Graphs containing toplogical H

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Abstract

Let H denote the tree with six vertices two of which are adjacent and of degree three. Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. We characterize those G that contain a topological H in which u_1, u_2 are of degree three, and a_1, a_2, a_3, a_4 are of degree one. This work was motivated by the Kelmans–Seymour conjecture that 5-connected nonplanar graphs contain topological K_5.

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1 Introduction

The work in this paper was motivated by the well known conjecture of Seymour [14] and Kelmans [6]: Every 5-connected nonplanar graph contains a topological K_5 (i.e., subdivision of K_5). Clearly, this would provide structural information that guarantees the existence of a topological K_5. Earlier, Dirac [3] conjectured an extremal function for the existence of a topological K_5: If G is a simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges then G contains a topological K_5. This conjecture was established by Mader [12]. Kézdy and McGuiness [7] showed that the Kelmans-Seymour conjecture if true would imply Mader's result. This Kelmans-Seymour conjecture is also related to a conjecture of Hajós (see [2]) that every graph containing no topological K_{k+1} is k-colorable. Hajós' conjecture is false for $k \geq 6$ [2] and true for $k = 1, 2, 3$, and remains open for the case $k = 4$ and $k = 5$.

An approach to the Kelmans-Seymour conjecture is to study the so called rooted K_4 problem: Given a graph G and four distinct vertices of G, when does G contain a topological K_4 in which x_1, x_2, x_3, x_4 are the vertices of degree three. This problem was solved for planar graphs, see [16]. Recently, Aigner-Horev and Krakovski [1] used this to prove Kelmans-Seymour conjecture for apex graphs. (A different and shorter proof was found by Ma, Thomas and Yu [9].)

One step in [16] is to solve the following rooted H problem for planar graphs: Let H represent the tree on six vertices two of which are adjacent and of degree 3. Let G be a graph

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and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. When does G contain a topological H in which u_1, u_2 are of degree 3 and a_1, a_2, a_3, a_4 are of degree 1? We say such a topological H is rooted at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$. For convenience, we use *quadruple* to denote (G, u_1, u_2, A) where u_1, u_2 are distinct vertice of a graph G, $A \subseteq V(G) - u_1, u_2\}$, and $|A| = 4$.

The main result of this paper is a characterization of graphs quadruples (G, u_1, u_2, A) that contain a topological H rooted at u_1, u_2, A. Since the statement of this result requires a fair amount of terminology, we defer it to Section 2, see Theorem 2.1.

We devote the rest of this section to notation and terminology. A *separation* in a graph G consists of a pair of subgraphs G_1, G_2, denoted as (G_1, G_2), such that $E(G_1 \cap G_2) = \emptyset$, $E(G_1) \cup V(G_1) \not\subseteq G_1 \cap G_2$, and $E(G_2) \cup V(G_2) \not\subseteq G_1 \cap G_2$. The *order* of this separation is $|V(G_1 \cap G_2)|$, and (G_1, G_2) is said to be a *k-sepration* if its order is k. Let G be a graph. A set $S \subseteq V(G)$ is a *k-cut* or a *cut* of size k in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$ and $V(G_i - S) \neq \emptyset$ for $i \in \{1, 2\}$. If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G.

Let G be a graph. If there is no confusion, we may write $S \subseteq G$ instead of $S \subseteq V(G)$ or $S \subseteq E(G)$, and write $x \in G$ instead of $x \in V(G)$ or $x \in E(G)$. Let $H \subseteq G$, $S \subseteq V(G)$, and T a set of 2-element subsets of $V(H) \cup S$; then $H + (S \cup T)$ denotes the graph with vertex set $V(H) \cup S$ and edge set $E(G) \cup T$. If $T = \{\{x, y\}\}$, we write $G + xy$ instead of $G + \{\{x, y\}\}$.

Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). We may view paths as sequences of vertices; thus if P is a path between x and y, Q is a path between y and z, and $P \cap Q = \{y\}$, then PyQ denotes the path $P \cup Q$. The *ends* of the path P are the vertices of the minimum degree in P, and all other vertices of P are its *internal* vertices. A path P with ends u and v is also said to be *from u to v* or *between u and v*. A colloection of paths are said to be *independent* if no vertex of any path is an internal vertex of any other path.

2 Obstructions

For convenience, we say that a quadruple (G, u_1, u_2, A) is *feasible* if G contains a topological H rooted at u_2, u_2, A. An *obstruction* is a quadruple that is not feasible. We now describe basic obstructions.

A quadruple (G, u_1, u_2, A) is of *type I* if G is the edge-disjoint union of subgraphs U_1, U_2, A_1 such that $|V(U_1 \cap A_1)| = 3$, $|V(U_2 \cap A_1)| = 4$, $V(U_1 \cap U_2) \subseteq A \cap V(A_1)$, $|V(U_1 \cap U_2)| = 2$, $A \subseteq A_1$, and for some $\in \{1, 2\}$, $u_i \in U_1 - A_1$ and $u_{3-i} \in U_2 - A_1$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then $J \cap U_1$ consists of three independent paths from u_i to $V(U_1 \cap A_1)$. Therefore, $J \cap U_2$ must have three independent paths from u_{3-i} to $(U_2 \cap A_1) - U_1$, a contradiction. So quadruples of type I are obstructions.

A quadruple (G, u_1, u_2, A) is of *type II* if there exist edge disjoint subgraphs U_1, U_2, A_1, A_2, A_3 such that $G = U_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3$, $|V(U_2 \cap A_3)| = |V(U_i \cap A_j)| = 1$ for $i \in \{1, 2\}$ and $j \in \{1, 2\}$, $|V(U_1 \cap A_3)| = 2$, $A_i \cap A_j \subseteq U_1 \cup U_2$, $U_1 \cap U_2 \subseteq A_1 \cup A_2 \cup A_3$, $|V(A_i) \cap A| = 1$ for $i = 1, 2$, $|V(A_3) \cap A| = 2$, if $a_i \in U_j$ then $a_i \in U_2$ then $a_i \in A_1 \cap A_2 \cap A_3$, if $a_i \in U_1 \cap (A_1 \cup A_2)$ then $a_i \in A_3$,—V(A$_i$)| = 1 for some $i \in \{1, 2\}$ then $A_i \subseteq A_j$ for all $j \neq i$, and for some $i \in \{1, 2\}$, $u_i \in U_1 - (A_1 \cup A_2 \cup A_3)$ and $u_{3-i} \in U_2 - (A_1 \cup A_2 \cup A_3)$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then $J \cap U_2$ consists of three independent paths from u_{3-i} to $V(A_1 \cup A_2) \cap A) \cup V(U_2 \cap A_3)$. Therefore, $J \cap U_1$ must have three independent paths from u_i to $V(U_1 \cap A_3)$, a contradiction. So quadruples of type II are obstructions.

A quadruple (G, u_1, u_2, A) is of *type III* if there exist edge disjoint subgraphs U_1, U_2, A_1, A_2 of G such that $G = U_1 \cup U_2 \cup A_1 \cup A_2$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_1)| = 1$, $|V(U_1 \cap A_2)| = |V(U_2 \cap A_2)| = 2$, $V(U_1 \cap U_2) \subseteq A_1 \cup A_2 \cup A_3$, $|V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, if $a_i \in U_j$ then $a_i \in A_\cap A_2$, and $u_i \in U_i - (A_1 \cup A_2)$ for $i = 1, 2$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then $J \cap (U_1 \cup A_1)$ has three independent paths from u_1 to the three vertices in $(V(A_1) \cap A) \cup V(U_1 \cap A_2)$. So $J \cap U_2$ has three independent paths from u_2 to $V(U_2 \cap A_2)$, a contradiction. So quadruples of type III are obstructions.

A quadruple (G, u_1, u_2, A) is of *type IV* if there exist edge-disjoint subgraphs $U_1, U_2, A_1, A_2, A_3, A_4$ such that $G/xy = U_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3 \cup A_4$, $|V(U_i \cap A_j)| = 1$ for $1 \leq i \leq 4$ and $j = 1, 2$, $V(U_1 \cap U_2) \subseteq A_1 \cup A_2 \cup A_3 \cup A_4$, $|V(A_i) \cap A| = 1$ for $1 \leq i \leq 4$, if $a_i \in U_j$ then $a_i \in A_1 \cap A_2 \cap A_3 \cap A_4 \cap U_{3-j}$, and $u_i \in U_i - (A_1 \cup A_2 \cup A_3 \cup A_4)$ for $i = 1, 2$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then the path in J between u_1 and u_2 must go through A_i for some $1 \leq i \leq 4$. But then J cannot use $V(A_i) \cap A$, a contradiction. So quadruples of type IV are obstructions.

A quadruple (G, u_1, u_2, A) is of *type V* if there exist edge disjoint subgraphs U_1, U_2, A_1, A_2 of G such that $G = U_1 \cup U_2 \cup A_1 \cup A_2$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = 1$, $|V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$, $V(U_1 \cap U_2) \subseteq A_1 \cup A_2$, $|V(A_1) \cap A| = 2 = |V(A_2) \cap A|$, and $u_i \in U_i - (A_1 \cup A_2)$ for $i = 1, 2$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then $J \cap U_i$ has three independent paths from u_i to the vertices in $V(U_i) \cap V(A_1 \cup A_2)$, respectively. So the path in J between u_1 and u_2 must go through A_1 or A_2, say A_1 by symmetry. Then J can only use one of $V(A_1) \cap A$, a contradiction. So quadruples of type V are obstructions.

A quadruple (G, u_1, u_2, A) is of *type VI* if there exist edge-disjoint subgraphs U_1, U_2, A_1 of G such that $GU_1 \cup U_2 \cup A_1$, $|V(U_i \cap A_1)| = 3$ for $i = 1, 2$, $V(U_1 \cap U_2) \subseteq A \cap V(A_1)$, $|V(U_1 \cap U_2)| = 1$, and $u_i \in U_i - A_1$ for $i = 1, 2$. Clearly, if G has a topological H rooted at u_1, u_2, A, say J, then $J \cap U_i$ consists of three independent paths from u_i to $V(U_i \cap A_1)$. Therefore, J contains a path from u_1 to u_2 and containing a vertex from A, a contradiction. So quadruples of type VI are obstructions.

We can now state our main result which chracterizes all feasible quadruples.

Theorem 2.1. *Let (G, u_1, u_2, A) be a quadruple and let $A := \{a_1, a_2, a_3, a_4\}$. Then one of the following holds.*

(i) *(G, u_1, u_2, A) is feasible.*

(ii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in G_1 - G_2$ and $A \cup \{u_{3-i}\} \subseteq G_2$.*

(iii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $u_1, u_2 \in G_1 - G_2$, and $A \subseteq G_2$.*

(iv) *(G, u_1, u_2, A) is an obstruction of type I-VI.*

The idea of our proof of Theorem 2.1 is to find an edge xy in $G - (A \cup \{u_1, u_2\})$ and consider the graph G/xy obtained from G by contracting xy. Clearly, if $(G/xy, u_1, u_2, A)$ is feasible then (G, u_1, u_2, A) is feasible. We will show that if $(G/xy, u_1, u_2, A)$ is an obstruction of one these six types, then (i), or (ii), or (iii), or (iv) holds. This is done in Section 4.

3 Disjoint paths

In this section we prove useful lemmas about disjoint paths. First, we state the following result of Perfect [13]; we will need the $k = 3$ case.

Lemma 3.1. (*Perfect*) *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G-u)$. Suppose there exist k independent paths from u to distinct $a_1, \ldots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \ldots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \ldots, P_n$ may be chosen so that $a_i \in P_i$ for $i = 1, \ldots, k$.*

We need structural information about graphs containing no cycle through three given edges. Lovász [8] proved the following.

Lemma 3.2. *(Lovász) Let G be a 3-connected graph and e_1, e_2, e_3 be distinct edges of G. Then G contains a cycle through e_1, e_2, e_3 iff $G - \{e_1, e_2, e_3\}$ is connected.*

We also need the following easy generalization of Lemma 3.2.

Lemma 3.3. *Let G be a connected graph and let $e_1, e_2, e_3 \in E(G)$ be distinct. Then one of the following holds.*

(i) $\{e_1, e_2, e_3\}$ is contained in a cycle in G.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 1$ and $E(G_i) \cap \{e_1, e_2, e_3\} \neq \emptyset$ for $i = 1, 2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 2$ and for some $i \in \{1, 2\}$, $|E(G_i) \cap \{e_1, e_2, e_3\}| \leq 1$ and $|V(G_i)| \geq 3$.

(iv) $G - \{e_1, e_2, e_3\}$ is not connected.

Proof. Suppose the assertion is false, and choose a counterexample G, e_1, e_2, e_3 such that $|V(G)|$ is minimum. Then G is not 3-connected, as otherwise (i) or (iv) holds by Lemma 3.2. So let (G_1, G_2) be a k-separation of G such that $k \in \{1, 2\}$, and $G_i - G_{3-i} \neq \emptyset$ for $i = 1, 2$.

If $k = 2$ then (iii) holds, a contradiction. So $k = 1$, and we may assume by symmetry that $\{e_1, e_2, e_3\} \subseteq G_1$ (or else (ii) would hold). By the minimality of G, we see that one of (i)–(iv) holds for G_1, e_1, e_2, e_3. Because $k = 1$, it is easy to check that one of (i)–(iv) holds for G, e_1, e_2, e_3, a contradiction.

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The problem for finding a cycle through three given edges is equivalent to the problem for finding two disjoint paths between two pairs of vertices and through a given edge. In general one could ask the problem for finding k disjoint paths between two k-sets (of vertices) and through a specified edge. We solve the $k = 3$ case here, which will be used many times in our proof of Theorem 2.1.

First, we introduce the concept of a bridge. For a subgraph H of a graph G, an H-*bridge* of G is a subgraph of G, say B, for which there exists a component D of $G - V(H)$ such that B is induced by the edges which are either contained in D or from D to H.

Lemma 3.4. *Let G be a graph, $A = \{a_1, a_2, a_3\} \subseteq V(G)$, $B = \{b_1, b_2, b_3\} \subseteq V(G)$, and $e \in E(G)$ such that $A \cap B = \emptyset$ and $V(e) \cap (A \cup B) = \emptyset$. Then one of the following statements holds.*

(i) G has three disjoint paths from A to B and through e.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $A \subseteq G_1$, and $B \subseteq G_2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 1$, $e \in G_1$, and $A \cup B \subseteq G_2$.

(iv) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 3$, $A \subseteq G_1$, and $B \subseteq G_2$.

(v) $G = G_1 \cup G_2 \cup G_3$ such that $G_1 \cap G_3 = \emptyset$, $e \in G_2$, $|V(G_1 \cap G_2)| \leq 1$, $|V(G_2 \cap G_3)| \leq 1$, $|V(G_1) \cap A| = 1 = |V(G_1) \cap B|$, and $|V(G_3) \cap A| = |V(G_3) \cap B| = 2$.

(vi) $G = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5$ such that $|V(G_i \cap G_j)| = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, $V(G_1 \cap G_2) \subseteq G_3 \cup G_4 \cup G_5$, $G_i \cap G_j \subseteq G_1 \cup G_2$ for $3 \leq i \neq j \leq 5$, $e \in G_1$, and either $A \subseteq G_2$ and $|V(G_j) \cap B| = 1$ for $j \in \{3, 4, 5\}$ or $B \subseteq G_2$ and $|V(G_j) \cap A| = 1$ for $j \in \{3, 4, 5\}$.

Proof. We may assume that G has three disjoint paths from A to B, or else (ii) follows from Menger's theorem. So let P_1, P_2, P_3 denote three disjoint paths in G from A to B, and let $P := \bigcup_{i=1}^{3} P_i$. If $e \in P$ then (i) holds. So we may assume that $e \notin P$ for any choice of P. Let H_P denote the P-bridge of G containing e. We choose P so that

(1) H_P is maximal.

Without loss of generality we may assume that P_i is from a_i to b_i for $i = 1, 2, 3$. Let $x_i, y_i \in V(P_i \cap H_P)$ (if not empty) such that $x_i P_i y_i$ is maximal. We may assume a_i, x_i, y_i, b_i occur on P_i in order. For conveneience, let $H' := H_P - P$, and let $H_i := G[H' \cup x_i P_i y_i]$ for $i = 1, 2, 3$.

(2) For any i with x_i, y_i defined, G has no P-bridge intersecting both $a_i P_i x_i - x_i$ and $x_i P_i b_i - x_i$, or both $a_i P_i y_i - y_i$ and $y_i P_i b_i - y_i$.

For, suppose G has a P-bridge J intersecting both $a_i P_i x_i - x_i$ and $x_i P_i b_i - x_i$. Then $J \neq H_P$, and J contains a path Q_i from $u_i \in V(a_i P_i x_i - x_i)$ to $v_i \in V(x_i P_i b_i - x_i)$ and internally disjoint from P. Let $P_i' := a_i P_i u_i Q_i v_i P_i b_i$, and $P' := (P - P_i) \cup P_i'$. Then the P'-bridge of G containing e contains $H_P + x_i$; so P' contradicts the choice of P.

(3) We may assume that for any i with x_i, y_i defined, H_i' has a separation (H_{i1}, H_{i2}) such that $|V(H_{i1} \cap H_{i2})| = 1$, $x_i, y_i \in H_{i1}$, and $e \in H_{i2}$; and we choose (H_{i1}, H_{i2}) so that H_{i2} is minimal, and let $w_i \in V(H_{i1} \cap H_{i2})$.

For, otherwise, it follows from Menger's theorem that H_i' contains path Q_i from x_i to y_i and through e. Let $P_i' := a_i P_i x_i Q_i y_i P_y b_i$. Then $P' := (P - P_i) \cup P_i'$ shows that (i) holds.

Note that if w_i, w_j are deinfed and $w_i = w_j$ then by the minimality of H_{i2}, H_{j2}, we have $H_{i2} = H_{j2}$.

(4) We may assume that w_1 and w_2 are defined and $w_1 \neq w_2$.

If x_i, y_i are defined for at most one i then, by (3), the separation $(H_{i2}, G - (H_{i2} - w_i))$ shows that (iii) holds. So we may assume that w_i, x_i, y_i are defined for $i = 1, 2$. If w_3, x_3, y_3 are not defined then we may assume $w_1 \neq w_2$ (or else the separation $(H_{12}, G - (H_{12} - w_1))$ shows that (iii) holds). So we may assume that w_3, x_3, y_3 are defined as well. Then by symmetry we may assume $w_1 \neq w_2$; for if $w_1 = w_2 = w_3$ then the separation $(H_{12}, G - (H_{12} - w_1))$ shows that (iii) holds.

By (4), $H_P - (P - \{w_1, w_2\})$ contains a path from w_1 to w_2, through e, and internally disjoint from P. So for $\{i, j\} = \{1, 2\}$, $H_P - P_3$ contains a path Q_{ij} from x_i to y_j, through e, and internally disjoint from P. Moreover, $H_P - P_3$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{w_1, w_2\}$, $e \in H_2$, and $H_{11} \cup H_{12} \subseteq H_2$.

(5) G has no P-bridge that is different from H_P and intersects both $a_1P_1y_1 - y_1$ and $x_2P_2b_2 - b_2$, or both $a_2P_2y_2 - y_2$ and $x_1P_1b_1 - b_1$.

For, suppose some P-bridge $J \neq H_P$ of G intersects both $a_1P_1y_1 - y_1$ and $x_2P_2b_2 - x_2$. Then J contains a path Q from $u \in V(a_1P_1y_1 - y_1)$ to $v \in V(x_2P_2b_2 - x_2)$ and internally disjoint from P. Now $a_1P_1uQvP_2b_2, a_2P_2x_2Q_{21}y_1P_1b_1, P_3$ show that (i) holds. Similarly, by using Q_{12}, (i) holds if some P-bridge of G (different from H_P) intersects both $a_2P_2y_2 - y_2$ and $x_1P_1b_1 - b_1$.

Case 1. w_3, x_3, y_3 are defined.

Then $G[H' + \{x_i, y_j\}]$ has a path Q_{ij} from x_i to y_j for any $1 \leq i \neq j \leq 3$. By (3), G has a separation (K, L) such that $V(K \cap L) = \{w_1, w_2, w_3\}$ and $L = H_{12} \cap H_{22} \cap H_{32}$.

Suppose $w_3 \notin \{w_1, w_2\}$. Then (5) holds for any $i \neq j$. Therefore, if $\{x_1, x_2, x_3\} \neq \{a_1, a_2, a_3\}$ or some P-bridge of G contains two of $\{x_1, x_2, x_3\}$, then G has separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, x_3\}$, $A \subseteq G_1$, and $B \subseteq G_2$; so (iv) holds. Thus we may asume that $\{x_1, x_2, x_3\} = \{a_1, a_2, a_3\}$ and no P-bridge of G contains two of $\{x_1, x_2, x_3\}$. Similarly, we may assume that $\{y_1, y_2, y_3\} = \{b_1, b_2, b_3\}$, and no P-bridge of G contains two of $\{y_1, y_2, y_3\}$. Now, let $G_1 = H_2$, $G_2 = B$, and $G_3 = G - (G_1 - \{w_1, w_2, w_3\}$. The we see that (vi) holds.

Thus, we may assume that by symmetry that $w_3 = w_2$. By the same argument as for (5), we may assume that no P-bridge of G intersects both $a_1P_1y_1 - y_1$ and $x_3P_3b_3 - x_3$ or both $a_3P_3y_3 - y_3$ and $x_1P_1b_1 - x_1$.

If no P-bridge of G intersecting P_1 intersects $P_2 \cup P_3$, then (v) holds with G_1 has the union of $P_1 \cup H_{11}$ and all P-bridges of G (different from H_P) intersecting P_1, $G_2 = H_2$, and $G_3 := G - G_1 - (G_2 - \{w_1, w_2\})$. Thus by symmetry we may assume that G has a path Q from $u_2 \in V(a_2P_2x_2)$ to $u_1 \in V(a_1P_1x_1 - y_1) \cup V(a_3P_3x_3 - y_3)$, and we choose Q to minimize $u_2P_2x_2$. Let $u_3 \in a_3P_3x_3$ with $u_3P_3x_3$ minimal such that $u_3 = a_3$, or some P-bridge of G containing u_3 intersects $(a_1P_1x_1 - y_1) \cup (a_2P_2x_2 - y_2)$.

If G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, u_2, u_3\}$, $Q \cup A \subseteq G_1$ and $B \subseteq G_2$, then (iv) holds. So we may assume that such a separation does not exist in G. Then there exists a path R in G from $r \in V(a_2P_2u_2 - u_2) \cup V(a_3P_3u_3 - u_3)$ to $t \in V(x_1P_1b_1 - x_1)$ and internally disjoint from $P \cup Q$. By symmetry, we may assume $r \in a_2P_2u_2 - u_2$.

When $u_1 \in a_3P_3x_3 - y_3$, the paths $a_1P_1x_1Q_{13}y_3P_3b_3, a_2P_2rRtP_1b_1, a_3P_3u_1Qu_2P_2b_2$ show that (i) holds. So we may assume $u_1 \in a_1P_1x_1 - y_1$. Then $a_1P_1u_1Qu_2P_2b_2, a_2P_2rRtP_1b_1, P_3$ contradict the choice of P (that H_P is maximal).

Case 2. w_3, x_3, y_3 are not defined.

Let $u \in V(P_3)$ with a_3P_3u maximal such that $u = a_3$ or u belongs to some P-bridge of G intersecting $(a_1P_1x_1 - x_1) \cup (a_2P_2x_2 - x_2)$. Similarly, let $v \in V(P_3)$ with b_3P_3v maximal such that $v = b_3$ or v belongs to some P-bridge of G intersecting $(y_1P_1b_1 - y_1) \cup (y_2P_2b_2 - y_2)$.

we may assume $\{x_1, x_2, u\} = \{a_1, a_2, a_3\}$ and $\{y_1, y_2, v\} = \{b_1, b_2, b_3\}$. For, otherwise, we may suppose $\{x_1, x_2, u\} \neq \{a_1, a_2, a_3\}$. If G has no path from $a_3P_3u - u$ to $(x_1P_1b_1 - x_1) \cup (x_2P_2b_2 - x_2)$ and internally disjoint from P then, by (5), G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, x_3\}$, $A \subseteq G_1$, and $B \subseteq G_2$, and (iv) holds. So we may assume that G has a path Q from $x \in V(a_3P_3u - u)$ to $y \in V(x_1P_1b_1 - x_1) \cup V(x_2P_2b_2 - x_2)$ and internally disjoint from P. Let R be a path in G from u to $z \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2x_2 - x_2)$ and internally disjoint from P, and by symmetry we may assume that $z \in a_2P_2x_2 - x_2$. If $y \in x_2P_2b_2 - x_2$ then $P_1, a_2P_2zRuP_3b_3, a_3P_3xQyP_2b_2$ are three disjoint paths that contradict the choice of P (with H_P maximal). So $y \in x_1P_1b_1 - x_1$. Then $a_1P_1x_1Q_{12}y_2P_2b_2, a_1P_2zRuP_3b_3, a_3P_3xQyP_1b_1$ show that (i) holds.

We may assume that some P-bridge of G intersects both P_2 and P_3 and some P-bridge of G intersects both P_1 and P_3. For, otherwise, we may assume by symmetry that no P-bridge of G intersecting P_3 also intersects P_1. Let G_1 denote the union of $P_2 \cup P_3$, H_{21}, and all P-bridges of G different from H_P and intersecting 4 $P_2 \cup P_3$. Let $G_2 = H_2$, and let G_3 be the union of P_1, H_{11}, and all P-bridges of G different from H_P and intersecting P_1. Then by (5) we see that G_1, G_2, G_3 satisfies (v).

Suppose G has a P-bridge J such that $J \cap P_i \neq \emptyset$ for $i = 1, 2, 3$. Then $J \neq H_P$ as w_3, x_3, y_3 are not defined. So by (5) and by symmetry, we may assume that $V(J \cap P_1) = \{a_1\}$ and $V(J \cap P_2) = \{a_2\}$. Let $u \in V(J \cap P_3)$ with a_3P_3u maximal. We may assume that G has a path Q from $x \in V(a_3P_3u - u)$ to $y \in V(P_1 - a_1) \cup V(P_2 - a_2)$; for otherwise G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_1, a_2, u\}$, $A \subseteq G_1$, and $B \subseteq G_2$, which implies (iv). Let Q_i denote paths in J from u to a_i, $i = 1, 2$, that are internally disjoint from P. If $y \in P_2$ then $P_1, Q_2uP_3b_3, a_3P_3xQyP_2b_2$ show that (i) holds; and if $y \in P_1$ then $Q_1uP_3b_3, Q_2, a_3P_3xQyP_1b_1$ show that (i) holds.

So we may assume that no P-bridge of G intersects P_i for all $i = 1, 2, 3$. If all P-bridges of G intersect P_3 in exactly one common vertex, say z, then we may assume $z \neq a_3$ (as $a_3 \neq b_3$); now G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_1, a_2, z\}$, $A \subseteq G_1$, and $B \subseteq G_2$, which implies (iv). So we may assume that G has P-bridges J_1 and J_2 such that $J_1 \cap P_1 \neq \emptyset$, $J_2 \cap P_2 \neq \emptyset$, and there exists $u_1 \in J_1 \cap P_3$ and $u_2 \in J_2 \cap P_3$ with $u_1 \neq u_2$. By symmetry let a_3, u_1, u_2, b_3 occur on P_3 in order. Note that $J_1 \neq J_2$.

Let $v_1 \in V(J_1 \cap P_1)$ with $a_1P_1v_1$ maximal, and let $v_2 \in V(J_2 \cap P_2)$ with $v_2P_2b_2$ maximal. For $i = 1, 2$, let Q_i be a path in J_i from u_i to v_i and internally disjoint from P. If $v_1 \neq a_1$ and $v_2 \neq b_2$, then $Q_{12}, a_2P_2v_2Q_2u_2P_3b_3, a_3P_3u_1Q_1v_1P_1b_1$ show that (i) holds. So we may assume by symmetry that $v_2 = b_2$. We may modify P_3 if necessary to make J_2 maximal. Then no P-bridge of G other than J_2 intersects both $a_3P_3u_2 - u_2$ and $u_2P_3b_3 - u_2$.

If there is no P-bridge of G different from J_2 intersecting $u_2P_3b_3 - u_2$, then G has a separation (G_1, G_2) with $V(G_1 \cap G_2) = \{b_1, b_2, u_2\}$, $A \subseteq V(G_1)$, and $B \subseteq V(G_2)$; so (iv) holds.

Hence, we may assume that some P-bridge of G different from J_2 intersects $u_2P_3b_3 - u_2$; hence, there is a path R_2 in G from $s_2 \in V(u_2P_3b_3 - u_2)$ to $t_2 \in V(P_1 - b_1) \cup V(P_2 - b_2)$ and internally disjoint from P.

If $t_2 \in P_1 - b_1$ then $a_1P_1t_2R_2s_2P_3b_3, Q_{21}, a_3P_3u_2Q_2b_2$ show that (i) holds. So we may asume $t_2 \in P_2 - b_2$. Then $P_1, a_1P_2t_2R_2s_2P_3b_3, a_3P_3u_2Q_2b_2$ show that (i) holds.

∎

As an application of Lemma 3.4 we prove the following lemma which will be used many times to deal with $(G/xy, u_1, u_2, A)$.

Lemma 3.5. *Let (G, u_1, u_2, A) be a quadruple and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose G has a separation (U_1, U_2) such that $|V(U_1 \cap U_2)| \le 3$, $|V(U_1 \cap U_2) \cap A| \ne 0$, $u_1 \in U_1 - U_2$, $u_2 \in U_2 - U_1$, and $A \subseteq U_1$. Then one of the following holds.*

(i) (G, u_1, u_2, A) is feasible;

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \le 2$ and for some $i \in \{1, 2\}$, $u_i \in G_1 - G_2$ and $A \cup \{u_{3-i}\} \subseteq G_2$;

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \le 4$, $u_1, u_2 \in G_1 - G_2$, and $A \subseteq G_2$;

(iv) (G, u_1, u_2, A) is an obstruction of type I or IV.

Proof. We may assume $|V(U_1 \cap U_2)| = 3$; as otherwise (ii) holds. So let $V(U_1 \cap U_2) = \{v_1, v_2, v_3\}$. If $V(U_1 \cap U_2) \subseteq A$ then u_1 and u_2 belong to different components of $G - A$; so (iii) holds. Thus we may assume that $v_3 \notin A$. Since $V(U_1 \cap U_2) \cap A \ne \emptyset$, we may assume that $v_1 = a_1$.

We may assume that U_2 has three independent paths from u_2 to a_1, v_2, v_3, respectively. Otherwise U_2 has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \le 2$, $u_2 \in U_{21} - U_{22}$ and $\{a_1, v_2, v_3\} \subseteq U_{22}$. Now $(U_{21}, U_{22} \cup U_1)$ is a separation in G showing that (ii) holds.

Suppose $v_2 \in A$. Without loss of generality, we may assume $v_2 = a_2$. Then G has a topological H rooted at u_1, u_2, A iff $U_1 - \{a_1, a_2\}$ has three independent paths from u_1 to a_3, a_4, v_3, respectively. Thus (i) holds, or U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \le 4$, $a_1, a_2 \in U_{11} \cap U_{12}$, $u_1 \in U_{11} - U_{12}$ and $\{a_3, a_4, v_3\} \subseteq U_{12}$. Now U_{11}, U_2, U_{12} show that (G, u_1, u_2, A) is an obstruction of type I, and (iv) holds.

So we may assume that $v_2 \notin A$. Then G has a topological H rooted at u_1, u_2, A iff $(U_1 - a_1) + v_2v_3$ has three independent paths from u_1 to a_2, a_3, a_4 and containging the edge v_2v_3. Let U_1' be obtained from $(U_1 - a_1) + v_2v_3$ by duplicating u_1 twice, as u_1', u_1''. We wish to see if U_1' has three disjoint paths from $\{u_1, u_1', u_1''\}$ to $\{a_2, a_3, a_4\}$ and containing v_2v_3. So we apply Lemma 3.4.

If Lemma 3.4(i) holds then U_1' has three disjoint paths from $\{u_1, u_1', u_1''\}$ to $\{a_2, a_3, a_4\}$ and containing v_2v_3. So $(U_1 - a_1) + v_2v_3$ has three independent paths from u_1 to a_2, a_3, a_4 and containging the edge v_2v_3. Hence, G has a topological H rooted at u_1, u_2, A, and (i) holds.

Suppose Lemma 3.4(ii) holds. Then U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \le 2$, $\{u_1, u_1', u_1''\} \subseteq U_{11}$, and $\{a_2, a_3, a_4\} \subseteq U_{12}$. If $v_2v_3 \in U_{12}$ then the separation

$(G[U_{11} - \{u_1', u_1''\}], U_{12})$ shows that (ii) holds. If $v_2v_3 \in U_{11}$ then the separation $(G[U_{11} - \{u_1', u_1''\}], U_{12})$ shows that (iii) holds.

If Lemma 3.4(iii) holds then U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 1$, $\{u_1, u_1', u_1''\} \subseteq U_{11}$ and $v_2, v_3 \in U_{12}$. Now the separation $(G[U_{11} - \{u_1', u_1''\}], G[V(U_{12})] \cup U_2)$ shows that then (ii) holds.

Suppose Lemma 3.4(iv) holds. Then U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| = 3$, $\{u_1, u_1', u_1''\} \subseteq U_{11}$ and $\{a_2, a_3, a_4\} \subseteq U_{12}$. If $v_2v_3 \in U_{11}$, then $G[V(U_{11}) - \{u_1', u_1''\} + \{a_1\}], U_2, G[U_{12} + a_1]$ show that (G, u_1, u_2, A) is an obstruction of type I, and (iv) holds. If $v_2v_3 \in U_{12}$ then the separation $(G[V(U_{12} + a_1)], G[U_{11} + a_1] \cup U_2)$ shows that (iii) holds.

Since u_1' and u_1'' are duplicates of u_1, Lemma 3.4(v) cannot occur. So we may assume Lemma 3.4(vi) holds. Again, since u_1' and u_1'' are duplicates of u_1, U_1' is the edge disjoint union of graphs G_i, $1 \leq i \leq 5$, such that $|V(G_i \cap G_j)| = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, $G_1 \cap G_2 \subseteq G_3 \cup G_4 \cup G_5$, $G_i \cap G_j \subseteq G_1 \cup G_2$ for $3 \leq i \neq j \leq 5$, $v_2v_3 \subseteq G_1$, $\{u_1, u_1', u_1''\} \subseteq G_2$, and $|V(G_j) \cap \{a_2, a_3, a_4\}| = 1$ for $j \in \{3, 4, 5\}$. Then $G[G_2 - \{u_1', u_1''\} + a_1], U_2 \cup G[V(G_1 + a_1)], \{a_1\}, G_3, G_4, G_5$ show that (G, u_1, u_2, A) is an obstruction of type IV, so (iv) holds. ∎

As an easy corollary of Lemma 3.5, we can deal with obstructions of type VI.

Corollary 3.6. *Let (G, u_1, u_2, A) be a quadruple, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exist $xy \in E(G)$ such that $x, y \in V(G) - A - \{u_1, u_2\}$ and $(G/xy, u_1, u_2, A)$ is of type VI. Then one of the following holds.*

(i) *(G, u_1, u_2, A) is feasible.*

(ii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, and for some $i \in \{1, 2\}$, $u_i \in G_1 - G_2$, and $A \cup \{u_{3-i}\} \subseteq L$.*

(iii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $A \subseteq G_1$ and $\{u_1, u_2\} \subseteq G_2 - G_1$.*

(iv) *(G, u_1, u_2, A) is an obstruction of types I, IV,or VI.*

Proof. Let G/xy be the edge-disjoint union of subgraphs U_1, U_2, A_1 such that $|V(U_1 \cap A_1)| = 3$, $|V(U_2 \cap A_1)| = 3$, $V(U_1 \cap U_2) \subseteq A \cap V(A_1)$, $|V(U_1 \cap U_2)| = 1$, $A \subseteq A_1$, and $u_1 \in U_1 - A_1$ and $u_2 \in U_2 - A_1$. Let v denote the vertex of G/xy resulting from the contraction of xy.

If $v \notin V(U_i \cap A_1)$ for $i = 1, 2$ then we see that (G, u_1, u_2, A) is an obstruction of type VI. Otherwise, we may assume by symmetry that $v \in U_2 \cap A_1$. Now $(U_1, A_1 \cup U_2)$ is a separation which alllows us use Lemma 3.5. So the assertion of the lemma holds. ∎

4 Contraction critical quadruples

In this section we prove lemmas to be used to deal with contraction critical quadruples (G, u_1, u_2, A): those such that for any $xy \in E(G - (A \cup \{u_1, u_2\})$, $(G/xy, u_1, u_2, A)$ is an obstruction.

Lemma 4.1. *Let (G, u_1, u_2, A) be a quadruple, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exist $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type I. Then one of the following holds.*

(i) (G, u_1, u_2, A) *is feasible.*

(ii) G *has a separation* (G_1, G_2) *such that* $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, *and* $A \cup \{u_2\} \subseteq G_2$.

(iii) G *has a separation* (G_1, G_2) *such that* $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, *and* $A \subseteq G_2$.

(iv) (G, u_1, u_2, A) *is an obstruction of types I, II or IV.*

Proof. Let G/xy be the edge disjoint union of $U_1 \cup U_2 \cup A_1$ such that $V(U_1 \cap U_2) = \{a_1, a_2\}$, $V(U_1 \cap A_1) = \{a_1, a_2, v_1\}$, $V(U_2 \cap A_1) = \{a_1, a_2, v_2, v_3\}$, $V(U_1 \cap U_2) = \{a_1, a_2\}$, $u_1 \in U_1 - A_1$, and $u_2 \in U_2 - A_1$. Let v denote the vertex resulting from the contraction of x, y.

We may assume $v = v_1$. For, suppose $v \neq v_1$. Then $(U_1, G - (U_1 - \{a_1, a_2, v_1\}))$ is a separation in G which allows us to apply Lemma 3.5; so (i) or (ii) or (iii) or (iv) holds.

Let U_1', A_1' be obtained from U_1, A_1, respectively, by uncontracting v to xy. Note the symmetry between U_1' and U_2. We choose U_1', U_2, A_1 so that, subject to $a_1, a_2 \in U_1' \cap U_2$, $U_1' \cup U_2$ is maximal. Then $xy, v_2v_3 \notin A_1'$. Moreover we may assume $a_3a_4 \notin A_1'$; otherwise, $(G - a_3a_4, G[\{a_3, a_4\}])$ shows that (iii) holds.

We may assume that for some permutation ij of $\{1, 2\}$, $U_1' - a_j$ has three independent paths from u_1 to a_i, x, y, respectively, and $U_2 - a_i$ has three independent paths from u_2 to a_j, v_2, v_3, respectively. To see this, let H be obtained from $U_1' \cup U_2$ by duplicating each u_i twice with u_i', u_i''. If H contains six disjoint paths from $\{u_i, u_i', u_i'' : i = 1, 2\}$ to $\{a_1, a_2, v_2, v_3, x, y\}$ then the desired permutation and six paths exist. So we may assume by Menger's theorem that H has a spearation (H_1, H_2) such that $|V(H_1 \cap H_2)| \leq 5$, $\{u_i, u_i', u_i'' : i = 1, 2\} \subseteq V(H_1)$ and $\{a_1, a_2, v_2, v_3, x, y\} \subseteq V(H_2)$. It is easy to see that $|V(H_1 \cap H_2) \cap V(U_1')| \leq 2$, or $|V(H_1 \cap H_2) \cap V(U_2)| \leq 2$, or $|V(H_1 \cap H_2) \cap V(U_1')| = 3$ and $V(H_1 \cap H_2) \cap V(U_1') \cap \{a_1, a_2\} \neq \emptyset$, or $|V(H_1 \cap H_2) \cap V(U_2)| = 3$ and $V(H_1 \cap H_2) \cap V(U_2) \cap \{a_1, a_2\} \neq \emptyset$. If the first two cases occur, $V(H_1 \cap H_2) \cap V(U_1') \leq 2$ or $|V(H_1 \cap H_2) \cap V(U_2)| \leq 2$ then (ii) holds. If the next two cases occur, then by Lemma 3.5 the assertion of the lemma holds.

Let J denote the union of the six paths in $U_1' - a_j$ and $U_2 - a_i$. If $A_1^* := (A_1' - \{a_1, a_2\}) + \{a_3a_4, v_2v_3, xy\}$ contains a cycle C through $\{a_3a_4, v_2v_3, xy\}$ then $C - \{a_3a_4, v_2v_3, xy\}$ and J form a topological H rooted at u_1, u_2, A, and (i) holds. So we may assume that such a cycle C does not exist in A_1^*. Then by Lemma 3.3, we have three cases to consider.

In the first case, A_1^* has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$ and $|E(A_{11}) \cap \{a_3a_4, v_2v_3, xy\}| = 1$. If $xy \in A_{11}$, then $U_1' \cup G[V(A_{11}) + \{a_1, a_2\}], U_2$ and $G[V(A_{12}) + \{a_1, a_2\}]$ show that (G, u_1, u_2, A) is an obstruction of type I. If $v_2v_3 \in A_{11}$ then $U_1', U_2 \cup G[V(A_{11}) + \{a_1, a_2\}], G[V(A_{12}) + \{a_1, a_2\}]$ show that (G, u_1, u_2, A) is an obstruction of type I. If $a_3a_4 \in A_{11}$ then $(G[V(A_{11}) + \{a_1, a_2\}], U_1' \cup U_2 \cup G[V(A_{12}) + \{a_1, a_2\}])$ show that (iii) holds.

In the second case, A_1^* has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| = 2$ and $|E(A_{11}) \cap \{a_3a_4, v_2v_3, xy\}| = 1$. If $xy \in A_{11}$ or $v_2v_3 \in A_{11}$, then $U_1' \cup G[V(A_{11}) + \{a_1, a_2\}] \cup U_2$ contradicts the maximality of $U_1' \cup U_2$. So $a_3a_4 \in A_{11}$. Then $(U_1' \cup U_2 \cup G[V(A_{12}) + \{a_1, a_2\}], G[V(A_{11}) + \{a_1, a_2\}])$ shows that (iii) holds.

Therefore, we may assume that $A_1^* - \{a_3a_4, v_2v_3, xy\}$ is not connected. Since $a_3a_4, v_2v_3, xy \notin A_1'$, A_1' consists of disjoint subgraphs A_{11}, A_{12} such that each of a_3a_4, v_2v_3, xy has one end in A_{11} and the other in A_{12}. Now $U_1', U_2, A_{11}, A_{12}, \{a_1\}, \{a_2\}$ show that (G, u_1, u_2, A) is an obstruction oy type II. ∎

Lemma 4.2. *Let (G, u_1, u_2, A) be a quatruple with $A = \{a_1, a_2, a_3, a_4\}$. Suppose there exist $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type II. Then one of the following holds.*

(i) (G, u_1, u_2, A) is feasible.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, and $A \cup \{u_2\} \subseteq G_2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$.

(iv) (G, u_1, u_2, A) is an obstruction of types I, II, III, IV.

Proof. Let G/xy be the edge-disjoint union of U_1, U_2, A_1, A_2, A_3 such that $V(U_1 \cap A_i) = \{v_i\}$ for $i = 1, 2$ and $V(U_1 \cap A_3) = \{v_3, v_4\}$, $V(U_2 \cap A_i) = \{w_i\}$ for $1 \leq i \leq 3$, $V(U_1 \cap U_2) \subseteq \{v_1, v_2, v_3, v_4\} \cap \{w_1, w_2, w_3\}$, $V(A_i \cap A_j) \subseteq V(U_1 \cap U_2)$ for $1 \leq i \neq j \leq 3$, $u_i \in U_i - (A_1 \cup A_2 \cup A_3 \cup A_4)$ for $i = 1, 2$, $a_i \in A_i$ for $i = 1, 2$ and $a_3, a_4 \in A_3$, if $|V(A_i)| = 1$ then $A_i \subseteq A_j$ for all $j \neq i$, and if $w_3 \in A$ then $w_3 \in U_2 \cap A_i$ for $i = 1, 2, 3$.

Let v denote the vertex resulting from the contraction of xy. If $v \notin \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\}$, then (G, u_1, u_2, A) is also an obstruction of type II. So we may assume that $v \in \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\}$. By symmetry, it suffices to consider four cases: $v = v_1$, $v = v_4$, $v = w_1$, and $v = w_3$.

Case 1. $v = v_1$.

Then by Lemma 3.5 we may assume that $\{w_1, w_2, w_3, v_2\} \cap A = \emptyset$. Let U_1', A_1' be obtained from U_1, A_1, respectively, by uncontracting v to xy.

We may assume that U_2 has three independent paths from u_2 to w_1, w_2, w_3, respectively. Otherwise, U_2 has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$, and $\{w_1, w_2, w_3\} \subseteq U_{22}$. Now the separation $(U_{21}, U_{22} \cup U_1' \cup A_1' \cup A_2 \cup A_3)$ in G shows that (ii) holds.

We may also assume that A_1' has disjoint paths from $\{x, y\}$ to $\{a_1, w_1\}$. For, otherwise, A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq A_{11}$ and $\{a_1, w_1\} \subseteq A_{12}$. Now $U_1 \cup A_{11}, U_2, A_{12}, A_2, A_3$ show that (ii) holds, or (G, u_1, u_2, A) is also an obstruction of type II.

We may assume that for each $i \in \{3, 4\}$, A_3 has disjoint paths from $\{w_3, v_i\}$ to $\{a_3, a_4\}$, which avoids v_{7-i} if $v_{7-i} \notin A$. For, suppose no such disjoint paths exist. Then A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 1$ (if $v_{7-i} \in A$), $|V(A_{31} \cap A_{32})| \leq 2$ and $v_{7-i} \in A_{31} \cap A_{32}$ (when $v_{7-i} \notin A$), $\{w_3, v_i\} \subseteq A_{31}$, and $\{a_3, a_4\} \subseteq A_{32}$. Now the separation $(G[V(A_{32} + \{a_1, a_2\}], U_1 \cup U_2' \cup A_1' \cup A_2 \cup G[V(A_{31} + \{a_1, a_2\}])$ shows that (iii) holds.

We may assume that A_2 has a path from w_2 to a_2 which avoids v_2 when $v_2 \neq a_2$. Otherwise, A_2 has a separation (A_{21}, A_{22}) such that $A_{21} \cap A_{22} = \emptyset$ (when $v_2 = a_2$) or $A_{21} \cap A_{22} = \{v_2\}$ (when $v_2 \neq a_2$), $a_2 \in A_{21}$, and $w_2 \in A_{22}$. Now the separation $(U_2 \cup A_{22}, U_1' \cup A_1' \cup A_{22} \cup A_3)$ shows that (iii) holds.

We may assume that if $\{v_3, v_4\} \neq \{a_3, a_4\}$ then $v_4 \notin \{a_3, a_4\}$.

Now if $U_1' - (A - \{v_3\})$ contains disjoint paths from u_1 to x, y, v_3, respectively, then (i) holds. Thus we may assume that $U_1' - (A - \{v_3\})$ has a separation (U_{11}, U_{12}) such that

$|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{x, y, v_3\} \subseteq U_{12}$. Choose this separation to minimize U_{12}.

We may assume $|V(U_{11} \cap U_{12})| = 2$. For, otherwise, we may assume $v_2, v_4 \in N(U_{11} - U_{12})$ (or else (ii) holds). Recall that $v_2 \notin A$. By Lemma 3.5 we may a;so assume $v_4 \notin A$; so $v_2, v_4 \in U_{11} - U_{12}$. Then $G[U_{11} + v_4], U_2, A_2, G[U_{12} + v_4] \cup A_1' \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type III. So let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$.

By the minmality of U_{12}, $U_{12} - A$ contains disjoint paths from $\{s_1, s_2\}$ to $\{x, y\}$. For, otherwise, $U_{12} - A$ has a separation (K, L) such that $|V(K \cap L)| \leq 1$, $\{s_1, s_2\} \subseteq K$, and $\{x, y\} \subseteq L$. Then $(U_{11} \cup G[K + v_3], G[L + v_3])$ is a separation in $U_1 - (A - \{v_3\})$, contradicting the minimality of U_{12}.

Suppose $v_4 \notin N(U_{11} - U_{12})$. If $v_2 \notin U_{11} - U_{12}$, then (ii) holds. So we may assume that $v_2 \notin U_{11} - U_{12}$. Then $U_{11}, U_2, A_2, G[U_{12} + v_3] \cup A_1' \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type III. So we may assume $v_4 \in N(U_{11} - U_{12})$.

We may assume that $G[U_{11} + v_4]$ has three independent paths from u_1 to s_1, s_2, v_4, respectively. Otherwise, $G[U_{11} + v_4]$ has a separation (K, L) such that $|V(K \cap L)| \leq 2$, $u_1 \in K - L$ and $\{s_1, s_2, v_4\} \subseteq L$. If $v_2 \notin K - L$ or $|V(K \cap L)| \leq 1$ then (ii) holds. So assume $v_2 \in K - L$ and $|V(K \cap L)| = 2$. Then $K, U_2, L \cup G[U_{12} + v_4] \cup A_1' \cup A_2 \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type III.

We may assume $v_4 \notin A$. For, otherwise, we have $v_3, v_4 \in A$. If $v_2 \notin U_{11} - U_{12}$ then $(G[U_{11} + v_4], G[U_{12} + v_4] \cup U_2 \cup A_1' \cup A_2 \cup A_3)$ allows us to apply Lemma 3.5; so the assertion of the lemma holds. So we may assume $v_2 \in U_{11} - U_{12}$. Then U_1 has three independent paths from u_1 to x, y, v_4, respectively; and (i) holds.

Thus we may assume that $v_4 \notin A$, and hence $v_4 \in U_{11} - U_{12}$. So U_{11} has three independent paths from u_1 to s_1, s_2, v_4, respectively; thus $U_1 - A$ has three independent paths from u_1 to x, y, v_4, respectively.

If $A_3' := A_3 - (\{v_3\} - A)$ has disjoint paths from $\{v_4, w_3\}$ to $\{a_3, a_4\}$, then (i) holds. So we may assume that A_3' has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 1$, $\{v_4, w_3\} \subseteq A_{31}$ and $\{a_3, a_4\} \subseteq A_{32}$. Now $V(A_{32}) = \{v_3\} \subseteq \{a_3, a_4\}$; otherwise (iii) holds. If $v_2 \notin U_{11} - U_{12}$ then $U_{11}, U_2, G[U_{12} + v_3], A_1' \cup A_2$ show that (G, u_1, u_2, A) is an obstruction of type III. So assume $v_2 \in U_{11} - U_{12}$. Then $U_{11}, U_2, A_1' \cup G[U_{12} + v_3], A_2, A_3 - v_3$ show that (G, u_1, u_2, A) is an obstruction of type II.

Case 2. $v = v_4$.

Let U_1', A_3' be obtained from U_1, A_3, respectively, by uncontracting v to xy. By Lemma 3.5, we may assume $\{v_1, v_2, w_1, w_2, w_3\} \cap A = \emptyset$.

We may assume that A_3' has three disjoint paths from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For, if such paths do not exist, then A_3' has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $\{v_3, x, y\} \subseteq A_{31}$, and $\{a_3, a_4, w_3\} \subseteq A_{32}$. Now $U_1' \cup A_{31}, U_2, A_1, A_2, A_{32}$ show that (G, u_1, u_2, A) is an obstruction of type II.

We may assume that U_2 has three independent paths from u_2 to w_1, w_2, w_3, respectively; or else (ii) holds. Also we may assume that, for $i = 1, 2$, A_i has a path from w_i to a_i; otherwise (ii) holds.

Thus if U_1' has three independent paths from u_1 to v_3, x, y, respectively, then (i) holds. So we may assume that U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11}\cap U_{12})| \leq 2$, $u_1 \in U_{11}-U_{12}$, and $\{v_3, x, y\} \subseteq U_{12}$.

If $v_1, v_2 \notin U_{11}-U_{12}$ then (ii) holds. So we may assume that $v_1 \in U_{11}-U_{12}$. If $v_2 \notin U_{11}-U_{12}$ then $U_{11}, U_2, A_1, U_{12} \cup A_2 \cup A_3'$ show that (G, u_1, u_2, A) is an obstrcution of type III. So we assume that $v_2 \in U_{11} - U_{12}$. Then $U_{11}, U_2, A_1, A_2, A_3' \cup U_{12}$ show that (G, u_1, u_2, A) is an obstruction of type II.

Case 3. $v = w_3$.

Let U_2', A_3' be obtained from U_2, A_3, respectively, by uncontracting v to xy. Note the symmetry between U_1 and U_2'. We choose U_1, U_2', A_1, A_2, A_3 to maximize $U_1 \cup U_2'$.

We may assume that A_3' contains three disjoint paths: one from $\{x, y\}$ to $\{v_3, v_4\}$ and the other two from $\{a_3, a_4\}$ to $\{v_3, v_4, x, y\}$. For, suppose not. Then $A_3'' := A_3' + \{a_3a_4, v_3v_4, xy\}$ contains no cycle through $S := \{a_3a_4, v_3v_4, xy\}$. So we may apply Lemma 3.3. First, suppose A_3'' has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 1$ and and $|E(A_{32}) \cap S| = 1$. If $xy \in A_{32}$ or $v_3v_4 \in A_{32}$ then we see that (G, u_1, u_2, A) is an obstruction of type II; and if $a_3a_4 \in A_{32}$ then we see that (iii) holds. Now, suppose A_3'' has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| = 2$, $|E(A_{32}) \cap S| = 1$, and $|V(A_{32})| \geq 3$. Then by the maximality of $U_1 \cup U_2'$, we see that $a_3a_4 \in S$, which shows (iii) holds. We may thus assume that S is an edge cut of A_3''. In this case, (G, u_1, u_2, A) is an obstruction of type IV.

We may assume that for any $i \in \{1, 2\}$, $U_2' - (A - \{w_i\})$ contains three independent paths from u_2 to w_i, x, y, respectively. For, suppose not. Then $U_2' - (A - \{w_i\})$ has separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$, and $\{w_i, x, y\} \subseteq U_{22}$. Choose this separation to minimze U_{22}. We may assume $w_{3-i} \in N(U_{21} - U_{22})$ and $|V(U_{21} \cap U_{22})| = 2$; or else (ii) holds. Then by Lemma 3.5, we may assume $w_{3-i} \notin A$ (and hence, we may also assume that $v_{3-i} \notin A$). So $w_{3-i} \in U_{21} - U_{22}$. By the minimality of U_{22} there are disjoint paths in $U_{22} - A$ from $V(U_{21} \cap U_{22})$ to $\{x, y\}$. We may further assume that U_{21} has three independent paths from u_1 to $V(U_{21} \cap U_{22}) \cup \{w_{3-i}\}$; for otherwise U_{21} has a separation (K, L) such that $|V(K \cap L)| \leq 2$, $U_{11} \cap U_{12} \subseteq L$, and $u_2 \in K - L$, which gives the separation $(L, G - (L - K))$ in G showing that (ii) holds. Thus $U_2' - (A - \{w_{3-i}\})$ has three independet paths from u_2 to w_{3-i}, x, y, respectively. If U_1 contains three independent paths from u_1 to v_i, v_3, v_4, respectively, then (i) holds. So we may asume that U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_i, v_3, v_4\} \subseteq U_{12}$. we may assume $|V(U_{11} \cap U_{12})| = 2$ and $v_i \in U_{11} - U_{12}$; or else (ii) holds. Then $U_{11}, U_{21}, A_2, U_{12} \cup U_{22} \cup A_1 \cup A_3'$ show that (G, u_1, u_2, A) is an obstruction of type III.

Similarly, we may assume that for any $i \in \{1, 2\}$, $U_1 - (A - \{v_i\})$ contains three independent paths from u_1 to v_i, v_3, v_4, respectively. Now it is easy to see that (i) holds.

Case 4. $v = w_1$.

Let U_2', A_1' be obtained from U_2, A_1, respectively, by uncontracting v to xy.

We may assume that A_1' has disjoint paths from $\{v_1, a_1\}$ to $\{x, y\}$. For otherwise, A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{v_1, a_1\} \subseteq A_{11}$ and $\{x, y\} \subseteq A_{12}$. Now $U_1, U_2 \cup A_{12}, A_{11}, A_2, A_3$ show that (G, u_1, u_2, A) is an obstruction of type II.

Subcase 4.1. $U_1 - \{v_2\} \cap A$ has three independent paths from u_1 to v_1, v_3, v_4, respectively.

We may assume that $A_3' := A_3 - (\{w_3\} - A)$ has disjoint paths from $\{v_3, v_4\}$ to $\{a_3, a_4\}$.

For, otherwise, A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $w_3 \in A_{31} \cap A_{32}$ if $w_3 \notin A$, $\{v_3, v_4\} \subseteq A_{31}$, and $\{a_3, a_4\} \subseteq A_{32}$. Then the separation $(G[A_{32} + \{a_1, a_2\}], A_{31} \cup U_1 \cup U_2 \cup A'_1 \cup A_2)$ show that (iii) holds.

If $U_2 - (A - \{w_2\})$ has three independent paths from u_2 to w_2, x, y, respectively, then (i) holds. So we may assume that $U_2 - (A - \{w_2\})$ has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$, and $\{w_2, x, y\} \subseteq U_{22}$. We choose this separation to minimize U_{22}.

We may assume $w_3 \in N(U_{21} - U_{22})$, or else (ii) holds. Thus we may assume by Lemma 3.5 that $w_3 \notin A$ and $V(U_{21} \cap U_{22}) \cap A = \emptyset$; so $w_3 \in U_{21} - U_{22}$. By the minimality of U_{22}, $U_{22} - \{w_2\} \cap A$ contains disjoint paths from $V(U_{21} \cap U_{22})$ to $\{x, y\}$. Thus, $U_2 - \{w_2\} \cap A$ has three independent paths from u_2 to w_3, x, y, respectively.

Suppose for some $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has three independent paths from u_1 to v_1, v_2, v_i, respectively. If $A''_3 := A_3 - (\{v_{3-i}\} - A)$ has disjoint paths from $\{v_i, w_3\}$ to $\{a_3, a_4\}$, then (i) holds. So we may assume that A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 1$ (or $|V(A_{31} \cap A_{32})| \leq 2$ and $v_{7-i} \in A_{31} \cap A_{32}$), $\{v_i, w_3\} \subseteq A_{31}$, and $\{a_3, a_4\} \subseteq A_{32}$. Now the separation $(G[A_{32} + \{a_1, a_2\}], U_1 \cup U'_2 \cup A'_1 \cup A_2 \cup G[A_{31} + \{a_1, a_2\}])$ show that (iii) holds.

Thus may assume that for any $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has no three independent paths from u_1 to $v_1, v_2.v_i$, respectively. Then for any $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_2, v_i\} \subseteq U_{12}$. If $v_{7-i} \in A$ then the separation $(G[U_{11} + v_{7-i}], G[U_{12} + v_{7-i}] \cup U_2 \cup A'_1 \cup A_2 \cup A_3)$ and Lemma 3.5 imply the assertion. So we may assume $v_3, v_44 \notin A$.

Clearly, $U_1 + \{v, vv_3, vv_4\}$ has no three independent paths from u_1 to v_1, v_2, v, respectively. So $U_1 + \{v, vv_3, vv_4\}$ has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_2, v\} \subseteq U_{12}$. If $v \notin U_{11} \cap U_{12}$ then $(U_{11}, G - (U_{11} - U_{12}))$ shows that (ii) holds. If $v \in U_{11} \cap U_{12}$ then $U_{11} - v, U_{21}, A_3, (U_{12} - v) \cup A'_1 \cup A_2$ show that (G, u_1, u_2, A) is an obstruction of type IV.

Subcase 4.2. $U_1 - \{v_2\} \cap A$ has no three independent paths from u_1 to v_1, v_3, v_4, respectively.

Then $U_1 - \{v_2\} \cap A$ has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_3, v_4\} \subseteq U_{12}$. Choose this separation so that U_{12} is minimal.

We may assume $|V(U_{11} \cap U_{12})| = 2$ and $v_2 \in N(U_{11} - U_{12})$; otherwsie (ii) holds. Let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$. By Lemma 3.5, we may assume $\{s_1, s_2, v_2, w_2\} \cap A = \emptyset$. Thus $v_2 \in U_{11} - U_{12}$.

We may further assume that U_{11} has three independent paths from u_1 to s_1, s_2, v_2, respectively; otherwise we have (ii). By the minimality of U_{12}, for any $i \in \{3, 4\}$, $U_{12} - (A - v_i)$ has disjoint paths from $\{s_1, s_2\}$ to $\{v_1, v_i\}$. So for any $i \in \{3, 4\}$, $U_1 - (A - v_i)$ has three independent paths from u_1 to v_1, v_2, v_i, respectively.

We may also assume that U_2 has three independent paths from u_2 to x, y, w_3, respectively. For, suppose not. Then U_2 has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$ and $\{x, y, w_3\} \subseteq U_{22}$. If $w_2 \notin U_{21} - U_{22}$ then (ii) holds. So assume $w_2 \in U_{21} - U_{22}$. Then $U_{11}, U_{21}, A_2, U_{12} \cup U_{22} \cup A'_1 \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type III.

Suppose $\{v_3, v_4\} = \{a_3, a_4\}$. If $A_3 - v_3$ has a path from w_3 to v_4 then (i) holds. So we may assume that A_3 has a separation (A_{31}, A_{32}) such that $A_{31} \cap A_{32} = \{v_3\}$, $w_3 \in A_{32}$, and

$v_4 \in A_{31}$. Now $U_1 \cup A_{31}, U_2, A_1, A_2, A_{32}$ show that (G, u_1, u_2, A) is an obstruction of type II.

So we may assume that $v_4 \notin A$. If $A_3 - v_4$ has disjoint paths from $\{v_3, w_3\}$ to $\{a_3, a_4\}$ then (i) holds. So we may assume that A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $v_4 \in A_{31} \cap A_{32}$, $\{v_3, w_3\} \subseteq A_{31}$, and $a_3, a_4\} \subseteq A_{32}$. Now the separation $(G[A_{32} + \{a_1, a_2\}], U_1 \cup U_2 \cup A_1' \cup A_2 \cup A_{31})$ shows that (iii) holds.

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Lemma 4.3. *Let (G, u_1, u_2, A) be a quadruple, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exists $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type III. Then one of the following holds.*

(i) (G, u_1, u_2, A) is feasible.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, and $A \cup \{u_2\} \subseteq G_2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$.

(iv) (G, u_1, u_2, A) is an obstruction of types I,II, III, IV, V.

Proof. Let G/xy be the edge disjoint union of U_1, U_2, A_1, A_2 such that $V(U_1 \cap A_1) = \{v_1\}$ and $V(U_2 \cap A_1) = \{w_1\}$, $V(U_1 \cap A_2) = \{v_2, v_3\}$ and $V(U_2 \cap A_2) = \{w_2, w_3\}$, $V(U_1 \cap U_2) \subseteq (\{v_1\} \cap \{w_1\}) \cup (\{v_2, v_3\} \cap \{w_2, w_3\})$, $a_1 \in A_1$, $a_2, a_3, a_4 \in A_2$, and $u_i \in U_i - (A_1 \cup A_2)$ for $i = 1, 2$.

Let v denote the vertex resulting from the contraction of xy. If $v \notin \{v_1, v_2, v_3, w_1, w_2, w_3\}$ then (G, u_1, u_2, A) is an obstruction of type III. So we may assume by symmetry that $v = v_1$ or $v = v_2$. By Lemma 3.5 we may assume that $\{w_1, w_2, w_3\} \cap A = \emptyset$.

We may assume that U_2 has three independent paths from u_2 to w_1, w_2, w_3, respectively; for otherwsie (ii) holds.

Case 1. $v = v_1$.

Let U_1', A_1' be obtained from U_1, A_1, respectively, by uncontracting v to xy. We may assume that A_1' has disjoint paths from $\{x, y\}$ to $\{a_1, w_1\}$. Otherwise, A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq A_{11}$, and $\{a_1, w_1\} \subseteq A_{12}$. Now $U_1' \cup A_{11}, U_2, A_{12}, A_2$ show that (G, u_1, u_2, A) is an obstruction of type III.

We may assume that for some $i \in \{2, 3\}$, $U_1' - (A - v_i)$ has three independent paths from u_1 to x, y, v_i, respectively. For, suppose not. Then $U_1' - (A - \{v_2\})$ has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{x, y, v_2\} \subseteq U_{12}$. Choose this separation to minimize U_{12}. Then $v_3 \in N(U_{11} - U_{12})$; otherwsie (ii) holds. So we may assume $v_3 \notin A$ by Lemma 3.5; hence $v_3 \in U_{11} - U_{12}$. Moreover, we may assume U_{11} has three independent paths form u_1 to $V(U_{11} \cap U_{12}) \cup \{v_3\}$; otherwise (ii) holds. Also by Lemma 3.5 we may assume $v_2 \notin A$ if $v_2 \in U_{11} \cap U_{12}$. So by the minimality of U_{12}, $U_{12} - A$ contains disjoint paths from $V(U_{11} \cap U_{12})$ to $\{x, y\}$. So $U_1' - (A - \{v_3\})$ has three independent paths from u_1 to x, y, v_3, respectively.

Thus we may assume that $U_1' - (A - v_2)$ has three independent paths from u_1 to x, y, v_2, respectively. If $A_2 - (\{v_3\} - A)$ has three disjoint paths from $\{a_2, a_3, a_4\}$ to $\{v_2, w_2, w_3\}$ then (i) holds. So we may assume that A_2 has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 2$,

$\{a_2, a_3, a_4\} \subseteq A_{21}$, and $\{v_2, w_2, w_3\} \subseteq A_{22}$, or $|V(A_{21} \cap A_{22})| \leq 3$, $v_3 \in A_{21} \cap A_{22} - A$, $\{a_2, a_3, a_4\} \subseteq A_{22}$, and $\{v_2, w_2, w_3\} \subseteq A_{21}$. Then the separation $(G[A_{22}+a_1], A_{21} \cup U_1' \cup A_1' \cup U_2)$ shows that (iii) holds.

Case 2. $v = v_3$.

Let U_1', A_2' be obtained from U_1, A_2, respectively, by uncontracting v to xy. We choose such U_1', U_2, A_1, A_2' to maximize $U_1' \cup U_2$. We may assume $v_1 \notin A$ by Lemma 3.5.

We may assume that U_1' has three independent paths from u_1 to v_2, x, y, respectively. For, otherwise, U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{x, y, v_3\} \subseteq U_{12}$. Then $v_1 \in U_{11} - U_{12}$; otherwise (ii) holds. So $U_{11}, U_2, A_1, U_{12} \cup A_2'$ show that (G, u_1, u_2, A) is of type III.

If $A_2'' := A_2' + w_2w_3$ has three disjoint paths from $\{v_2, x, y\}$ to $\{a_2, a_3, a_4\}$ and through w_2w_3, then (i) holds. So we may assume that such paths do not exist, and apply Lemma 3.4.

First, suppose Lemma 3.4(ii) holds. Then A_2'' has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 2$, $\{v_2, x, y\} \subseteq A_{21}$, $\{a_2, a_3, a_4\} \subseteq A_{22}$. If $w_2w_3 \in A_{21}$ then $U_1', U_2 \cup A_{22}, A_1, G[A_{21} - w_2w_3]$ contradict the choice of U_1', U_2, A_1, A_2' (maximality of $U_1' \cup U_2$). So $w_2w_3 \in A_{22}$. Then $U_1' \cup A_{21}, U_2, A_1, G[A_{22} - w_2w_3]$ show that (G,u$_1, u_2, A$) is an obstruction of type III.

Now suppose Lemma 3.4(iii) holds. Then A_2'' has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 1$, $\{x, y, v_3\} \cup \{a_2, a_3, a_4\} \subseteq A_{21}$, and $\{w_2, w_3\} \subseteq A_{22}$. So the separation $(U_1' \cup A_{21} \cup A_1, U_2 \cup G[A_{22} - w_2w_3])$ shows that (ii) holds.

Suppose Lemma 3.4(iv) holds. Then A_2'' has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| = 3$, $\{x, y, v_3\} \subseteq A_{21}$, and $\{a_2, a_3, a_4\} \subseteq A_{22}$. If $w_2w_3 \in A_{22}$ then $U_1' \cup A_{21}, U_2, A_1, G[A_{22} - w_2w_3]$ contradict the choice of U_1', U_2, A_1, A_2' (the maximality of $U_1' \cup U_2$). So $w_2w_3 \in A_{21}$. Now the separation $(G[A_{22} + a_1], U_1' \cup U_2 \cup A_1 \cup G[A_{21} - w_2w_3])$ shows that (iii) holds.

Suppose Lemma 3.4(v) holds. Then $A_2'' = G_1 \cup G_2 \cup G_3$ such that $G_1 \cap G_3 = \emptyset$, $w_2w_3 \in G_2$, $|V(G_1 \cap G_2)| \leq 1$, $|V(G_2 \cap G_3)| \leq 1$, $|V(G_1) \cap \{a_2, a_3, a_4\}| = 1 = |V(G_1) \cap \{v_2, x, y\}|$, and $|V(G_3) \cap \{a_2, a_3, a_4\}| = |V(G_3) \cap \{v_2, x, y\}| = 2$. Then $U_1', U_2 \cup G[G_2 - w_2w_3], A_1, G_1, G_3$ show that (G, u_1, u_2, A) is an obstruction of type IV.

Finally, assume that Lemma 3.4(vi) holds. Then $A_2'' = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5$ such that $|V(G_i \cap G_j)| = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, $V(G_1 \cap G_2) \subseteq G_3 \cup G_4 \cup G_5$, $G_i \cap G_j \subseteq G_1 \cup G_2$ for $3 \leq i \neq j \leq 5$, $w_2w_3 \in G_1$, and either $\{a_2, a_3, a_4\} \subseteq G_2$ and $|V(G_j) \cap \{v_2, x, y\}| = 1$ for $j \in \{3, 4, 5\}$ or $\{v_2, x, y\} \subseteq G_2$ and $|V(G_j) \cap \{a_2, a_3, a_4\}| = 1$ for $j \in \{3, 4, 5\}$. In the former case, $(G[G_2 + a_1], G[G_1 - w_2w_3] \cup U_1' \cup U_2 \cup A_1' \cup G_3 \cup G_4 \cup G_5)$ shows that (iii) holds. Thus, we may assume the latter case. Then $U_1' \cup G_2, U_2 \cup G[G_1 - w_2w_3], A_1, G_3, G_4, G_5$ show that (G, u_1, u_2, A) is an obstruction of type IV. ∎

Lemma 4.4. *Let (G, u_1, u_2, A) be a quadruple, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exists $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type IV. Then one of the following holds.*

(i) (G, u_1, u_2, A) is feasible.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, and $A \cup \{u_2\} \subseteq G_2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$.

(iv) (G, u_1, u_2, A) *is an obstruction of types I, II, IV.*

Proof. Let G/xy be the edge disjoint union of $U_1, U_2, A_1, A_2, A_3, A_4$ such that $V(U_1 \cap A_i) = \{v_i\}$ and $V(U_2 \cap A_i) = \{w_i\}$ for $1 \leq i \leq 4$, $V(U_1 \cap U_2) \subseteq \bigcup_{i=1}^{4}(\{v_i\} \cap \{w_i\})$, $a_i \in A_i$ for $i = 1, 2, 3, 4$, and $u_i \in U_i - (A_1 \cup A_2 \cup A_3 \cup A_4)$ for $i = 1, 2$.

Let v denote the vertex resulting from the contraction of xy. If $v \notin \{v_i, w_i : 1 \leq i \leq 4\}$ then (G, u_1, u_2, A) is an obstruction of type IV, and (iv) holds. So by symmetry we may assume that $v = v_1$. Let U_1', A_1' be obtained from U_1, A_1, respectively, by uncontracting v to xy.

We may assume that A_1' contains disjoint paths from $\{x, y\}$ to $\{a_1, w_1\}$. For, if such paths do not exist, then A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq A_{11}$, and $\{a_1, w_1\} \subseteq A_{12}$. Now $U_1' \cup A_{11}, U_2, A_{12}, A_2, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type IV, and (iv) holds.

Moreover, for each $i \in \{2, 3, 4\}$, if $A_i \neq \{a_i\}$ then we may assume $a_i \notin \{v_i, w_i\}$, and $A_i - v_i$ (respectively, $A_i - w_i$) has a path between w_i (respectively, v_i) and a_i. (Otherwise, we can enlarge U_1' or U_2.

Case 1. There exist two $i \in \{2, 3, 4\}$ such that $J_i := U_1' - (A - \{v_i\})$ has no three independent paths from u_1 to x, y, v_i, respectively.

First, suppose J_2 contains no three independent paths from u_1 to x, y, v_2, respectively. Then J_2 has a separation (J_{21}, J_{22}) such that $|V(J_{21} \cap J_{22})| \leq 2$, $u_1 \in J_{21} - J_{22}$, and $\{x, yv_2\} \subseteq J_{22}$. We choose (J_{21}, J_{22}) so that $|V(J_{21} \cap J_{22})|$ is minimum and then J_{21} is minimal.

If $\{v_3, v_4\} \cap N(J_{21} - J_{22}) = \emptyset$ then the separation $(J_{21}, G - (J_{21} - J_{22}))$ shows that (ii) holds. So we may assume by symmetry that $v_3 \in N(J_{21} - J_{22})$. We may also assume $|V(J_{21} \cap J_{22})| \neq 0$; otherwise, the separation $(G[J_{21} + \{v_3, v_4\}], G - (J_{21} - J_{22} - \{v_3, v_4\}))$ shows that (ii) holds.

Suppose $|V(J_{21} \cap J_{22})| = 1$. Then we may assume that $v_4 \in N(J_{21} - J_{22})$; otherwise, the separation $(G[J_{21} + v_3], G - (J_{21} - J_{22} - v_3))$ shows that (ii) holds. Moreover, the separation $(G[J_{21} + \{v_3, v_4\}], G - (J_{21} - J_{22} - \{v_3, v_4\}))$ allows us to use Lemma 3.5 to assume $v_3, v_4 \notin A$. Hence, $v_3, v_4 \in J_{21}$. Then $J_{21}, U_2, A_3, A_4, J_{22} \cup A_1' \cup A_2$ show that (G, u_1, u_2, A) is an obstruction of type II.

So we may assume that $|V(J_{21} \cap J_{22})| = 2$. Let $V(J_{21} \cap J_{22}) = \{s_1, s_2\}$. So by the minimality of $|V(J_{21} \cap J_{22})|$, $J_{22} - (A - \{v_2\}$ contains disjoint paths from $\{s_1, s_2\}$ to $\{x, y\}$.

By the minimality of J_{21}, we see that $G[J_{21} + v_3]$ has three independent paths from u_1 to s_1, s_2, v_3, respectively. So J_3 has three independent paths from u_1 to x, y, v_3, respectively. Similarly, if $v_4 \in N(J_{21} - J_{22})$ then J_4 has three independent paths from u_1 to x, y, v_4, respectively. Thus we may assume that $v_4 \notin N(J_{21} - J_{22})$. Then by Lemma‘3.5 we may assume $v_3 \notin A$; and hence we may assume $w_3 \notin A$.

If U_2 has three independent paths from u_2 to w_1, w_2, w_4, respectively, then we see that (i) holds. So we may assume that U_2 has a separation $(U_{21}, U_{22}) such that —V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$, and $\{w_1, w_2, w_4\} \subseteq U_{22}$. Then we may assume that $|V(U_{21} \cap U_{22})| = 2$ and $w_3 \in U_{21} - U_{22}$ as otherwise (ii) holds. Now $J_{21}, U_{21}, A_3, J_{22} \cup U_{22} \cup A_1' \cup A_2 \cup A_4$ show that (G, u_1, U_2, A) is an obstruction of type II.

Case 2. There exist two $i \in \{2, 3, 4\}$ such that $J_i := U_1' - (A - \{v_i\})$ has three inpendent paths from u_1 to x, y, v_i, respectively.

Without loss of generality, we may assume that for $i = 2, 3$, J_i has three independent paths from u_1 to x, y, v_i, respectively.

If $U_2' := U_2 - \{w_2\} \cap A$ has three independent paths from u_2 to w_1, w_3, w_4, respectively, then (i) holds. So we may assume that U_2' has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$, and $\{w_1, w_3, w_4\} \subseteq U_{22}$. Choose (U_{21}, U_{22}) so that $|V(U_{21} \cap U_{22})|$ is minimum and then U_{22} is minimal. Thus, $U_{22} - (A \cap \{w_3\})$ has disjoint paths from $V(U_{21} \cap U_{22})$ to $\{w_1, w_4\}$.

We may assume $w_2 \in N(U_{21} - U_{22})$ and $|V(U_{21} \cap U_{22})| = 2$, as otherwise (ii) holds. Thus by Lemma 3.5 we may assume that $w_2 \notin A$ and $V(U_{21} \cap U_{22}) \cap A = \emptyset$. So $w_2 \in U_{21} - U_{22}$. Hence, U_{21} has three independent paths from u_2 to $V(U_{21} \cap U_{22}) \cup \{w_2\}$. Therefore, $U_2 - (A \cap \{w_3\})$ has three independent paths from u_2 to w_1, w_2, w_4, respectively. Again, (G, u_1, u_2, A) is feasible, and (i) holds. ∎

Lemma 4.5. *Let G be a graph, let $u_1, u_2, a_1, a_2, a_3, a_4$ be distintc vertices of G, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exist $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type V. Then one of the following holds.*

(i) (G, u_1, u_2, A) is feasible.

(ii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, and $A \cup \{u_2\} \subseteq G_2$.

(iii) G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$.

(iv) (G, u_1, u_2, A) is an obstruction of types I, II, III, IV or V.

Proof. Let G/xy be the edge-disjoint union of U_1, U_2, A_1, A_2 such that $V(U_1 \cap A_1) = \{v_1\}$, $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_1) = \{w_1, w_2\}$, $V(U_2 \cap A_2) = \{w_3\}$, $V(U_1 \cap U_2) \subseteq (\{v_1\} \cap \{w_1, w_2\}) \cup (\{v_2, v_3\} \cap \{w_3\})$, $a_1, a_2 \in A_1$, $a_3, a_4 \in A_2$, and $u_i \in U_i - (A_1 \cup A_2)$ for $i = 1, 2$.

Let v denote the vertex resulting from the contraction of xy. If $v \notin \{v_i, w_i : 1 \leq i \leq 3\}$ then it is easy to see that (G, u_1, u_2, A) is also an obstruction of type V, and (iv) holds. Thus, we may assume $v \in \{v_i, w_i : 1 \leq i \leq 3\}$. By symmetry, we need to consider only two cases: $v = v_1$ or $v = v_2$. By Lemma 3.5 we may assume that $\{w_1, w_2, w_3\} \cap A = \emptyset$.

We may assume that U_2 contains three independent paths from u_2 to w_1, w_2, w_3, respectively; for otherwise Menger's theorem shows that (ii) holds.

Case 1. $v = v_2$.

Let U_1', A_2' be obtained from U_1, A_2 by uncontracting v to xy. We may assume that A_2' contains three disjoint paths from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For if such three paths do not exist then A_2' has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 2$, $\{a_3, a_4, w_3\} \subseteq A_{22}$ and $\{v_3, x, y\} \subseteq A_{21}$. Then $U_1' \cup A_{21}, U_2, A_1, A_{22}$ show that (G, u_1, u_2, A) is an obstruction of type V.

We may assume $v_1 \notin A$. For, suppose $v_1 \in A$, say $v_1 = a_1$. Then $(A_1 \cup U_2, A_2' \cup U_1' + a_2)$ is a separation in G, and hence by Lemma 3.5, the assertion of the lemma holds.

We may assume that $A_1 - v_1$ contains disjoint paths from $\{w_1, w_2\}$ to $\{a_1, a_2\}$. For, otherwise, A_1 has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 2$, $v_1 \in A_{11} \cap A_{12}$,

$\{w_1, w_2\} \subseteq A_{12}$ and $\{a_1, a_2\} \subseteq A_{12}$. Then the separation $(G[A_{12}+\{a_3, a_4\}], A_{11}\cup A_2'\cup U_1'\cup U_2)$ shows that (iii) holds.

If U_1' contains three independent paths from u_1 to v_3, x, y, then (i) holds. So we may assume that U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11}\cap U_{12})| \leq 3$, $u_1 \in U_{11} - U_{12}$, and $\{v_3, x, y\} \subseteq U_{12}$. If $v_1 \notin U_{11} - U_{12}$ then (ii) holds. So assume $v_1 \in U_{11} - U_{12}$. Then $U_{11}, U_2, A_1, A_2' \cup U_{12}$ show that (G, u_1, u_2, A) is an obstruction of type V.

Case 2. $v = v_1$.

Let U_1', A_1' be obtained from U_1, A_1, respectively, by uncontracting v to xy. We choose U_1', U_2, A_1', A_2 to maximize $U_1' \cup U_2$.

We may assume that $A_1^* := A_1' + \{a_1a_2, w_1w_2, xy\}$ contains a cycle through a_1a_2, w_1w_2, xy. For, suppose not. Then by Lemma 3.3 there are three possibilities. First, suppose A_1^* has a separation (K, L) such that $|V(K\cap L)| \leq 1$ and $|E(K)\cap\{a_1a_2, w_1w_2, xy\}| = 1$. If $w_1w_2 \in K$, then the separation $(U_1'\cup L\cup A_2, K\cup U_2)$ shows that (ii) holds. If $xy \in K$ then $U_1'\cup K, U_2, L, A_2$ show that (G, u_1, u_2, A) is an obstruction of type V. If $a_1a_2 \in K$ then $(G[K + \{a_3, a_4\}], L \cup U_1' \cup U_2 \cup A_2)$ shows that (iii) holds. Now, suppose A_1^* has a separation (K, L) such that $|V(K\cap L)| = 2$, $|E(K)\cap\{a_1a_2, w_1w_2, xy\}| = 1$, and $|V(K)| \geq 3$. If $w_1w_2 \in K$ or $xy \in K$ then $U_1'\cup K, U_2, L, A_2$ or $U_1', U_2\cup L, K, A_2$ contradicts the choice of U_1', U_2, A_1', A_2 (maximality of $U_1' \cup U_2$). If $a_1a_2 \in K$ then the separation $(G[K + \{a_3, a_4\}], U_1' \cup U_2 \cup L \cup A_2)$ shows that (iii) holds. Finally, $\{a_1a_2, w_1w_2, xy\}$ is an edge cut in A_1^*. Then it is easy to check that (G, u_1, u_2, A) is an obstruction of type II, and (iv) holds.

We may assume that for any $i \in \{2, 3\}$, $A_2 - \{\{v_{5-i}\} - A)$ contains disjoint paths from $\{w_3, v_i\}$ to $\{a_3, a_4\}$. For suppose the contrary. Then by symmetry we may assume that $A_2 - (\{v_3\} - A)$ contains no disjoint paths from $\{w_3, v_2\}$ to $\{a_3, a_4\}$. So Menger's theorem implies that A_2 has a separation (A_{21}, A_{22}) such that $|V(A_{21}\cap A_{22})| \leq 1$ (when $v_3 \notin A$), $|V(A_{21}\cap A_{22})| \leq 2$ and $v_3 \in A_{21}\cap A_{22}$ (when $v_3 \notin A$), $\{a_3, a_4\} \subseteq A_{21}$ and $\{w_3, v_2\} \subseteq A_{22}$. We may assume that $V(A_{21}) = V(A_{21}\cap A_{22}) \cup \{v_3\} = \{a_3, a_4\}$, or else the separation $(G[A_{21} + \{a_1, a_2\}], A_{22} \cup U_1' \cup U_2 \cup A_1')$ shows that (iii) holds. As $\{w_1, w_2, v_2\}$ separates u_2 from $A \cup \{u_1\}$ in G, we may assume by Lemma 3.5 that $v_2 \notin A$. If U_1' has three independent paths from u_1 to x, y, v_3, respectively, then we see that (i) holds. So we may assume that U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11}\cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$ and $\{x, y, v_3\} \subseteq U_{12}$. If $v_2 \notin U_{11} - U_{12}$ then (ii) holds. So assume $v_2 \in U_{11} - U_{12}$. Then $U_{11}, U_2, U_{12} \cup A_1', A_2 - v_3$ show that (G, u_1, u_2, A) is an obstruction of type III, and (iv) holds.

We may assume that $U_1' - (A - \{v_3\})$ has no three independent paths from u_1 to x, y, v_3, respectively. For, such paths together with disjoint paths in A_2 from $\{v_3, w_3\}$ to $\{a_3, a_4\}$, three paths in U_2 from u_2 to w_1, w_2, w_3, and $C - \{a_1a_2, w_1w_2, xy\}$, give a topological H in G rooted at u_1, u_2, A; so (i) holds.

Thus, $U_1' - (A - \{v_3\})$ has a separation (U_{11}, U_{12}) such that $|V(U_{11}\cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{x, y, v_3\} \subseteq U_{12}$. We choose U_{11}, U_{12} so that $|V(U_{11}\cap U_{12})|$ is minimum and then U_{12} is minimum.

We may assume that $v_2 \in N(U_{11} - U_{12})$ and $|V(U_{11}\cap U_{12})| = 2$; or else (ii) holds. So by Lemma 3.5 we may assume $v_2 \notin A$. So $v_2 \in U_{11} - U_{12}$. By the minimality of $|V(U_{11}\cap U_{12})|$, U_{11} has three independent paths from u_1 to x, y, v_2, respectively. By the minimality of U_{12}, $U_{12} - (\{v_3\} \cap A)$ has disjoint paths from $V(U_{11}\cap U_{12})$ to $\{x, y\}$, respectively. Thus, $U_1' - (A -$

$\{v_2\})$ has three independent paths from u_1 to x, y, v_2, respectively. So these paths, disjoint paths in $A_2 - (\{v_3\} - A)$ from $\{v_2, w_3\}$ to $\{a_3, a_4\}$, three paths in U_2 from u_2 to w_1, w_2, w_3, and $C - \{a_1a_2, w_1w_2, xy\}$, give a topological H in G rooted at u_1, u_2, A; so (i) holds. ∎

5 Proof of main theorem

Proof. Suppose this is not true. Let (G, u_1, u_2, A) be a counterexample with $|V(G)|$ minimum.

We claim that no cut of size at most 4 in G is disjoint from $\{u_1, u_2\}$, and separates $\{u_1, u_2\}$ from A. For, suppose G has a cut S such that $|S| \leq 4$, $S \cap \{u_1, u_2\} = \emptyset$, and S separates $\{u_1, u_2\}$ from A. Then $|S| = 4$ for any such choice of S; otherwise, (iii) holds. But this shows that G admits a good 4-separation, a contradiction.

We also claim that u_1 is not adjacent to u_2. For, suppose $u_1u_2 \in E(G)$. Then let G' be obtained from G by duplicating u_1 and u_2, and let u_i', $i = 1, 2$, denote the dupplicate of u_i. Now by (2), G' contains four disjoint paths from $\{u_1, u_1', u_2, u_2'\}$ to A. These paths and u_1u_2 form a topological H in G rooted at u_1, u_2, A, a contradiction.

We further claim that $N(u_1) \cap N(u_2) \subseteq A$. Now let $u \in N(u_1) \cap N(u_2) - A$. Let G' be obtained from $G - u$ by duplicating u_i (with duplicate u_i') for $i = 1, 2$. By (2), G' contains four disjoint paths from $\{u_1, u_1', u_2, u_2'\}$ to A. These paths together with u_1uu_2 form a topological H in G rooted at u_1, u_2, A, a contradiction.

We now show that there exists an edge $xy \in E(G)$ such that $x, y \notin A \cup \{u_1, u_2\}$, and if $d(u_i) = 3$ then $\{x, y\} \not\subseteq N(u_i)$. If $V(G) = A \cup \{u_1, u_2\}$ then, since $u_1u_2 \notin E(G)$, u_1 and u_2 are the components of $G - A$, so (G, u_1, u_2, A) may be viewed as an obstruction of type IV. Thus, we may assume $V := V(G) - (A \cup \{u_1, u_2\}) \neq \emptyset$. We may assume that $G[V]$ contains no edge, as any edge in $G[V]$ gives the desired edge. Therefore, since $N(u_1) \cap N(u_2) \subseteq A$, $V(G) - A$ can be partitioned into two sets V_1, V_2, such that $u_i \in V_i$ for $i = 1, 2$. Now $G[V_1], G[V_2], a_1, a_2, a_3, a_4$ show that (G, u_1, u_2, A) is an obstruction of type IV.

By the choice of G, $(G/xy, u_1, u_2, A)$ satisfies (i) or (ii) or (iii) or (iv). If $(G/xy, u_1, u_2, A)$ satisfies (i) then (G, u_1, u_2, A) also satisfies (i).

Suppose $(G/xy, u_1, u_2, A)$ satisfies (ii). Let (G_1, G_2) be a separation in G such that $V(G_1 \cap G_2)| \leq 2$, $u_i \in G_1 - G_2$, and $A \cup \{u_{3-i}\} \subseteq G_2$. By the minimality of G, $G_1 - G_2 = \{u_i\}$. Thus $x, y \in N(u_i)$, a contradiction. So $(G/xy, u_1, u_2, A)$ cannot satisfy (ii).

Suppose $(G/xy, u_1, u_2, A)$ satisfies (iv). Then (G, u_1, u_2, A) satisfies (i)–(iv) by Lemmas 3.6, 4.1, 4.2, 4.3, 4.4, and 4.5.

So we may assume that $(G/xy, u_1, u_2, A)$ satisfies (iii). Let (G_1, G_2) be a separation in G such that $|V(G_1 \cap G_2)| = 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$. Let v denote the vertex resulting from the contraction of xy. If $v \notin G_1 \cap G_2$ for one such separation, then (iii) also holds for (G, u_1, u_2, A). Thus we may assume that $v \in G_1 \cap G_2$ for all such separations. So G_2 has four disjoint paths from $A' := V(G_1 \cap G_2)$ to A. We choose (G_1, G_2) to minimize G_1.

Let $A' = \{a_1', a_2', a_3', v\}$. By the minimality of (G, u_1, u_2, A), (G_1, u_1, u_2, A') is not a counterexample. Thus, (G_1, u_1, u_2, A') satisfies (i) – (iv). If (G_1, u_1, u_2, A') satisfies (i) then (G, u_1, u_2, A) also satisfies (i).

If (G_1, u_1, u_2, A') satisfies (ii) then G_1 has a separaion (K, L) such that $|V(K \cap L)| \leq 2$,

$u_i \in K-L$ and $A' \cup \{u_{3-i}\} \subseteq L$. If $v \notin K \cap L$ or $|V(K \cap L)| \leq 1$ then (ii) holds for (G, u_1, u_2, A). If $v \in K \cap L$ and $|V(K \cap L)| = 2$ then by the minimality of G, $V(K-L) = \{u_i\}$. This shows that $x, y \in N(u_i)$, a contradiction.

Now suppose (G_1, u_1, u_2, A') satisfies (iii) then G_1 has a separaion (K, L) such that $|V(K \cap L)| = 4$, $\{u_1, u_2\} \subseteq K - L$ and $A' \subseteq L$. So $v \in K \cap L$. But this contradics the minimality of G_1.

Therefore, (G_1, u_1, u_2, A') satisfies (iv).

(4) G contains no 5-cut S such that u_1, u_2 belong to different components of $G - S$, and the components of $G - S$ containing u_1 or u_2 are disjoint from A.

Otherwise, let S be a 5-cut in G and U_1 and U_2 be components of $G - S$ such that for $i = 1, 2$, $u_i \in U_i$ and $U_i \cap A = \emptyset$.

We now apply Lemma **??**. Lemma **??**(i) cannot occur; otherwise G would satisfy (ii). By (2), Lemma **??**(ii) cannot occur. So Lemma **??**(iii) occurs. Thus for any $v \in N(U_1) \cap N(U_2)$ with $v \notin A$ and for $i = 1, 2$, $G[U_i \cup N(U_i)]$ contains three paths P_1^i, P_2^i, P_3^i from u_i to $N(U_i) \cap S$ such that $P_j^i \cap P_k^i = \{u_i\}$ whenever $j \neq k$, $v \in P_3^1 \cap P_3^2$, and each vertex in $S - \{v\}$ belongs to precisely one of these paths.

If $G - (U_1 \cup U_2 \cup \{v\})$ has four disjoint paths from $S - \{v\}$ to A, then these paths and P_j^i, $i = 1, 2$ and $j = 1, 2, 3$, form a topological H in G rooted at $u_1, u_2, a_1, a_2, a_3, a_4$, a contradiction. Thus such paths do not exist. So $G - (U_1 \cup U_2 \cup \{v\})$ has a cut T with $|T| \leq 3$ separating $S - \{v\}$ from A. Hence $T \cup \{v\}$ is a cut in G separating A from $\{u_1, u_2\}$, contradicting (2).

Thus for any $v \in N(U_1) \cap N(U_2) - A$, $G - (U_1 \cup U_2 \cup \{v\})$ has a cut T with $|T| \leq 3$ and separating $S - \{v\}$ from A. If $|T| \leq 2$ then $T \cup \{v\}$ shows that (iii) holds, a contradiction. So $|T| = 3$, which shows that (v) holds, a contradiction.

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