamma = \gamma(\varepsilon)$ with $\lim_{\varepsilon \to 0} \gamma(\varepsilon) = 0$ such that

$$e(G[V_1, \cdots, V_r]) > e(T_r(n)) - \gamma n^2. \tag{10}$$

Let $\beta = \beta(\varepsilon) := \max(2\sqrt{\varepsilon}, \sqrt[3]{4\gamma})$. We may assume that ε is small so that $\beta < (r-1)^{-2}$. Let $B_i = \{x \in V_i : |N(v) \cap V_i| > \beta n\}$ for $i \in [r]$. Let $B = \cup_{i=1}^{r} B_i$ and let $U_i = V_i \setminus B$. Because of (9) and $\beta \ge 2\sqrt{\varepsilon}$, we get

$$|B| < \frac{2\varepsilon n^2}{\beta n} \le \frac{\beta}{2} n.$$

The next claim further bounds the size of B from above by an absolute constant. Recall the definition of $\sigma(H)$ in Section 2.

Claim 4.2. *There exists some positive constant K depending only on β and H such that $|B| \le K(\sigma(H) - 1)$. In particular, if H is edge-critical, then $B = \emptyset$.*

Proof. Note that $V_1, \cdots, V_r$ is a partition of $V(G)$ such that $\sum_{i=1}^{r} e(V_i)$ is minimized. So for any $v \in V_i$ and $j \ne i$, we have $|N(v) \cap V_j| \ge |N(v) \cap V_i|$. This together with the definition of B show that for any $v \in B$ and $i \in [r]$, $|N(v) \cap V_i| \ge \beta n$. Since $U_i = V_i \setminus B$ and $|B| < \frac{\beta}{2} n$, it follows that $|N(v) \cap U_i| > \frac{\beta}{2} n$.

Consider an arbitrarily but fixed $v \in B$. Let $S_i \subset N(v) \cap U_i$ be a set of size $\frac{\beta}{2} n$ for each $i \in [r]$. The inequality (10) tells that $G[V_1, \cdots, V_r])$ misses at most γn^2 edges, so for all $i \ne j$ we have $e(G[S_i, S_j]) > |S_i||S_j| - \gamma n^2 \ge (1-\beta)\beta^2 n^2/4$. (Here, we used $\beta \ge \sqrt[3]{4\gamma}$.) Thus the edge-density of $G[\cup_{i=1}^{r} S_i]$ is at least

$$\frac{\binom{r}{2}(1-\beta)\beta^2 n^2/4}{\binom{r\beta n/2}{2}} \ge \frac{r-1}{r}(1-\beta) > \frac{r-2}{r-1},$$

where the last inequality holds because of that $\beta < (r-1)^{-2}$. So we can apply the super-saturation theorem of Erdős and Simonovits [12] and conclude that the graph $G[\cup_{i=1}^{r} S_i]$ contains at least cn^{br} copies of the b-blowup $K_r(b)$, where $b := |V(H)|$ and $c := c(\beta, H) > 0$ is a constant.

Let $\mathcal{X}$ be the set of all copies of $K_r(b)$ in $G[\cup_{i=1}^{r} S_i]$. So $|\mathcal{X}| \le n^{br}$. We then define an auxiliary bipartite graph $\mathcal{G}$ with the bipartition $(\mathcal{X}, B)$, where $R \in \mathcal{X}$ and $v \in B$ are adjacent in $\mathcal{G}$ if and only if $V(R) \subseteq N_G(v)$. By the previous paragraph, we see $d_{\mathcal{G}}(v) \ge cn^{br}$ for all $v \in B$. We point out that $d_{\mathcal{G}}(R) \le \sigma(H) - 1$ for all $R \in \mathcal{X}$, as otherwise it will lead to an H-copy by the definition of $\sigma(H)$. Therefore, $|B|cn^{br} \le e(\mathcal{G}) \le (\sigma(H) - 1) \cdot n^{br}$. This shows that $|B| \le K(\sigma(H) - 1)$, where $K = 1/c$. □

Claim 4.3. *There exists some $\theta = \theta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\theta(\varepsilon) = 0$ such that $||V_i| - \frac{n}{r}| < \theta n$ for all $i \in [r]$.*

Proof. By symmetry, it suffices for us to prove for $i = 1$. Let $p := |V_1|/n \in [0, 1]$. Let $K_{V_1,\cdots,V_r}$ be the complete r-partite graph with parts $V_1, \cdots, V_r$. Each K_m-copy in G either contains some edge in $\cup_{i=1}^r E(G[V_i])$ or is contained in $G[V_1, \cdots, V_r] \subseteq K_{V_1,\cdots,V_r}$. By (7) and (9), it follows that

$$\mathcal{N}(T_r(n), K_m) \le \mathcal{N}(G, K_m) \le \varepsilon n^2 \cdot n^{m-2} + \mathcal{N}(K_{V_1,\cdots,V_r}, K_m)$$

Since K_m-copy in $K_{V_1,\cdots,V_r}$ either contains exactly one vertex in V_1 or is contained in $K_{V_2,\cdots,V_r}$, we have

$$\mathcal{N}(K_{V_1,\cdots,V_r}, K_m) \le |V_1| \cdot \mathcal{N}(K_{V_2,\cdots,V_r}, K_{m-1}) + \mathcal{N}(K_{V_2,\cdots,V_r}, K_m).$$

By Theorem 1.1, we also have that for $j \in \{m-1, m\}$

$$\mathcal{N}(K_{V_2,\cdots,V_r}, K_j) \le \mathcal{N}(T_{r-1}(n - |V_1|), K_j).$$

Putting the above inequalities together, it holds that

$$\mathcal{N}(T_r(n), K_m) \le \varepsilon n^m + |V_1| \cdot \mathcal{N}(T_{r-1}(n - |V_1|), K_{m-1}) + \mathcal{N}(T_{r-1}(n - |V_1|), K_m).$$

By Proposition 2.2, this yields

$$\binom{r}{m}\left(\frac{n}{r}\right)^m \le pn\binom{r-1}{m-1}\left(\frac{(1-p)n}{r-1}\right)^{m-1} + \binom{r-1}{m}\left(\frac{(1-p)n}{r-1}\right)^m + 2\varepsilon n^m.$$

After some simplifications, it gives that

$$f(p) := m(r-1)p(1-p)^{m-1} + (r-m)(1-p)^m - r\left(1 - \frac{1}{r}\right)^m \ge -2\varepsilon.$$

One can easily verify that $f(p)$ increases in $[0, \frac{1}{r}]$ and decreases in $[\frac{1}{r}, 1]$, where $f(\frac{1}{r}) = 0$. So by the continuity of f, there exists some $\theta = \theta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\theta(\varepsilon) = 0$ such that $|p - \frac{1}{r}| < \theta$. This proves Claim 4.3. □

Claim 4.4. *There exists $\eta = \eta(\varepsilon)$ with $\lim_{\varepsilon\to 0}\eta(\varepsilon) = 0$ such that*

$$|N(v) \cap U_j| > \left(\frac{1}{r} - \eta\right) n$$

for every $v \in U_i$ and every $j \neq i$.

Proof. Fix a vertex $v \in U_i$ and some $j \neq i$. We will show this claim by estimating $d_G^{(m)}(v)$.

First let us estimate the number of K_m-copies containing v in $G[V_1, \cdots, V_r]$. Such copies may contain some vertex in V_j or not. By Claim 4.3, we have $|V_k| < \frac{n}{r} + \theta n$ for each $k \in [r]$. So the number of such copies containing some vertex in V_j is at most

$$|N(v) \cap V_j| \cdot \binom{r-2}{m-2}\left(\frac{n}{r} + \theta n\right)^{m-2},$$

and the number of such copies containing no vertex in V_j is at most

$$\binom{r-2}{m-1}\left(\frac{n}{r} + \theta n\right)^{m-1}.$$

For each copy of K_m in G that contains v, if it is not in $G[V_1, \cdots, V_r]$, then it contains either some neighbor of v in V_i or an edge in $G[V_k]$ for some $k \neq i$. The number of K_m-copies of the former kind is at most $|N(v) \cap V_i| \cdot n^{m-2} \leq \beta n^{m-1}$, and in view of (9), the number of the latter kind is at most $\varepsilon n^2 \cdot n^{m-3} \leq \varepsilon n^{m-1}$. This shows that

$$d_G^{(m)}(v) \leq (\beta+\varepsilon)n^{m-1} + |N(v) \cap V_j| \cdot \binom{r-2}{m-2}\left(\frac{n}{r}+\theta n\right)^{m-2} + \binom{r-2}{m-1}\left(\frac{n}{r}+\theta n\right)^{m-1}.$$

Also by Claim 4.1, we may assume that

$$\delta^{(m)}(G) \geq \delta^{(m)}(T_r(n)) = \binom{r-1}{m-1}\left(\frac{n}{r}\right)^{m-1} + O(n^{m-2}).$$

Putting the above two inequalities together, we have

$$\binom{r-1}{m-1}\frac{1}{r^{m-1}} \leq \beta + 2\varepsilon + \frac{|N(v) \cap V_j|}{n}\binom{r-2}{m-2}\left(\frac{1}{r}+\theta\right)^{m-2} + \binom{r-2}{m-1}\left(\frac{1}{r}+\theta\right)^{m-1}.$$

It then follows that there exists some $\xi = \xi(\beta, \varepsilon, \theta)$ with $\lim_{\beta,\varepsilon,\theta \to 0} \xi(\beta, \varepsilon, \theta) = 0$ such that $|N(v) \cap V_j| > (\frac{1}{r} - \xi)n$. Finally, recall that $|B_j| \leq |B| < \frac{\beta}{2}n$ (or use Claim 4.2 instead). Thus by letting $\eta(\varepsilon) := \xi(\beta, \varepsilon, \theta) + \frac{\beta}{2}$, we get that $|N(v) \cap U_j| \geq |N(v) \cap V_j| - |B_j| > (\frac{1}{r} - \eta)n$, completing the proof of Claim 4.4. $\square$

Claim 4.5. *For every $i \in [r]$, $e(U_i) \leq \mathrm{biex}(n, H)$. In particular, if H is edge-critical, then $U_1, \cdots, U_r$ are all independent sets.*

Proof. Suppose for a contradiction that say, $e(U_1) > \mathrm{biex}(n, H)$. Then $G[U_1]$ contains some $F \in \mathcal{F}_H$. Let $b = |V(H)|$. We assert that we can find $X_2, \cdots, X_r$ with $X_i \subset U_i$ and $|X_i| = b$ such that $G[V(F), X_2, \cdots, X_r]$ is a complete r-partite graph. If so, then clearly $G[V(F) \cup X_2 \cup \cdots \cup X_r]$ contains a copy of H, a contradiction.

To do this, suppose inductively that for some $i \in \{1, ..., r-1\}$, we have obtained $X_2, \cdots, X_i$ such that $G[V(F), X_2, \cdots, X_i]$ is complete i-partite. (For $i = 1$, we just view it as the set $V(F)$.) Then the number of common neighbors of $L_i := V(F) \cup X_2 \cup \cdots \cup X_i$ in U_{i+1} is at least

$$\left(\sum_{v \in L_i} |N(v) \cap U_{i+1}|\right) - (|L_i| - 1)|U_{i+1}| > |L_i|(\frac{1}{r} - \eta)n - (|L_i| - 1)(\frac{1}{r} + \theta)n$$
$$\geq \left(\frac{1}{r} - |L_i|(\eta+\theta)\right) \cdot n \geq \left(\frac{1}{r} - br(\eta+\theta)\right) \cdot n.$$

Here, the first inequality follows from Claim 4.4 and the fact $|U_{i+1}| \leq |V_{i+1}| < (\frac{1}{r}+\theta)n$ (by Claim 4.3), and the last inequality holds as $|L_i| \leq bi \leq br$. Since η and θ are sufficiently small and n is sufficiently large, we can find the desired set $X_{i+1} \subset U_{i+1}$ with $|X_{i+1}| = b$, proving Claim 4.5. $\square$

We are ready to prove the upper bound (8) of $\mathcal{N}(G, K_m)$. It is clear that every copy of K_m in G either is contained in $G[U_1, \cdots, U_r]$, or contains some edge in $\cup_{i=1}^r E(G[U_i])$, or contains some vertex in B. Since $G[U_1, \cdots, U_r]$ is K_{r+1}-free, by Theorem 1.1, we have

$$\mathcal{N}(G[U_1, \cdots, U_r], K_m) \leq \mathcal{N}(T_r(n), K_m).$$

Since $\sum_{i=1}^r e(U_i) \leq r \cdot \mathrm{biex}(n, H)$ (by Claim 4.5) and every edge can be contained in at most n^{m-2} copies of K_m, the number of copies of K_m that contain some edge in $\cup_{i=1}^r E(G[U_i])$ is at most

$$r \cdot \mathrm{biex}(n, H) \cdot n^{m-2} = \mathrm{biex}(n, H) \cdot O(n^{m-2}).$$

Lastly, since each vertex can be contained at most n^{m-1} copies of K_m, the number of copies of K_m that contain some vertex in B is at most

$$|B| \cdot n^{m-1} \leq K(\sigma(H)-1) \cdot n^{m-1} \leq \mathrm{biex}(n,H) \cdot O(n^{m-2}),$$

where the first inequality follows by Claim 4.2 and the last inequality holds because of Proposition 2.1. Putting the above together, we obtain the desired upper bound

$$\mathcal{N}(G,K_m) \leq \mathcal{N}(T_r(n),K_m) + \mathrm{biex}(n,H) \cdot O(n^{m-2}).$$

The proof of Theorem 1.2 is completed.

Now suppose H is edge-critical. By Claim 4.2, $B=\emptyset$ and so $V(G)=U_1\dot{\cup}\cdots\dot{\cup}U_r$. By Claim 4.5, we see that $U_1,\cdots,U_r$ are all independent sets, implying that G is r-partite and thus K_{r+1}-free. Hence by Theorem 1.1, it holds that $\mathcal{N}(G,K_m) \leq \mathcal{N}(T_r(n),K_m)$, with the equality holds if and only if $G=T_r(n)$. This proves Corollary 1.3. □

5 Counting complete multipartite graphs

Throughout this section let $r \geq 2$ and $t \geq s$ be fixed integers. Let $K_{s,t}^{(r)}$ denote the complete r-partite graph with one part of size t and the other $r-1$ parts of size s. It is easy to see that Theorem 1.5 will follow from the coming result.

Theorem 5.1. *Let $r \geq 2$ and $t \geq s$ be positive integers. Then the following hold:*

(a) If $t < s + \frac{1}{2} + \sqrt{2s+\frac{1}{4}}$, then for sufficiently large n, the unique n-vertex K_{r+1}-free graph which maximizes the number of copies of $K_{s,t}^{(r)}$ is the Turán graph $T_r(n)$.

(b) If $t = s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$, then $\mathrm{ex}(n,K_{s,t}^{(r)},K_{r+1}) = (1+o(1))\cdot\mathcal{N}(T_r(n),K_{s,t}^{(r)})$. Moreover, in case of $r=2$, $\mathrm{ex}(n,K_{s,t},K_3) \geq \mathcal{N}(T_2(n),K_{s,t}) + \Omega(n^{s+t-2})$.

(c) If $t > s+\frac{1}{2}+\sqrt{rs+\frac{1}{4}}$, then there exists a constant $c=c(r,s,t)>0$ such that $\mathrm{ex}(n,K_{s,t}^{(r)},K_{r+1}) \geq (1+c)\cdot\mathcal{N}(T_r(n),K_{s,t}^{(r)})$.

In this section we will prove Theorem 5.1, by assuming Lemmas 5.2 and 5.3 (see below; their proofs will be postponed to the next section). Before introducing the lemmas, we will need to give some notations.

Definition 5.1. *For integers $a \leq n$, let $G_{a,n}^r$ be the complete r-partite graph G on n vertices with parts $V_1,V_2,...,V_r$ such that $G[V_2\cup\cdots\cup V_r]=T_{r-1}(a)$. Let $F_{r,s,t}(a,n)$ be the number of copies of $K_{s,t}^{(r)}$ in $G_{a,n}^r$ each of which contains a fixed vertex in V_1.*

Let $\lambda_{s,t}$ be $\frac{1}{2}$ if $s=t$ and 1 otherwise. Then $F_{r,s,t}(a,n)$ can be expressed as

$$\lambda_{s,t}\cdot\left[\binom{n-1-a}{s-1}\cdot\mathcal{N}(T_{r-1}(a),K_{s,t}^{(r-1)}) + \binom{n-1-a}{t-1}\cdot\mathcal{N}(T_{r-1}(a),K_{s,s}^{(r-1)})\right]. \quad (11)$$

In case that $a=\lfloor\frac{r-1}{r}n\rfloor$, we see that $G_{a,n}^r=T_r(n)$ and $G_{a,n}^r\backslash\{v\}=T_r(n-1)$ for any $v\in V_1$. Hence we have

$$\mathcal{N}(T_r(n),K_{s,t}^{(r)}) - \mathcal{N}(T_r(n-1),K_{s,t}^{(r)}) = F_{r,s,t}\left(\left\lfloor\frac{r-1}{r}n\right\rfloor,n\right). \quad (12)$$

Lemma 5.2. *(i) If $s \leq t < s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$, then the following holds for sufficiently large n. If $F_{r,s,t}(\lfloor\frac{r-1}{r}n\rfloor,n) \leq F_{r,s,t}(d,n)$, then $d \geq \lfloor\frac{r-1}{r}n\rfloor$.*

(ii) If $t = s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$, then for any $\varepsilon > 0$, there exists a real $\eta > 0$ such that the following holds for sufficiently large n. If $F_{r,s,t}(\lfloor \frac{r-1}{r}n \rfloor, n) \le F_{r,s,t}(d,n) + \eta n^{(r-1)s+t-1}$, then $d \ge \frac{r-1}{r}n - \varepsilon n$.

Lemma 5.3. *For $s \le t \le s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$ and sufficiently large n, let G be an n-vertex r-partite graph which maximizes the number of copies of $K_{s,t}^{(r)}$. Then G is a complete r-partite graph with each part of size $\frac{n}{r}+o(n)$. Moreover, if $t < s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$, then $G = T_r(n)$ is unique.*

Now we are in a position to prove Theorem 5.1.

Proof of Theorem 5.1 (Assuming Lemmas 5.2 and 5.3). We first prove the "moreover" part of (b) and the case (c), by indicating that some complete r-partite graphs have more copies of $K_{s,t}^{(r)}$ than the Turán graphs $T_r(n)$. For the "moreover" part of (b), we have $t = s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$ and $r = 2$. By some tedious but straightforward calculations, one can show for $x = \Theta(\sqrt{n})$ that

$$\mathcal{N}(K_{\frac{n}{2}-x,\frac{n}{2}+x}, K_{s,t}) - \mathcal{N}(T_2(n), K_{s,t}) = 2st\left(\frac{n}{2}\right)^{s+t-3}x^2 - \frac{2st}{3}\left(\frac{n}{2}\right)^{s+t-4}x^4 + o(n^{s+t-2}).$$

By letting $x = \frac{\sqrt{3n}}{2} + o(\sqrt{n})$, the desired inequality follows by

$$\mathrm{ex}(n, K_{s,t}, K_3) \ge \mathcal{N}(K_{\frac{n}{2}-x,\frac{n}{2}+x}, K_{s,t}) \ge \mathcal{N}(T_2(n), K_{s,t}) + \left(\frac{3st}{2}+o(1)\right)\left(\frac{n}{2}\right)^{s+t-2}.$$

For the case (c), let $t > s+\frac{1}{2}+\sqrt{rs+\frac{1}{4}}$ and consider $K_{x_1 n,\cdots,x_r n}$, where $x_i = x \in (0,\frac{1}{r-1})$ for $i \in [r-1]$ and $x_r = 1-(r-1)x$. It is not hard to see that

$$\mathcal{N}(K_{x_1 n,\cdots,x_r n}, K_{s,t}^{(r)}) = \sum_{i=1}^{r}\binom{x_i n}{t}\prod_{j\neq i}\binom{x_j n}{s} = \frac{e^{F(x)}+o(1)}{t!(s!)^{r-1}}n^{(r-1)s+t},$$

where $F(x) = s\log[x^{r-1}-(r-1)x^r] + \log[(r-1)x^{t-s}+(1-(r-1)x)^{t-s}]$. In particular,

$$\mathcal{N}(T_r(n), K_{s,t}^{(r)}) = \frac{e^{F(\frac{1}{r})}+o(1)}{t!(s!)^{r-1}}n^{(r-1)s+t}.$$

Therefore to prove the case (c), it suffices to show that $\frac{1}{r}$ is not a maximum point of $F(x)$ in the interval $(0,\frac{1}{r-1})$; and further, it is enough to show $F''(\frac{1}{r}) > 0$. This indeed is the case, as by some routine calculations one can show that

$$F''\left(\frac{1}{r}\right) = r^2(r-1)\cdot[(t-s)^2 - t - s(r-1)] > 0,$$

where the inequality holds by $r \ge 2$ and $t > s+\frac{1}{2}+\sqrt{rs+\frac{1}{4}}$.

In the rest of the proof we assume $s \le t \le s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$. We will apply induction on r to prove the remaining statements of Theorem 5.1, namely for sufficiently large n,

(a). $T_r(n)$ uniquely attains the maximum $\mathrm{ex}(n, K_{s,t}^{(r)}, K_{r+1})$ if $s \le t < s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$;

(b). $\mathrm{ex}(n, K_{s,t}^{(r)}, K_{r+1}) = \mathcal{N}(T_r(n), K_{s,t}^{(r)}) + o(n^{(r-1)s+t})$ if $t = s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$.

For the case $r = 1$, we view $K_{s,t}^{(r)}$ and $T_r(n)$ as graphs with empty edge set on t vertices and n vertices respectively, and then items (a) and (b) holds trivially. Now suppose that these two items hold for the case $r-1$.

Let n be sufficiently large, $\varepsilon > 0$ be sufficiently small, and η be obtained from Lemma 5.2 (ii) such that $\frac{r-1}{r} - \varepsilon > \frac{3r-4}{3r-1}$. Let G be an n-vertex K_{r+1}-free graph which maximizes the number of copies of $K_{s,t}^{(r)}$. So we have

$$\mathcal{N}(G, K_{s,t}^{(r)}) = \mathrm{ex}(n, K_{s,t}^{(r)}, K_{r+1}) \geq \mathcal{N}(T_r(n), K_{s,t}^{(r)}). \tag{13}$$

We then recursively define a sequence of graphs G_i's as following. Let $G_n := G$. For $i \leq n$, if there is some vertex $v_i \in V(G_i)$ with $d_{G_i}(v_i, K_{s,t}^{(r)}) \leq \delta_i - 1$, where

$$\delta_i := \mathcal{N}(T_r(i), K_{s,t}^{(r)}) - \mathcal{N}(T_r(i-1), K_{s,t}^{(r)}),$$

then let $G_{i-1} = G_i \backslash \{v_i\}$ and continue; otherwise, terminate. Suppose this recursive process stops at $H := G_\ell$ for some $\ell \leq n$. Then H has ℓ vertices with $\delta(H, K_{s,t}^{(r)}) \geq \delta_\ell$ and

$$\mathcal{N}(H, K_{s,t}^{(r)}) = \mathcal{N}(G, K_{s,t}^{(r)}) - \sum_{i=\ell+1}^{n} d_{G_i}(v_i, K_{s,t}^{(r)}) \tag{14}$$

$$\geq \mathcal{N}(T_r(n), K_{s,t}^{(r)}) - \sum_{i=\ell+1}^{n} \delta_i + (n-\ell) = \mathcal{N}(T_r(\ell), K_{s,t}^{(r)}) + (n-\ell). \tag{15}$$

Assume that $n \geq n_0 + n_0^{(r-1)s+t}$ for some sufficiently large n_0. We claim that $\ell \geq n_0$; as otherwise $n_0 > \ell$, from which it follows that

$$n_0^{(r-1)s+t} > \mathcal{N}(H, K_{s,t}^{(r)}) \geq \mathcal{N}(T_r(\ell), K_{s,t}^{(r)}) + (n-\ell) \geq n - n_0 \geq n_0^{(r-1)s+t},$$

a contradiction.

Let $v \in V(H)$ have minimum degree d_v in H. We claim that $d_H(v, K_{s,t}^{(r)})$ is at most

$$\lambda_{s,t} \cdot \left[\binom{\ell-1-d_v}{s-1} \cdot \mathrm{ex}(d_v, K_{s,t}^{(r-1)}, K_r) + \binom{\ell-1-d_v}{t-1} \cdot \mathrm{ex}(d_v, K_{s,s}^{(r-1)}, K_r) \right]. \tag{16}$$

Note that as H is K_{r+1}-free, $H[N_H(v)]$ is K_r-free. Every $K_{s,t}^{(r)}$-copy T in H containing v must contain either $(r-2)s+t$ vertices in $N_H(v)$ which induce a copy of $K_{s,t}^{(r-1)}$, or $(r-1)s$ vertices in $N_H(v)$ which induce a copy of $K_{s,s}^{(r-1)}$. Moreover, if the former case occurs, then the other $s-1$ vertices of T must be in $V(H) \backslash (N_H(v) \cup \{v\})$, as otherwise it will lead to a copy of K_r in $H[N_H(v)]$; similarly, if the later one occurs, then the other $t-1$ vertices of T must be in $V(H) \backslash (N_H(v) \cup \{v\})$. This justifies the claim.

Let $\mu = 0$ if $s \leq t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, and $\mu = 1$ otherwise. By (12), we have

$$d_H(v, K_{s,t}^{(r)}) \geq \delta(H, K_{s,t}^{(r)}) \geq \delta_\ell = F_{r,s,t}\left(\left\lfloor \frac{r-1}{r} \ell \right\rfloor, \ell \right) \geq \Omega(\ell^{(r-1)s+t-1}). \tag{17}$$

Then by (16), $\ell^{s-1} d_v^{(r-2)s+t} + \ell^{t-1} d_v^{(r-1)s} \geq d_H(v, K_{s,t}^{(r)}) \geq \Omega(\ell^{(r-1)s+t-1})$, which implies that $d_v = \Omega(\ell) = \Omega(n_0)$ is sufficiently large. By our induction, it follows that

$$\mathrm{ex}(d_v, K_{s,t}^{(r-1)}, K_r) = \mathcal{N}(T_{r-1}(d_v), K_{s,t}^{(r-1)}) + \mu \cdot o(d_v^{(r-2)s+t-1}).$$

This, together with (16) and (11) (i.e., the definition of $F_{r,s,t}$), implies that

$$d_H(v, K_{s,t}^{(r)}) \leq F_{r,s,t}(d_v, \ell) + \mu \cdot o(\ell^{(r-1)s+t-1}).$$

By (17), for sufficiently large ℓ (as $\ell \geq n_0$), we have

$$F_{r,s,t}\left(\left\lfloor \frac{r-1}{r}\ell \right\rfloor, \ell\right) \leq F_{r,s,t}(d_v, \ell) + \mu \cdot \eta \cdot \ell^{(r-1)s+t-1},$$

where η is obtained from Lemma 5.2 (ii). Applying Lemma 5.2, we obtain that the minimum degree $\delta(H) = d_v \geq (\frac{r-1}{r} - \varepsilon)\ell > \frac{3r-4}{3r-1}\ell$. As H is an ℓ-vertex K_{r+1}-free graph, by Theorem 2.6 we see that H is r-partite. Then Lemma 5.3 shows that

$$\mathcal{N}(H, K_{s,t}^{(r)}) \leq \mathcal{N}(T_r(\ell), K_{s,t}^{(r)}) + \mu \cdot o(\ell^{(r-1)s+t}),$$

where the equality holds for $\mu = 0$ if and only if $H = T_r(\ell)$. By (13) and (14), we have

$$\begin{aligned}\mathcal{N}(T_r(n), K_{s,t}^{(r)}) &\leq \mathcal{N}(G, K_{s,t}^{(r)}) = \mathcal{N}(H, K_{s,t}^{(r)}) + \sum_{i=\ell+1}^{n} d_{G_i}(v_i, K_{s,t}^{(r)}) \\ &\leq \mathcal{N}(T_r(\ell), K_{s,t}^{(r)}) + \sum_{i=\ell+1}^{n} \delta_i + \mu \cdot o(\ell^{(r-1)s+t}) - (n-\ell) \\ &= \mathcal{N}(T_r(n), K_{s,t}^{(r)}) + \mu \cdot o(\ell^{(r-1)s+t}) - (n-\ell).\end{aligned}$$

If $s \leq t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$ (that is, $\mu = 0$), then it is easy to see that $n = \ell$, $G = H$ and $\mathcal{N}(H, K_{s,t}^{(r)}) = \mathcal{N}(T_r(n), K_{s,t}^{(r)})$; and in this case Lemma 5.3 also shows that $G = H = T_r(n)$ is unique. For the case $t = s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, it is also easy to see that $\mathcal{N}(G, K_{s,t}^{(r)}) = \mathcal{N}(T_r(n), K_{s,t}^{(r)}) + o(n^{(r-1)s+t})$. The proof of Theorem 5.1 is completed. ☐

6 Two Lemmas

Here we prove Lemmas 5.2 and 5.3. Throughout this section, let r, s, t be fixed integers such that $r \geq 2$ and $s \leq t \leq s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, and let n be sufficiently large.

6.1 Proof of Lemma 5.2

Recall the definition of $\lambda_{s,t}$, and let $\tilde{\lambda}_{s,t,r} = r - 1$ if $t \neq s$ and 1 otherwise. One can easily obtain the following.

Proposition 6.1. $\mathcal{N}(T_r(n), K_{s,t}^{(r)}) = (1 + o(1)) \frac{\tilde{\lambda}_{s,t,r+1}}{(s!)^{r-1} t!} (\frac{n}{r})^{(r-1)s+t}$.

Proposition 6.2. $F_{r,s,t}\left(\left\lfloor \frac{r-1}{r} n \right\rfloor, n\right) = (1 + o(1)) \frac{\lambda_{s,t}(s\tilde{\lambda}_{s,t,r}+t)}{(s!)^{r-1} t!} \left(\frac{n}{r}\right)^{(r-1)s+t-1}$.

From now on we will often write $F(a)$ instead of $F_{r,s,t}(a, n)$ for short.

Proposition 6.3. *There exist $\eta_0 > 0$ and $\gamma > 0$ such that the following holds. For any $\eta \in [0, \eta_0)$, if $F(\lfloor \frac{r-1}{r} n \rfloor) \leq F(d) + \eta n^{(r-1)s+t-1}$, then $d \geq \gamma n$.*

Proof. By Proposition 6.2, there is some $c > 0$ such that $F\left(\left\lfloor \frac{r-1}{r} n \right\rfloor\right) > c n^{(r-1)s+t-1}$. Let $\eta_0 := \frac{c}{2}$. Suppose $0 \leq \eta \leq \eta_0$ and $F(\lfloor \frac{r-1}{r} n \rfloor) \leq F(d) + \eta n^{(r-1)s+t-1}$. Then by the definition of F, we have that

$$n^{s-1} d^{(r-2)s+t} + n^{t-1} d^{(r-1)s} \geq F(d) \geq F\left(\left\lfloor \frac{r-1}{r} n \right\rfloor\right) - \eta n^{(r-1)s+t-1} \geq \frac{c}{2} n^{(r-1)s+t-1}.$$

This yields some $\gamma = \gamma(r, s, t) > 0$ such that $d \geq \gamma n$. ☐

The following two propositions assert some properties on $F(a)$. We leave the technical details of their proofs in the Appendix A.

Proposition 6.4. *For any $\gamma, \varepsilon > 0$ with $\gamma + \varepsilon < \frac{r-1}{r}$, the following hold.*

(i) If $t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, then $F(a+1) > F(a)$ for all integers $a \in [\gamma n, \lfloor \frac{r-1}{r} n \rfloor]$.

(ii) If $t = s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, then $F(a+1) > F(a)$ for all integers $a \in [\gamma n, (\frac{r-1}{r} - \varepsilon)n]$.

Proposition 6.5. *For any $\varepsilon \in (0, \frac{r-1}{r})$, there exists $\xi = \xi(\varepsilon, r, s, t) > 0$ such that $F(\lfloor \frac{r-1}{r} n \rfloor) - F(\lfloor (\frac{r-1}{r} - \varepsilon)n \rfloor) > \xi n^{(r-1)s+t-1}$.*

We have collected all propositions needed for the proof of Lemma 5.2.

Proof of Lemma 5.2. First we consider the case (i) that $s \le t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$. Suppose that $F\left(\lfloor \frac{r-1}{r} n \rfloor\right) \le F(d)$ (and n is assumed to be sufficiently large throughout this section). By Proposition 6.3, there exists some $\gamma > 0$ such that $d \ge \gamma n$. We may assume $\gamma < \frac{r-1}{r}$, as otherwise we are done. Then by Proposition 6.4 (i), $F\left(\lfloor \frac{r-1}{r} n \rfloor\right)$ is the unique maximum of $F(a)$ in $[\gamma n, \lfloor \frac{r-1}{r} n \rfloor]$. This yields that $d \ge \lfloor \frac{r-1}{r} n \rfloor$.

Now we consider the case (ii) that $t = s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$. For any $\varepsilon > 0$, let η_0 and ξ be obtained from Propositions 6.3 and 6.5 respectively. Let $\eta := \min\{\eta_0, \xi\} > 0$ and write $v = (r-1)s + t - 1$. Now suppose that $F\left(\lfloor \frac{r-1}{r} n \rfloor\right) \le F(d) + \eta n^v$. Our goal is to show $d \ge \frac{r-1}{r} n - \varepsilon n$.

Suppose to the contrary that $d < \frac{r-1}{r} n - \varepsilon n$. By Lemma 6.3, there exists some $\gamma > 0$ such that $d \ge \gamma n$. So $\gamma n \le d < (\frac{r-1}{r} - \varepsilon)n$. Putting Proposition 6.4 (ii) and Proposition 6.5 together, we have $F(d) \le F\left(\lfloor (\frac{r-1}{r} - \varepsilon)n \rfloor\right) < F\left(\lfloor \frac{r-1}{r} n \rfloor\right) - \xi n^v \le F\left(\lfloor \frac{r-1}{r} n \rfloor\right) - \eta n^v$, which is a contradiction to the assumption. This proves Lemma 5.2. □

6.2 Proof of Lemma 5.3

Let G be an n-vertex r-partite graph with the maximum number of $K_{s,t}^{(r)}$-copies. It is clear that G must be a complete r-partite graph. So we may assume that $G = K_{a_1,\cdots,a_r}$ with $n = a_1 + ... + a_r$ and $a_r \ge \cdots \ge a_1 \ge s$ (where $a_1 \ge s$ is because $\mathcal{N}(G, K_{s,t}^{(r)}) \ge 1$).

For any vector $\vec{x} = (x_1, ..., x_r)$ with positive integers x_i's, write $K_{\vec{x}} = K_{x_1,\cdots,x_r}$ and let

$$g(\vec{x}) = \sum_{i=1}^{r} \binom{x_i}{t} \prod_{j \ne i} \binom{x_j}{s}, \quad *\vec{x} = (x_1+1, x_2, ..., x_{r-1}, x_r - 1), \text{ and } \Delta g(\vec{x}) = g(*\vec{x}) - g(\vec{x}).$$

Therefore, if $t \ne s$, then $g(\vec{x}) = \mathcal{N}(K_{\vec{x}}, K_{s,t}^{(r)})$; otherwise, $g(\vec{x}) = r\mathcal{N}(K_{\vec{x}}, K_{s,t}^{(r)})$.

We present a sequence of propositions as following.

Proposition 6.6. *Let $\vec{a} = (a_1, ..., a_r)$. Then we have $\Delta g(\vec{a}) \le 0$.*

Proof. This clearly follows by the maximality of $\mathcal{N}(G, K_{s,t}^{(r)})$. □

Proposition 6.7. *There exists some $\gamma > 0$ such that $a_1 \ge \gamma n$.*

Proof. We have $\mathcal{N}(T_r(n), K_{s,t}^{(r)}) \le \mathcal{N}(G, K_{s,t}^{(r)}) \le \mathcal{N}(K_{a_1,n,...,n}, K_{s,t}^{(r)})$. Thus there exists some $c > 0$ such that $cn^{(r-1)s+t} \le \mathcal{N}(G, K_{s,t}^{(r)}) \le (r-1)a_1^s n^{(r-2)s+t} + a_1^t n^{(r-1)s}$. This implies that $a_1 \ge \gamma n$ for some constant $\gamma > 0$. □

For a vector $\vec{x} = (x_1, ..., x_r)$, let $h(\vec{x}) = x_r t!/s! \prod_{i=1}^{r} \binom{x_i}{s}$.

Proposition 6.8. *Let $q = t - s$. The product $h(\vec{a})\Delta g(\vec{a})$ is equal to*

$$\frac{sa_r - t(a_1+1)}{a_1+1-s}(a_r - s)_q + \frac{ta_r - s(a_1+1)}{a_1+1-t}(a_1 - s)_q + \frac{s(a_r - a_1 - 1)}{a_1+1-s}\sum_{i=2}^{r-1}(a_i - s)_q.$$

Proof. The proof is straightforward and we just give some computations here. By routine calculations, we have $\Delta g(\vec{a}) = g(*\vec{a}) - g(\vec{a}) = A \cdot \prod_{i=2}^{r-1}\binom{a_i}{s} + B \cdot \prod_{j\neq 1,i,r}\binom{a_j}{s}$, where

$$A = \binom{a_1+1}{s}\binom{a_r-1}{t} + \binom{a_1+1}{t}\binom{a_r-1}{s} - \binom{a_1}{s}\binom{a_r}{t} - \binom{a_1}{t}\binom{a_r}{s},$$

$$B = \binom{a_1+1}{s}\binom{a_r-1}{s} - \binom{a_1}{s}\binom{a_r}{s}.$$

Using the formula $\binom{a_1+1}{x}\binom{a_r-1}{y} = \frac{a_1+1}{a_1+1-x} \cdot \frac{a_r-y}{a_r} \cdot \binom{a_1}{x}\binom{a_r}{y}$, one can derive that

$$\frac{\Delta g(\vec{a})}{\prod_{i=1}^{r}\binom{a_i}{s}} = \frac{sa_r - t(a_1+1)}{(a_1+1-s)a_r}\frac{\binom{a_r}{t}}{\binom{a_r}{s}} + \frac{ta_r - s(a_1+1)}{(a_1+1-t)a_r}\frac{\binom{a_1}{t}}{\binom{a_1}{s}} + \frac{s(a_r-a_1-1)}{(a_1+1-s)a_r}\sum_{i=2}^{r-1}\frac{\binom{a_i}{t}}{\binom{a_i}{s}}.$$

Now it follows easily by $h(\vec{a}) = \frac{a_r t!}{s!\prod_{i=1}^{r}\binom{a_i}{s}}$ and the formula $\binom{a_i}{t} = \frac{s!}{t!}\binom{a_i}{s}(a_i - s)_q$. □

For reals $x > 0, \alpha \geq 0$ and an integer $k \geq 1$, let $(x)_k = \prod_{i=0}^{k-1}(x - i)$ and $H(x,\alpha) = H_1(x,\alpha) + H_2(x,\alpha) + H_3(x,\alpha)$, where

$$\begin{cases} H_1(x,\alpha) &= \left(s\alpha - q - \frac{qs+t}{x}\right)\left(1 + \frac{1-q}{x}\right)\frac{(x+\alpha x)_q}{x^q}, \\ H_2(x,\alpha) &= \left(t\alpha + q + \frac{qs-s}{x}\right)\left(1 + \frac{1}{x}\right)\frac{(x)_q}{x^q}, \\ H_3(x,\alpha) &= (r-2)s\left(\alpha - \frac{1}{x}\right)\left(1 + \frac{1-q}{x}\right)\frac{(x)_q}{x^q}. \end{cases}$$

Proposition 6.9. *Let $\hat{x} = a_1 - s$ and $\hat{\alpha} = \frac{a_r - a_1}{a_1 - s}$. If $a_r \geq a_1 + 1$, then $H(\hat{x}, \hat{\alpha}) \leq 0$.*

Proof. Assume that $a_r \geq a_1 + 1$. Let $p(x) = \frac{x^{q+2}}{(x+1)(x+1-q)}$. We first show that

$$h(\vec{a})\Delta g(\vec{a}) \geq H(\hat{x}, \hat{\alpha}) \cdot p(\hat{x}). \tag{18}$$

One can rewrite the first two terms of $h(\vec{a})\Delta g(\vec{a})$ in Proposition 6.8 as the following

$$\frac{sa_r - t(a_1+1)}{a_1+1-s}(a_r - s)_q = \frac{(s\hat{\alpha}\hat{x} - q\hat{x} - qs - t)(\hat{x} + \hat{\alpha}\hat{x})_q}{\hat{x}+1} = H_1(\hat{x},\hat{\alpha}) \cdot p(\hat{x}),$$
$$\frac{ta_r - s(a_1+1)}{a_1+1-t}(a_1 - s)_q = \frac{(t\hat{\alpha}\hat{x} + q\hat{x} + qs - s)(\hat{x})_q}{\hat{x}+1-q} = H_2(\hat{x},\hat{\alpha}) \cdot p(\hat{x}).$$

Thus to prove (18), it suffices to show that the third term of $h(\vec{a})\Delta g(\vec{a})$ in Proposition 6.8 is at least $H_3(\hat{x},\hat{\alpha})p(\hat{x})$. Indeed, since $\frac{s(a_r-a_1-1)}{a_1+1-s} \geq 0$ and $\sum_{i=2}^{r-1}(a_i - s)_q \geq (r-2)(a_1 - s)_q = (r-2)(\hat{x})_q$, this follows by $\frac{s(a_r-a_1-1)}{a_1+1-s}\sum_{i=2}^{r-1}(a_i - s)_q \geq \frac{(r-2)s(\hat{\alpha}\hat{x}-1)(\hat{x})_q}{\hat{x}+1} = H_3(\hat{x},\hat{\alpha}) \cdot p(\hat{x})$.

Next we use (18) to show $H(\hat{x},\hat{\alpha}) \leq 0$. Since n is sufficiently large and $a_1 \geq \gamma n$ (by Proposition 6.7), it holds that $p(\hat{x}) = p(a_1 - s) > 0$ and $h(\vec{a}) > 0$; also by Proposition 6.6, we have $\Delta g(\vec{a}) \leq 0$. Therefore one can easily derive from (18) that $H(\hat{x},\hat{\alpha}) \leq 0$. □

We also need the following properties on $H(x,\alpha)$, whose technical proofs can be found in Appendix B.

Proposition 6.10. *(i) For any fixed $C > \varepsilon > 0$, there exists x_0 such that the following holds. If $x \geq x_0$ and $C \geq \alpha \geq \varepsilon$, then $H(x, \alpha) > 0$.*
(ii) If $t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, then there exist ε_0 and x_1 such that the following holds. If $x \geq x_1$ and $\varepsilon_0 \geq \alpha \geq \frac{2}{x}$, then $H(x, \alpha) > 0$.

Now we can finish the proof of Lemma 5.3.

Proof of Lemma 5.3. By Proposition 6.7, there exists some $\gamma > 0$ such that $a_1 \geq \gamma n$. Let n be sufficiently large, $\hat{x} = a_1 - s$, and $\hat{\alpha} = \frac{a_r - a_1}{a_1 - s}$.

First we prove that $a_r - a_1 = o(n)$, which would imply that $a_i = n/r + o(n)$. Suppose to the contrary that $a_r - a_1 \geq \varepsilon n$ for some $\varepsilon > 0$. As n is sufficiently large, it follows that $2/\gamma \geq \hat{\alpha} \geq \varepsilon$. Let x_0 be obtained from Proposition 6.10 (i) by applying with $C = 2/\gamma$ and ε. Since $\hat{x} = a_1 - s \geq \gamma n - s \geq x_0$, by Proposition 6.10 (i) we get $H(\hat{x}, \hat{\alpha}) > 0$, which contradicts Proposition 6.9.

Next we assume $t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$ and aim to show that $G = T_r(n)$, or equivalently $a_r - a_1 \leq 1$. Assume that $a_r - a_1 \geq 2$. Let ε_0 and x_1 be obtained from Proposition 6.10 (ii). As we just prove $a_r - a_1 = o(n)$, for sufficiently large n we have $a_r - a_1 \leq \frac{\gamma \varepsilon_0}{2} n$. This implies that $\varepsilon_0 \geq \hat{\alpha} \geq \frac{2}{\hat{x}}$. Also we have $\hat{x} \geq \gamma n - s \geq x_1$, so by Proposition 6.10 (ii), we obtain $H(\hat{x}, \hat{\alpha}) > 0$, again a contradiction to Proposition 6.9. Now the proof of Lemma 5.3 is completed. □

7 Concluding remarks

In this paper we consider the generalized Turán numbers $\mathrm{ex}(n, T, H)$ for graphs T, H with $\chi(T) < \chi(H)$. In the case that T is a clique, Theorem 1.2 gives a sharp estimate. A natural question will be to consider for non-clique T. Theorem 5.1 provides some answers for complete multipartite graphs T. However, even for this case there lacks of evidences to speculate extremal graphs in general. A special problem which we encounter with is that if, for $(T, H) = (K_{s,t}, K_3)$ and $t \geq s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, the extremal graphs are always bipartite. If this is the case then one may expect to solve the problem similar as in Lemma 5.3. It also seems plausible to ask the extremal graphs for $\mathrm{ex}(n, T, K_r)$ for edge-critical graphs T (in particular, for $\mathrm{ex}(n, C_{2k+1}, K_r)$ where $r \geq 4$). Our attempt to generalize Theorem 5.1 is limited by our capability of computation, therefore it will be interesting to see if there exists some novel approach which can work for general problems.

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A Proofs of Propositions 6.4 and 6.5

We begin by defining some functions: (let $q = t - s$ and $1/C = (s!)^{r-1} t!(r-1)^{(r-2)s+t-1}$)

$$\Delta(a) = (F(a+1) - F(a))/\lambda_{s,t}\ , \quad M(a) = Ca^{(r-2)s+t}(n-a)^{s-2}\ , \text{ and}$$

$$H(z) = s\lambda_{s,t}(s\tilde{\lambda}_{s,t,r-1} + t)z - (s^2 - s)\tilde{\lambda}_{s,t,r}/(r-1) + st(r-1)^q z^{q+1} - (t^2 - t)(r-1)^{q-1} z^q.$$

First we will need to prove the following two claims.

Claim A.1. *For $\gamma n \le a \le (1-\varepsilon)\, n$, it holds for sufficiently large n that $\Delta(a) = M(a) \cdot \left[H\left(\frac{n-a}{a}\right) + o(1)\right]$, where $o(1)$ tends to 0 as n goes to infinity.*

Proof. We need to compute $\Delta(a)$. Write $N_1(a) = \mathcal{N}(T_{r-1}(a), K_{s,t}^{(r-1)})$ and $N_2(a) = \mathcal{N}(T_{r-1}(a), K_{s,s}^{(r-1)})$. By the definition of the function F, we have

$$\frac{F(a)}{\lambda_{s,t}} = \binom{n-a-1}{s-1} N_1(a) + \binom{n-a-1}{t-1} N_2(a), \tag{19}$$

By (12), $N_1(a+1) = N_1(a) + \delta_1$ and $N_2(a+1) = N_2(a) + \delta_2$, where $\delta_1 = F_{r-1,s,t}(\lfloor \frac{r-2}{r-1}(a+1)\rfloor, a+1)$ and $\delta_2 = F_{r-1,s,s}(\lfloor \frac{r-2}{r-1}(a+1)\rfloor, a+1)$. So we can obtain that

$$\frac{F(a+1)}{\lambda_{s,t}} = \binom{n-a-2}{s-1}(N_1(a) + \delta_1) + \binom{n-a-2}{t-1}(N_2(a) + \delta_2). \tag{20}$$

By (19) and (20), it follows that

$$\Delta(a) = \binom{n-a-2}{s-1}\delta_1 - \binom{n-a-2}{s-2} N_1(a) + \binom{n-a-2}{t-1}\delta_2 - \binom{n-a-2}{t-2} N_2(a).$$

Applying Propositions 6.1 and 6.2 to $N_1(a), N_2(a), \delta_1$ and δ_2, one can derive that

$$\begin{aligned}
\Delta(a) &= (n-a)^{s-1}\left[\tfrac{\lambda_{s,t}(s\tilde{\lambda}_{s,t,r-1}+t)}{(s-1)!(s!)^{r-2}t!} + o(1)\right]\left(\tfrac{a}{r-1}\right)^{(r-2)s+t-1} \\
&\quad -(n-a)^{s-2}\left[\tfrac{\tilde{\lambda}_{s,t,r}}{(s-2)!(s!)^{r-2}t!} + o(1)\right]\left(\tfrac{a}{r-1}\right)^{(r-2)s+t} \\
&\quad +(n-a)^{t-1}\left[\tfrac{s}{(t-1)!(s!)^{r-1}} + o(1)\right]\left(\tfrac{a}{r-1}\right)^{(r-1)s-1} \\
&\quad -(n-a)^{t-2}\left[\tfrac{1}{(t-2)!(s!)^{r-1}} + o(1)\right]\left(\tfrac{a}{r-1}\right)^{(r-1)s}.
\end{aligned}$$

Let $z=\frac{n-a}{a}$. After some simplifications, one can obtain that

$$\begin{aligned}\Delta(a) &= Ca^{(r-1)s+t-2}\cdot\left[s\lambda_{s,t}(s\tilde{\lambda}_{s,t,r-1}+t)z^{s-1}-\tfrac{s(s-1)\tilde{\lambda}_{s,t,r}}{r-1}z^{s-2}+st(r-1)^q z^{t-1}\right.\\ &\left.\quad-(t^2-t)(r-1)^{q-1}z^{t-2}+o(1)\right]=M(a)\cdot[H(z)+o(1)].\end{aligned}$$

This proves Claim A.1. □

Claim A.2. *$H(z)$ is strictly increasing in $[\frac{1}{r-1},+\infty)$ and $H(\frac{1}{r-1})\geq 0$. Moreover, $H(\frac{1}{r-1})=0$ if and only if $r=2$ and $t=s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$.*

Proof. If $t=s$, then $H(z)=s\cdot\frac{1}{2}(s+s)z-\frac{s^2-s}{r-1}+s^2z-\frac{s^2-s}{r-1}=2s^2\left(z-\frac{1}{r-1}+\frac{1}{s(r-1)}\right)$. It is obvious that $H(\frac{1}{r-1})>0$ and $H(z)$ is strictly increasing.

Next we consider $s<t\leq s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$. Then $q\geq 1$ and $2s+q-q^2\geq 0$, where $2s+q-q^2=0$ if and only if $t=s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$. In this case we have

$$H(z)=(s^2(r-2)+st)z-(s^2-s)+st(r-1)^q z^{q+1}-(t^2-t)(r-1)^{q-1}z^q. \tag{21}$$

This implies that $H\left(\frac{1}{r-1}\right)=\frac{sr+q-q^2}{r-1}\geq 0$, where the equality holds if and only if $r=2$ and $t=s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$. It remains to show $H(z)$ is strictly increasing in $[\frac{1}{r-1},+\infty)$. To do so, it suffices to prove $H'(z)$ is strictly increasing in $[\frac{1}{r-1},+\infty)$ and $H'(\frac{1}{r-1})\geq 0$.

By (21), one can obtain

$$H'(z)=s^2(r-2)+st+st(r-1)^q(q+1)z^q-(t^2-t)(r-1)^{q-1}qz^{q-1}, \tag{22}$$
$$H''(z)=tq(r-1)^{q-1}z^{q-2}[s(r-1)(q+1)z-(t-1)(q-1)]. \tag{23}$$

So for $z>0$, $H''(z)\geq 0$ is equivalent to that $h(z):=s(r-1)(q+1)z-(t-1)(q-1)\geq 0$. Since $h(z)$ is strictly increasing and $h\left(\frac{1}{r-1}\right)=(2s+q-q^2)+(q-1)\geq 0$, we infer that $h(z)>0$ for $z>\frac{1}{r-1}$. This also yields that $H''(z)>0$ for $z>\frac{1}{r-1}$. Therefore $H'(z)$ is strictly increasing in $[\frac{1}{r-1},+\infty)$. Lastly, it follows from (22) that $H'\left(\frac{1}{r-1}\right)=(r-2)s^2+t(2s+q-q^2)\geq 0$. Now the proof of Claim A.2 is completed. □

We are ready to prove Propositions 6.4 and 6.5.

Proof of Proposition 6.4. We will only prove the case (i), and the case (ii) can be proved analogously. Suppose that $t<s+\frac{1}{2}+\sqrt{2s+\frac{1}{4}}$. Observe that in this case $H(\frac{1}{r-1})>0$. We need to show $\Delta(a)>0$ for all $a\in[\gamma n,\lfloor\frac{r-1}{r}n\rfloor]$. Let $z=\frac{n-a}{a}$. Then $z\in[\frac{1}{r-1},\frac{1-\gamma}{\gamma}]$. By Claims A.1 and A.2, it holds for sufficiently large n that $\frac{\Delta(a)}{M(a)}=H(z)+o(1)\geq H\left(\frac{1}{r-1}\right)+o(1)>0$. Since $M(a)>0$, this proves $\Delta(a)>0$. □

Proof of Proposition 6.5. Let $\beta=\frac{r-1}{r}-\varepsilon$ and $\tilde{C}=\lambda_{s,t}\cdot C$. By the definition of $\Delta(a)$ and Claim A.1, we see that $F\left(\lfloor\frac{r-1}{r}n\rfloor\right)-F\left(\lfloor\left(\frac{r-1}{r}-\varepsilon\right)n\rfloor\right)$ equals

$$\lambda_{s,t}\cdot\sum_{a=\lfloor\beta n\rfloor}^{\lfloor\frac{r-1}{r}n\rfloor-1}\Delta(a)=\lambda_{s,t}\cdot\sum_{a=\lfloor\beta n\rfloor}^{\lfloor\frac{r-1}{r}n\rfloor-1}M(a)\left[H\left(\frac{n-a}{a}\right)+o(1)\right], \tag{24}$$

where $\lambda_{s,t} \cdot M(a) = \tilde{C} a^{(r-2)s+t}(n-a)^{s-2}$. Then the equation (24) becomes

$$\tilde{C} \sum_{a=cn}^{\lfloor \frac{r-1}{r} n \rfloor - 1} a^{(r-2)s+t}(n-a)^{s-2} H\left(\frac{n-a}{a}\right) + o(n^{(r-1)s+t-1}).$$

We use Riemann integral to estimate the above summation as following

$$\begin{aligned}
&\frac{1}{n^{(r-1)s+t-1}} \cdot \sum_{a=\lfloor \beta n \rfloor}^{\lfloor \frac{r-1}{r} n \rfloor - 1} a^{(r-2)s+t}(n-a)^{s-2} H\left(\frac{n-a}{a}\right) \\
= \quad & \sum_{a=\lfloor \beta n \rfloor}^{\lfloor \frac{r-1}{r} n \rfloor - 1} \left[\left(\frac{a}{n}\right)^{(r-2)s+t} \left(\frac{n-a}{n}\right)^{s-2} H\left(\frac{n-a}{a}\right) \cdot \frac{1}{n}\right] \\
\xrightarrow{n\to\infty} \quad & \int_{\beta}^{\frac{r-1}{r}} x^{(r-2)s+t}(1-x)^{s-2} H\left(\frac{1-x}{x}\right) dx = \int_{\frac{1}{r-1}}^{\frac{1}{\beta}-1} \frac{z^{s-2} H(z)}{(1+z)^{(r-1)s+t}} dz.
\end{aligned}$$

Let I denote the above integral. By Claim A.2, $H(z) > 0$ for $z \in (\frac{1}{r-1}, \frac{1}{\beta} - 1)$. So $I > 0$. Putting everything together, one can obtain that $F\left(\lfloor \frac{r-1}{r} n \rfloor\right) - F\left(\lfloor \left(\frac{r-1}{r} - \varepsilon\right) n \rfloor\right) = (\tilde{C} I + o(1)) \cdot n^{(r-1)s+t-1}$. Let $\xi = \frac{\tilde{C} I}{2} > 0$. Then it holds for sufficiently large n that $F\left(\lfloor \frac{r-1}{r} n \rfloor\right) - F\left(\lfloor \left(\frac{r-1}{r} - \varepsilon\right) n \rfloor\right) > \xi n^{(r-1)s+t-1}$. This proves Proposition 6.5. ☐

B Proof of Proposition 6.10

First we prove two claims. Let $q = t - s$ and $f(z) = (sz - q)(1+z)^q + (t + (r-2)s)z + q$.

Claim B.1. *There exists a polynomial $P(\alpha)$ with $P(0) = 0$ such that the following holds. For any fixed $C > 0$, if $\alpha \in [0, C]$, then $H(x, \alpha) = f(\alpha) + (q^2 - q - rs + P(\alpha))/x + O(x^{-2})$, where the absolute value of the constant term in $O(x^{-2})$ is bounded by C, r, s and t.*

Proof. Recall that $H(x, \alpha) = \sum_{i=1}^{3} H_i(x, \alpha)$. So we need to estimate each H_i.

Let $C > 0$ be fixed and $\alpha \in [0, C]$. Write $(z)_q = z^q + A z^{q-1} + g(z)$, where $g(z)$ is a polynomial of degree at most $q-2$. Then we have $(x + \alpha x)_q = (1+\alpha)^q x^q + A(1+\alpha)^{q-1} x^{q-1} + O(x^{q-2})$. From the definition of H_1 it follows that

$$H_1(x, \alpha) = \left(s\alpha - q - \frac{qs + t}{x}\right)\left(1 + \frac{1-q}{x}\right)\left[(1+\alpha)^q + \frac{A(1+\alpha)^{q-1}}{x} + O(x^{-2})\right].$$

Expanding this multiplication, since $\alpha \in [0, C]$ is bounded, we obtain $H_1(x, \alpha) = (s\alpha - q)(1+\alpha)^q + \tilde{P}_1(\alpha)/x + O(x^{-2})$, where $\tilde{P}_1(\alpha) = -(qs+t)(1+\alpha)^q + (s\alpha - q)(1-q)(1+\alpha)^q + (s\alpha - q)A(1+\alpha)^{q-1}$. Define $P_1(\alpha) = \tilde{P}_1(\alpha) - \tilde{P}_1(0)$, which is a polynomial with $P_1(0) = 0$. Then we have

$$H_1(x, \alpha) = (s\alpha - q)(1+\alpha)^q + \frac{-(qs+t) - q(1-q) - qA + P_1(\alpha)}{x} + O(x^{-2}).$$

By similar arguments one can write H_2 and H_3 as

$$\begin{aligned}
H_2(x, \alpha) &= t\alpha + q + \frac{qs - s + q + qA + P_2(\alpha)}{x} + O(x^{-2}), \\
H_3(x, \alpha) &= (r-2)s\alpha + \frac{-(r-2)s + P_3(\alpha)}{x} + O(x^{-2}),
\end{aligned}$$

where P_i is a polynomial with $P_i(0) = 0$ for $i \in \{2, 3\}$. Summing up the above we obtain

$$H(x, \alpha) = f(\alpha) + \frac{q^2 - q - rs + P(\alpha)}{x} + O(x^{-2}),$$

where $P(\alpha) = \sum_{i=1}^{3} P_i(\alpha)$ is a polynomial with $P(0) = 0$. This proves Claim B.1. ☐

Claim B.2. *The function $f(z)$ is strictly increasing in $[0, +\infty)$ with $f(0) = 0$ and $f'(0) = sr + q - q^2$.*

Proof. It is easy to verify $f(0) = 0$ and obtain

$$f'(z) = s(q+1)(1+z)^q - tq(1+z)^{q-1} + t + (r-2)s,$$

$$f''(z) = q(1+z)^{q-2}[s(q+1)z + 2s + q - q^2].$$

So $f'(0) = sr + q - q^2$. Next we show $f(z)$ is strictly increasing. If $q = 0$, then $f(z) = sz + (t + (r-2)s)z = rsz$, which is obviously increasing. So we may assume $q \geq 1$. Since $s \leq t \leq s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, we have $2s + q - q^2 \geq 0$. This shows that $f'(0) = sr + q - q^2 \geq 0$ and $f''(z) > 0$ for $z > 0$, implying that $f'(z) > f'(0) \geq 0$ for $z > 0$ and thus $f(z)$ is strictly increasing in $[0, +\infty)$. This completes the proof. □

We are ready to prove Proposition 6.10

Proof of Proposition 6.10. First, we consider the case (i). Suppose that $C > \varepsilon > 0$ are fixed and $\alpha \in [\varepsilon, C]$. By Claim B.1 there exists a polynomial $P(\alpha)$ such that $H(x, \alpha) = f(\alpha) + (q^2 - q - rs + P(\alpha))/x + O(x^{-2})$. By Claim B.2 we have $f(\alpha) \geq f(\varepsilon) > 0$. Since $|P(\alpha)|$ is bounded (as $\alpha \in [\varepsilon, C]$), there exists a large $x_0 > 0$ such that for $x \geq x_0$

$$|H(x,\alpha) - f(\alpha)| = \left| \frac{q^2 - q - sr + P(\alpha)}{x} + O(1/x^2) \right| \leq \frac{1}{2} f(\varepsilon).$$

Now it follows that $H(x, \alpha) \geq f(\varepsilon) - \frac{1}{2} f(\varepsilon) > 0$.

Next we consider the case (ii). We have $s \leq t < s + \frac{1}{2} + \sqrt{2s + \frac{1}{4}}$, which shows that $2s + q - q^2 > 0$. By Claim B.2, $f(0) = 0$ and $f'(0) = rs + q - q^2 \geq 2s + q - q^2 > 0$. So there exists $\varepsilon_1 > 0$ such that for $\alpha \in (0, \varepsilon_1]$ it holds that $|\frac{f(\alpha)}{\alpha} - f'(0)| \leq f'(0)/5$. This implies that $f(\alpha) \geq 4f'(0)\alpha/5$ for $\alpha \in [0, \varepsilon_1]$. Also since $P(\alpha)$ is a polynomial with $P(0) = 0$, there exists $\varepsilon_2 > 0$ such that for $\alpha \in [0, \varepsilon_2]$, $|P(\alpha)| \leq f'(0)/4$. Applying Claim B.1 with C being $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, for $\alpha \in [0, \varepsilon_0]$ we have

$$H(x,\alpha) = f(\alpha) + \frac{P(\alpha) - f'(0)}{x} + O(x^{-2}) \geq \frac{4f'(0)\alpha}{5} - \frac{5f'(0)}{4x} - \frac{D}{x^2},$$

where $D > 0$ is bounded by ε_0, r, s, t. Let $x_1 = 4D/f'(0)$. Then for $x \geq x_1$ and $\alpha \in [\frac{2}{x}, \varepsilon_0]$,

$$H(x,\alpha) \geq \frac{4f'(0)\alpha}{5} - \frac{5f'(0)}{4x} - \frac{f'(0)}{4x} = \left(\frac{4}{5}\alpha - \frac{3}{2x} \right) f'(0) \geq \frac{f'(0)}{10x} > 0.$$

This completes the proof of Proposition 6.10. □