Graphs containing topological H

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Abstract

Let H denote the tree with six vertices two of which are adjacent and of degree three. Let G be a graph and u₃, u₄, a₁, a₂, a₃, a₄ be distinct vertices of G. We characterize those G that contain a topological H in which u₁, u₂ are of degree three, and a₁, a₂, a₃, a₄ are of degree one. This work was motivated by the Kelmans-Seymour conjecture that 5-connected nonplanar graphs contain topological K₅.

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1 Introduction

The work in this paper was motivated by the well known conjecture of Seymour [14] and Kelmans [6]: Every 5-connected nonplanar graph contains a topological K₅ (i.e., subdivision of K₅). Clearly, this would provide structural information that guarantees the existence of a topological K₅. Earlier, Dirac [3] conjectured an extremal function for the existence of a topological K₅: If G is a simple graph with n ≥ 3 vertices and at least 3n − 5 edges then G contains a topological K₅. This conjecture was established by Mader [12]. Kézdy and McGuiness [7] showed that the Kelmans-Seymour conjecture if true would imply Mader’s result. This Kelmans-Seymour conjecture is also related to a conjecture of Hajós (see [2]) that every graph containing no topological Kₖ₊₁ is k-colorable. Hajós’ conjecture is false for k ≥ 6 [2] and true for k = 1, 2, 3, and remains open for the case k = 4 and k = 5.

An approach to the Kelmans-Seymour conjecture is to study the so called rooted K₄ problem: Given a graph G and four distinct vertices of G, when does G contain a topological K₄ in which x₁, x₂, x₃, x₄ are the vertices of degree three. This problem was solved for planar graphs, see [16]. Recently, Aigner-Horev and Krakovski [1] used this to prove Kelmans-Seymour conjecture for apex graphs. (A different and shorter proof was found by Ma, Thomas and Yu [9].)

One step in [16] is to solve the following rooted H problem for planar graphs: Let H denote the tree on six vertices two of which are adjacent and of degree 3. Let G be a graph and u₁, u₂, a₁, a₂, a₃, a₄ be distinct vertices of G. When does G contain a topological H in

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which \( u_1, u_2 \) are of degree 3 and \( a_1, a_2, a_3, a_4 \) are of degree 1? We say such a topological \( H \) is rooted at \( u_1, u_2, \{a_1, a_2, a_3, a_4\} \). For convenience, we use quadruple to denote \((G, u_1, u_2, A)\) where \( u_1, u_2 \) are distinct vertices of a graph \( G \), \( A \subseteq V(G) - u_1, u_2 \), and \(|A| = 4\).

The main result of this paper is a characterization of graphs quadruples \((G, u_1, u_2, A)\) that contain a topological \( H \) rooted at \( u_1, u_2, A \). Since the statement of this result requires a fair amount of terminology, we defer it to Section 2, see Theorem 2.1.

We devote the rest of this section to notation and terminology. A separation in a graph \( G \) consists of a pair of subgraphs \( G_1, G_2 \), denoted as \((G_1, G_2)\), such that \( E(G_1 \cap G_2) = \emptyset \), \( E(G_1) \cup V(G_1) \subseteq G_1 \cap G_2 \), and \( E(G_2) \cup V(G_2) \subseteq G_1 \cap G_2 \). The order of this separation is \(|V(G_1 \cap G_2)|\), and \((G_1, G_2)\) is said to be a \( k \)-separation if its order is \( k \). Let \( G \) be a graph. A set \( S \subseteq V(G) \) is a \( k \)-cut or a cut of size \( k \) in \( G \), where \( k \) is a positive integer, if \(|S| = k \) and \( G \) has a separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = S \) and \( V(G_i - S) \neq \emptyset \) for \( i \in \{1, 2\} \). If \( v \in V(G) \) and \( \{v\} \) is a cut of \( G \), then \( v \) is said to be a cut vertex of \( G \).

Let \( G \) be a graph. If there is no confusion, we may write \( S \subseteq G \) instead of \( S \subseteq V(G) \) or \( S \subseteq E(G) \), and write \( x \in G \) instead of \( x \in V(G) \) or \( x \in E(G) \). Let \( H \subseteq G \), \( S \subseteq V(G) \), and \( T \) a set of 2-element subsets of \( V(H) \cup S \); then \( H + (S \cup T) \) denotes the graph with vertex set \( V(H) \cup S \) and edge set \( E(G) \cup T \). If \( T = \{x, y\} \), we write \( G + xy \) instead of \( G + \{x, y\} \).

Given a path \( P \) in a graph and \( x, y \in V(P) \), \( xPy \) denotes the subpath of \( P \) between \( x \) and \( y \) (inclusive). We may view paths as sequences of vertices; thus if \( P \) is a path between \( x \) and \( y \), \( Q \) is a path between \( y \) and \( z \), and \( P \cap Q = \{y\} \), then \( PyQ \) denotes the path \( P \cup Q \). The ends of the path \( P \) are the vertices of the minimum degree in \( P \), and all other vertices of \( P \) are its internal vertices. A path \( P \) with ends \( u \) and \( v \) is also said to be from \( u \) to \( v \) or between \( u \) and \( v \). A collection of paths are said to be independent if no vertex of any path is an internal vertex of any other path.

## 2 Obstructions

For convenience, we say that a quadruple \((G, u_1, u_2, A)\) is feasible if \( G \) contains a topological \( H \) rooted at \( u_2, u_2, A \). An obstruction is a quadruple that is not feasible. We now describe basic obstructions.

A quadruple \((G, u_1, u_2, A)\) is of type I if \( G \) is the edge-disjoint union of subgraphs \( U_1, U_2, A_1 \) such that \(|V(U_1 \cap A_1)| = 3, |V(U_2 \cap A_1)| = 4, V(U_1 \cap U_2) \subseteq A \cap V(A_1), |V(U_1 \cap U_2)| = 2, A \subseteq A_1 \), and for some \( i \in \{1, 2\} \), \( u_i \in U_1 - A_1 \) and \( u_{3-i} \in U_2 - A_1 \). Clearly, if \( G \) has a topological \( H \) rooted at \( u_1, u_2, A \), say \( J \), then \( J \cap U_1 \) consists of three independent paths from \( u_i \) to \( V(U_1 \cap A_j) \). Therefore, \( J \cap U_2 \) must have three independent paths from \( u_{3-i} \) to \((U_2 \cap A_j) - U_1 \), a contradiction. So quadruples of type I are obstructions.

A quadruple \((G, u_1, u_2, A)\) is of type II if there exist edge disjoint subgraphs \( U_1, U_2, A_1, A_2, A_3 \) such that \( G = U_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3 \), \( |V(U_2 \cap A_3)| = |V(U_1 \cap A_j)| = 1 \) for \( i \in \{1, 2\} \) and \( j \in \{1, 2\} \), \( |V(U_1 \cap A_3)| = 2, A_i \cap A_j \subseteq U_1 \cup U_2, U_1 \cap U_2 \subseteq A_1 \cup A_2 \cup A_3, |V(A_i) \cap A| = 1 \) for \( i = 1, 2, |V(A_2) \cap A| = 2 \) if \( a_i \in U_j \) then \( a_i \in U_2 \) then \( a_i \in A_1 \cap A_2 \cap A_3 \), if \( a_i \in U_1 \cap (A_1 \cup A_2) \) then \( a_i \in A_3 \cap V(A_i) = 1 \) for some \( i \in \{1, 2\} \) then \( A_i \subseteq A_j \) for all \( j \neq i \), and for some \( i \in \{1, 2\} \), \( u_i \in U_1 - (A_1 \cup A_2 \cup A_3) \) and \( u_{3-i} \in U_2 - (A_1 \cup A_2 \cup A_3) \). Clearly, if \( G \) has a topological \( H \) rooted at \( u_1, u_2, A \), say \( J \), then \( J \cap U_2 \) consists of three independent paths from
from Theorem 2.1. Consider the graph $G$ of quadruples of type IV are obstructions. Let $G/u_1 = U_2 \cup A_2 \cup A_3 \cup A_4$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = |V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$, $|V(U_1 \cap U_2)| = 1$, $|V(U_2 \cap A_1)| = 2$, $|V(U_1 \cap U_2)| = 1$. So quadruples of type III are obstructions. A quadruple $(G, u_1, u_2, A)$ is of type IV if there exist edge disjoint subgraphs $U_1, U_2, A_1, A_2, A_3, A_4$ of $G$ such that $G = U_1 \cup U_2 \cup A_1 \cup A_2, |V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = |V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$, $V(U_1 \cap U_2) \subseteq A_1 \cup A_2 \cup A_3 \cup A_4$, $|V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, $|V(A_1) \cap A| = 3$. If $a_i \in U_j$ then $i \in A_i \cap A_j$, and $u_i \in U_i - (A_1 \cup A_2)$. Clearly, if $G$ has a topological $H$ rooted at $u_1, u_2$, say $J$, then $J \cap U_1 \cap A_1$ has three independent paths from $u_1$ to the three vertices in $(V(A_1) \cap A) \cup (V(U_1 \cap U_2))$. So $J \cap U_2$ has three independent paths from $u_2$ to $V(U_2 \cap A_2)$, a contradiction. So quadruples of type III are obstructions.

A quadruple $(G, u_1, u_2, A)$ is of type V if there exist edge disjoint subgraphs $U_1, U_2, A_1, A_2, A_3, A_4$ such that $G/xy = U_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3 \cup A_4$, $|V(U_1 \cap A_j)| = 1$ for $1 \leq i \leq 4$, $j = 1, 2$, $|V(U_1 \cap U_2)| = 2$, $V(U_1 \cap U_2) \subseteq A_1 \cup A_2 \cup A_3 \cup A_4, |V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, $|V(A_1) \cap A| = 3$. If $a_i \in U_j$ then $i \in A_i \cap A_j$, and $u_i \in U_i - (A_1 \cup A_2)$. Clearly, if $G$ has a topological $H$ rooted at $u_1, u_2$, say $J$, then $J \cap U_1$ has three independent paths from $u_1$ to the three vertices in $V(U_1) \cup V(A_1 \cup A_2)$, respectively. So the path in $J$ between $u_1$ and $u_2$ must go through $A_i$ for some $1 \leq i \leq 4$. But then $J$ cannot use $V(A_i) \cap A$, a contradiction. So quadruples of type IV are obstructions.

A quadruple $(G, u_1, u_2, A)$ is of type VI if there exist edge disjoint subgraphs $U_1, U_2, A_1, A_2, A_3, A_4$ of $G$ such that $G = U_1 \cup U_2 \cup A_1 \cup A_2 \cup A_3 \cup A_4$, $|V(U_1 \cap A_1)| = 3$, $|V(U_2 \cap A_1)| = 1$, and $u_i \in U_i - A_i$ for $i = 1, 2$. Clearly, if $G$ has a topological $H$ rooted at $u_1, u_2$, say $J$, then $J \cap U_1$ consists of three independent paths from $u_1$ to $V(U_1 \cap A_1)$. Therefore, $J$ contains a path from $u_1$ to $u_2$ and containing a vertex from $A$, a contradiction. So quadruples of type VI are obstructions.

We can now state our main result which characterizes all feasible quadruples.

**Theorem 2.1.** Let $(G, u_1, u_2, A)$ be a quadruple and let $A := \{a_1, a_2, a_3, a_4\}$. Then one of the following holds.

(i) $(G, u_1, u_2, A)$ is feasible.

(ii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in G_1 - G_2$ and $A \cup \{u_{3-i}\} \subseteq G_2$.

(iii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 4$, $u_1, u_2 \in G_1 - G_2$, and $A \subseteq G_2$.

(iv) $(G, u_1, u_2, A)$ is an obstruction of type I-VI.

The idea of our proof of Theorem 2.1 is to find an edge $xy$ in $G - (A \cup \{u_1, u_2\})$ and consider the graph $G/xy$ obtained from $G$ by contracting $xy$. Clearly, if $(G/xy, u_1, u_2, A)$ is
feasible then \((G, u_1, u_2, A)\) is feasible. We will show that if \((G/xy, u_1, u_2, A)\) is an obstruction of one these six types, then (i), or (ii), or (iii), or (iv) holds. This is done in Section 4.

### 3 Disjoint paths

In this section we prove useful lemmas about disjoint paths. First, we state the following result of Perfect [13]; we will need the \(k = 3\) case.

**Lemma 3.1.** (Perfect) Let \(G\) be a graph, \(u \in V(G)\), and \(A \subseteq V(G - u)\). Suppose there exist \(k\) independent paths from \(u\) to distinct \(a_1, \ldots, a_k \in A\), respectively, and otherwise disjoint from \(A\). Then for any \(n \geq k\), if there exist \(n\) independent paths \(P_1, \ldots, P_n\) in \(G\) from \(u\) to \(n\) distinct vertices in \(A\) and otherwise disjoint from \(A\) then \(P_1, \ldots, P_n\) may be chosen so that \(a_i \in P_i\) for \(i = 1, \ldots, k\).

We need structural information about graphs containing no cycle through three given edges. Lovász [8] proved the following.

**Lemma 3.2.** (Lovász) Let \(G\) be a 3-connected graph and \(e_1, e_2, e_3\) be distinct edges of \(G\). Then \(G\) contains a cycle through \(e_1, e_2, e_3\) iff \(G - \{e_1, e_2, e_3\}\) is connected.

We also need the following easy generalization of Lemma 3.2.

**Lemma 3.3.** Let \(G\) be a connected graph and let \(e_1, e_2, e_3 \in E(G)\) be distinct. Then one of the following holds.

(i) \(\{e_1, e_2, e_3\}\) is contained in a cycle in \(G\).

(ii) \(G\) has a separation \((G_1, G_2)\) such that \(\lvert V(G_1 \cap G_2)\rvert = 1\) and \(E(G_i) \cap \{e_1, e_2, e_3\} \neq \emptyset\) for \(i = 1, 2\).

(iii) \(G\) has a separation \((G_1, G_2)\) such that \(\lvert V(G_1 \cap G_2)\rvert = 2\) and for some \(i \in \{1, 2\}\), \(\lvert E(G_i) \cap \{e_1, e_2, e_3\}\rvert \leq 1\) and \(\lvert V(G_i)\rvert \geq 3\).

(iv) \(G - \{e_1, e_2, e_3\}\) is not connected.

**Proof.** Suppose the assertion is false, and choose a counterexample \(G, e_1, e_2, e_3\) such that \(\lvert V(G)\rvert\) is minimum. Then \(G\) is not 3-connected, as otherwise (i) or (iv) holds by Lemma 3.2. So let \((G_1, G_2)\) be a \(k\)-separation of \(G\) such that \(k \in \{1, 2\}\), and \(G_i - G_{3-i} \neq \emptyset\) for \(i = 1, 2\).

If \(k = 2\) then (iii) holds, a contradiction. So \(k = 1\), and we may assume by symmetry that \(\{e_1, e_2, e_3\} \subseteq G_1\) (or else (ii) would hold). By the minimality of \(G\), we see that one of (i)–(iv) holds for \(G_1, e_1, e_2, e_3\). Because \(k = 1\), it is easy to check that one of (i)–(iv) holds for \(G, e_1, e_2, e_3\), a contradiction.

The problem for finding a cycle through three given edges is equivalent to the problem for finding two disjoint paths between two pairs of vertices and through a given edge. In general one could ask the problem for finding \(k\) disjoint paths between two \(k\)-sets (of vertices) and
Let \( G \) be minimal, and let \( e \in E(G) \) such that \( A \cap B = \emptyset \) and \( V(e) \cap (A \cup B) = \emptyset \). Then one of the following statements holds.

(i) \( G \) has three disjoint paths from \( A \) to \( B \) and through \( e \).

(ii) \( G \) has a separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| \leq 2 \), \( A \subseteq G_1 \), and \( B \subseteq G_2 \).

(iii) \( G \) has a separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| \leq 1 \), \( e \in G_1 \), and \( A \cup B \subseteq G_2 \).

(iv) \( G \) has a separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| = 3 \), \( A \subseteq G_1 \), and \( B \subseteq G_2 \).

(v) \( G = G_1 \cup G_2 \cup G_3 \) such that \( G_1 \cap G_2 = \emptyset \), \( e \in G_2 \), \( |V(G_1 \cap G_2)| \leq 1 \), \( |V(G_2 \cap G_3)| \leq 1 \), \( |V(G_1) \cap A| = 1 = |V(G_1) \cap B| \), and \( |V(G_3) \cap A| = |V(G_3) \cap B| = 2 \).

(vi) \( G = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5 \) such that \( |V(G_i \cap G_j)| = 1 \) for \( i \in \{1, 2\} \) and \( j \in \{3, 4, 5\} \), \( V(G_1 \cap G_2) \subseteq G_3 \cup G_4 \cup G_5 \), \( G_i \cap G_j \subseteq G_1 \cup G_2 \) for \( 3 \leq i \neq j \leq 5 \), \( e \in G_1 \), and either \( A \subseteq G_2 \) and \( |V(G_j) \cap B| = 1 \) for \( j \in \{3, 4, 5\} \) or \( B \subseteq G_2 \) and \( |V(G_j) \cap A| = 1 \) for \( j \in \{3, 4, 5\} \).

Proof. We may assume that \( G \) has three disjoint paths from \( A \) to \( B \), or else (ii) follows from Menger’s theorem. So let \( P_1, P_2, P_3 \) denote three disjoint paths in \( G \) from \( A \) to \( B \), and let \( P := \bigcup_{i=1}^{3} P_i \). If \( e \in P \) then (i) holds. So we may assume that \( e \notin P \) for any choice of \( P \). Let \( H_P \) denote the \( P \)-bridge of \( G \) containing \( e \). We choose \( P \) so that

1. \( H_P \) is maximal.

Without loss of generality we may assume that \( P_i \) is from \( a_i \) to \( b_i \) for \( i = 1, 2, 3 \). Let \( x_i, y_i \in V(P_i \cap H_P) \) (if not empty) such that \( x_iP_iy_i \) is maximal. We may assume \( a_i, x_i, y_i, b_i \) occur on \( P_i \) in order. For convenience, let \( H' := H_P - P \), and let \( H_i := G[H' \cup x_iP_iy_i] \) for \( i = 1, 2, 3 \).

2. For any \( i \) with \( x_i, y_i \) defined, \( G \) has no \( P \)-bridge intersecting both \( a_iP_ix_i \) and \( x_iP_ib_i - x_i \), or both \( a_iP_iy_i - y_i \) and \( y_iP_ib_i - y_i \).

For, suppose \( G \) has a \( P \)-bridge \( J \) intersecting both \( a_iP_ix_i - x_i \) and \( x_iP_ib_i - x_i \). Then \( J \neq H_P \), and \( J \) contains a path \( Q_i \) from \( u_i \in V(a_iP_ix_i - x_i) \) to \( v_i \in V(x_iP_ib_i - x_i) \) and internally disjoint from \( P \). Let \( P'_i := a_iP_iu_iQ_iP_ib_i \), and \( P' := (P - P_i) \cup P'_i \). Then the \( P' \)-bridge of \( G \) containing \( e \) contains \( H_P + x_i \); so \( P' \) contradicts the choice of \( P \).

3. We may assume that for any \( i \) with \( x_i, y_i \) defined, \( H'_i \) has a separation \((H_{i1}, H_{i2})\) such that \( |V(H_{i1} \cap H_{i2})| = 1 \), \( x_i, y_i \in H_{i1} \), and \( e \in H_{i2} \); and we choose \((H_{i1}, H_{i2})\) so that \( H_{i2} \) is minimal, and let \( w_i \in V(H_{i1} \cap H_{i2}) \).

For, otherwise, it follows from Menger’s theorem that \( H'_{i1} \) contains path \( Q_i \) from \( x_i \) to \( y_i \) and through \( e \). Let \( P'_i := a_iP_ix_iQ_iP_ib_i \). Then \( P' := (P - P_i) \cup P'_i \) shows that (i) holds.
Note that if \( w_i, w_j \) are defined and \( w_i = w_j \) then by the minimality of \( H_{i2}, H_{j2} \), we have \( H_{i2} = H_{j2} \).

(4) We may assume that \( w_1 \) and \( w_2 \) are defined and \( w_1 \neq w_2 \).

If \( x_i, y_i \) are defined for at most one \( i \) then, by (3), the separation \((H_{i2}, G - (H_{i2} - w_i))\) shows that (iii) holds. So we may assume that \( w_i, x_i, y_i \) are defined for \( i = 1, 2 \). If \( w_3, x_3, y_3 \) are not defined then we may assume \( w_1 \neq w_2 \) (or else the separation \((H_{12}, G - (H_{12} - w_1))\) shows that (iii) holds). So we may assume that \( w_3, x_3, y_3 \) are defined as well. Then by symmetry we may assume \( w_1 \neq w_2 \); for if \( w_1 = w_2 = w_3 \) then the separation \((H_{12}, G - (H_{12} - w_1))\) shows that (iii) holds.

By (4), \( H_P - (P - \{w_1, w_2\}) \) contains a path from \( w_1 \) to \( w_2 \), through \( e \), and internally disjoint from \( P \). So for \( \{i, j\} = \{1, 2\} \), \( H_P - P_3 \) contains a path \( Q_{ij} \) from \( x_i \) to \( y_j \), through \( e \), and internally disjoint from \( P \). Moreover, \( H_P - P_3 \) has a separation \((H_1, H_2)\) such that \( V(H_1 \cap H_2) = \{w_1, w_2\}, e \in H_2 \), and \( H_{11} \cup H_{12} \subseteq H_2 \).

(5) \( G \) has no \( P \)-bridge that is different from \( H_P \) and intersects both \( a_1P_1y_1 - y_1 \) and \( x_2P_2b_2 - b_2 \), or both \( a_2P_2y_2 - y_2 \) and \( x_1P_1b_1 - b_1 \).

For, suppose some \( P \)-bridge \( J \neq H_P \) of \( G \) intersects both \( a_1P_1y_1 - y_1 \) and \( x_2P_2b_2 - b_2 \). Then \( J \) contains a path \( Q \) from \( u \in V(a_1P_1y_1 - y_1) \) to \( v \in V(x_2P_2b_2 - b_2) \) and internally disjoint from \( P \). Now \( a_1P_1uQ\cap P_2b_2, a_2P_2x_2Q_21y_1P_1b_1, P_3 \) show that (i) holds. Similarly, by using \( Q_{12} \), (i) holds if some \( P \)-bridge of \( G \) (different from \( H_P \)) intersects both \( a_2P_2y_2 - y_2 \) and \( x_1P_1b_1 - b_1 \).

Case 1. \( w_3, x_3, y_3 \) are defined.

Then \( G[H' + \{x_i, y_j\}] \) has a path \( Q_{ij} \) from \( x_i \) to \( y_j \) for any \( 1 \leq i \neq j \leq 3 \). By (3), \( G \) has a separation \((K, L)\) such that \( V(K \cap L) = \{w_1, w_2, w_3\} \) and \( L = H_{12} \cap H_{22} \cap H_{32} \).

Suppose \( w_3 \notin \{w_1, w_2\} \). Then (5) holds for any \( i \neq j \). Therefore, if \( \{x_1, x_2, x_3\} \neq \{a_1, a_2, a_3\} \) or some \( P \)-bridge of \( G \) contains two of \( \{x_1, x_2, x_3\} \), then \( G \) has separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{x_1, x_2, x_3\}, A \subseteq G_1 \), and \( B \subseteq G_2 \); so (iv) holds. Thus we may assume that \( \{x_1, x_2, x_3\} = \{a_1, a_2, a_3\} \) and no \( P \)-bridge of \( G \) contains two of \( \{x_1, x_2, x_3\} \). Similarly, we may assume that \( \{y_1, y_2, y_3\} = \{b_1, b_2, b_3\} \), and no \( P \)-bridge of \( G \) contains two of \( \{y_1, y_2, y_3\} \).

Now, let \( G_1 = H_2, G_2 = B \), and \( G_3 = G - (G_1 - \{w_1, w_2, w_3\}) \). The we see that (vi) holds.

Thus, we may assume that by symmetry that \( w_3 = w_2 \). By the same argument as for (5), we may assume that no \( P \)-bridge of \( G \) intersects both \( a_1P_1y_1 - y_1 \) and \( x_3P_3b_3 - x_3 \) or both \( a_3P_3y_3 - y_3 \) and \( x_1P_1b_1 - x_1 \).

If no \( P \)-bridge of \( G \) intersecting \( P_1 \) intersects \( P_2 \cup P_3 \), then (v) holds with \( G_1 \) has the union of \( P_1 \cup H_{11} \) and all \( P \)-bridges of \( G \) (different from \( H_P \)) intersecting \( P_1, G_2 = H_2, \) and \( G_3 := G - G_1 - (G_2 - \{w_1, w_2\}) \). Thus by symmetry we may assume that \( G \) has a path \( Q \) from \( u_2 \in V(a_2P_2x_2) \) to \( u_1 \in V(a_1P_1x_1 - y_1) \cup V(a_3P_3x_3 - y_3) \), and we choose \( Q \) to minimize \( u_2P_2x_2 \). Let \( u_3 \in a_3P_3x_3 \) with \( u_3P_3x_3 \) minimal such that \( u_3 = a_3 \), or some \( P \)-bridge of \( G \) containing \( u_3 \) intersects \( a_1P_1y_1 - y_1 \cup (a_2P_2x_2 - y_2) \).

If \( G \) has a separation \((G_1, G_2)\) such that \( V(G_1 \cap G_2) = \{x_1, w_2, w_3\}, Q \cup A \subseteq G_1 \) and \( B \subseteq G_2 \), then (iv) holds. So we may assume that such a separation does not exist in \( G \). Then there exists a path \( R \) in \( G \) from \( r \in V(a_2P_2x_2 - u_2) \cup V(a_3P_3x_3 - u_3) \) to \( t \in V(x_1P_1b_1 - x_1) \) and internally disjoint from \( P \cup Q \). By symmetry, we may assume \( r \in a_2P_2u_2 - u_2 \).

When \( u_1 \in a_3P_3x_3 - y_3 \), the paths \( a_1P_1x_1Q_{13}y_3P_3b_3, a_2P_2RtP_1b_1, a_3P_3u_1Q_{u_2}P_2b_2 \) show
that (i) holds. So we may assume $u_1 \in a_1P_1x_1 - y_1$. Then $a_1P_1u_1Qu_2P_2b_2, a_2P_2RtP_1b_1, P_3$ contradict the choice of $P$ (that $H_P$ is maximal).

Case 2. $w_3, x_3, y_3$ are not defined.

Let $u \in V(P_3)$ with $a_3P_3u$ maximal such that $u = a_3$ or $u$ belongs to some $P$-bridge of $G$ intersecting $(a_1P_1x_1 - x_1) \cup (a_2P_2x_2 - x_2)$. Similarly, let $v \in V(P_3)$ with $b_3P_3v$ maximal such that $v = b_3$ or $v$ belongs to some $P$-bridge of $G$ intersecting $(y_1P_1b_1 - y_1) \cup (y_2P_2b_2 - y_2)$.

We may assume $\{x_1, x_2, u\} = \{a_1, a_2, a_3\}$ and $\{y_1, y_2, v\} = \{b_1, b_2, b_3\}$. For, otherwise, we may suppose $\{x_1, x_2, u\} \neq \{a_1, a_2, a_3\}$. If $G$ has no path from $a_3P_3u - u$ to $(x_1P_1b_1 - x_1) \cup (x_2P_2b_2 - x_2)$ and internally disjoint from $P$ then, by (5), $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{x_1, x_2, x_3\}, A \subseteq G_1$, and $B \subseteq G_2$, and (iv) holds. So we may assume that $G$ has a path $Q$ from $x \in V(a_3P_3u - u)$ to $y \in V(x_1P_1b_1 - x_1) \cup V(x_2P_2b_2 - x_2)$ and internally disjoint from $P$. Let $R$ be a path in $G$ from $u$ to $z \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2x_2 - x_2)$ and internally disjoint from $P$, and by symmetry we may assume that $z \in a_2P_2x_2 - x_2$. If $y \in x_2P_2b_2 - x_2$ then $P_1, a_2P_2RzP_3b_3, a_3P_3QyPb_2$ are three disjoint paths that contradict the choice of $P$ (with $H_P$ maximal). So $y \in x_1P_1b_1 - x_1$. Then $a_1P_1x_1Q_1y_2P_2b_2, a_1P_2zRzP_3b_3, a_3P_3QyPb_2$ show that (i) holds.

We may assume that some $P$-bridge of $G$ intersects both $P_2$ and $P_3$ and some $P$-bridge of $G$ intersects both $P_1$ and $P_3$. For, otherwise, we may assume by symmetry that no $P$-bridge of $G$ intersecting $P_3$ also intersects $P_1$. Let $G_1$ denote the union of $P_2 \cup P_3$, $H_3$, and all $P$-bridges of $G$ different from $H_P$ and intersecting $P_2 \cup P_3$. Then by (5) we see that $G_1, G_2, G_3$ satisfies (v).

Suppose $G$ has a $P$-bridge $J$ such that $J \cap P_1 \neq \emptyset$ for $i = 1, 2, 3$. Then $J \neq H_P$ as $w_3, x_3, y_3$ are not defined. So by (5) and by symmetry, we may assume that $V(J \cap P_1) = \{a_1\}$ and $V(J \cap P_2) = \{a_2\}$. Let $u \in V(J \cap P_3)$ with $a_3P_3u$ maximal. We may assume that $G$ has a path $Q$ from $x \in V(a_3P_3u - u)$ to $y \in V(P_1 - a_1) \cup V(P_2 - a_2)$; for otherwise $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{a_1, a_2, u\}, A \subseteq G_1$, and $B \subseteq G_2$, which implies (iv). Let $Q_i$ denote paths in $J$ from $u$ to $a_i, i = 1, 2$, that are internally disjoint from $P$. If $y \in P_2$ then $P_1, Q_2uP_3b_3, a_3P_3QyPb_2$ show that (i) holds; and if $y \in P_1$ then $Q_1uP_3b_3, Q_2, a_3P_3QyPb_2$ show that (i) holds.

So we may assume that no $P$-bridge of $G$ intersects $P_i$ for all $i = 1, 2, 3$. If all $P$-bridges of $G$ intersect $P_3$ in exactly one common vertex, say $z$, then we may assume $z \neq a_3$ (as $a_3 \neq b_3$); now $G$ has a separation $(G_1, G_2)$ such that $V(G_1 \cap G_2) = \{a_1, a_2, z\}, A \subseteq G_1$, and $B \subseteq G_2$, which implies (iv). So we may assume that $G$ has $P$-bridges $J_1$ and $J_2$ such that $J_1 \cap P_1 \neq \emptyset, J_2 \cap P_2 \neq \emptyset$, and there exists $u_1 \in J_1 \cap P_3$ and $u_2 \in J_2 \cap P_3$ with $u_1 \neq u_2$. By symmetry let $a_3, u_1, u_2, b_3$ occur on $P_3$ in order. Note that $J_1 \neq J_2$.

Let $v_1 \in V(J_1 \cap P_1)$ with $a_1P_1v_1$ maximal, and let $v_2 \in V(J_2 \cap P_2)$ with $v_2P_2b_2$ maximal. For $i = 1, 2$, let $Q_i$ be a path in $J_i$ from $u_i$ to $v_i$ and internally disjoint from $P$. If $v_1 \neq a_1$ and $v_2 \neq b_2$, then $Q_1a_2P_2Q_2a_3P_3b_3, a_3P_3u_1Q_1v_1P_1b_1$ show that (i) holds. So we may assume by symmetry that $v_1 = b_2$. We may modify $P_3$ if necessary to make $J_2$ maximal. Then no $P$-bridge of $G$ other than $J_2$ intersects both $a_3P_3u_2 - u_2$ and $u_2P_3b_3 - u_2$.

If there is no $P$-bridge of $G$ different from $J_2$ intersecting $u_2P_3b_3 - u_2$, then $G$ has a separation $(G_1, G_2)$ with $V(G_1 \cap G_2) = \{b_1, b_2, u_2\}, A \subseteq V(G_1)$, and $B \subseteq V(G_2)$; so (iv) holds.
Hence, we may assume that some $P$-bridge of $G$ different from $J_2$ intersects $u_2P_3b_3 - u_2$; hence, there is a path $R_2$ in $G$ from $s_2 \in V(u_2P_3b_3 - u_2)$ to $t_2 \in V(P_1 - b_1) \cup V(P_2 - b_2)$ and internally disjoint from $P$.

If $t_2 \in P_1 - b_1$ then $a_1P_1t_2R_2s_2P_3b_3, Q_{21}, a_3P_3w_2Q_2b_2$ show that (i) holds. So we may assume $t_2 \in P_2 - b_2$. Then $P_1, a_1P_2t_2R_2s_2P_3b_3, a_3P_3w_2Q_2b_2$ show that (i) holds.

As an application of Lemma 3.4 we prove the following lemma which will be used many times to deal with $(G/xy, u_1, u_2, A)$.

**Lemma 3.5.** Let $(G, u_1, u_2, A)$ be a quadruple and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose $G$ has a separation $(U_1, U_2)$ such that $|V(U_1 \cap U_2)| \leq 3$, $|V(U_1 \cap A)| \neq 0$, $u_1 \in U_1 - U_2$, $u_2 \in U_2 - U_1$, and $A \subseteq U_1$. Then one of the following holds.

(i) $(G, u_1, u_2, A)$ is feasible;

(ii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in G_1 - G_2$ and $A \cup \{a_{3-i}\} \subseteq G_2$;

(iii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 4$, $u_1, u_2 \in G_1 - G_2$, and $A \subseteq G_2$;

(iv) $(G, u_1, u_2, A)$ is an obstruction of type I or IV.

**Proof.** We may assume $|V(U_1 \cap U_2)| = 3$; as otherwise (ii) holds. So let $V(U_1 \cap U_2) = \{v_1, v_2, v_3\}$. If $V(U_1 \cap U_2) \subseteq A$ then $u_1$ and $u_2$ belong to different components of $G - A$; so (iii) holds. Thus we may assume that $v_3 \notin A$. Since $V(U_1 \cap U_2) \cap A \neq \emptyset$, we may assume that $v_1 = a_1$.

We may assume that $U_2$ has three independent paths from $u_2$ to $a_1, v_2, v_3$, respectively. Otherwise $U_2$ has a separation $(U_{21}, U_{22})$ such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$ and $\{a_1, v_2, v_3\} \subseteq U_{22}$. Now $(U_{21}, U_{22} \cup U_1)$ is a separation in $G$ showing that (ii) holds.

Suppose $v_2 \in A$. Without loss of generality, we may assume $v_2 = a_2$. Then $G$ has a topological $H$ rooted at $u_1, u_2, A$ iff $U_1 - \{a_1, a_2\}$ has three independent paths from $u_1$ to $a_3, a_4, v_3$, respectively. Thus (i) holds, or $U_1$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 4$, $a_1, a_2 \in U_{11} \cap U_{12}$, $u_1 \in U_{11} - U_{12}$ and $\{a_3, a_4, v_3\} \subseteq U_{12}$. Now $U_{11}, U_{12}$ show that $(G, u_1, u_2, A)$ is an obstruction of type I, and (iv) holds.

So we may assume that $v_2 \notin A$. Then $G$ has a topological $H$ rooted at $u_1, u_2, A$ iff $(U_1 - a_1) + v_2v_3$ has three independent paths from $u_1$ to $a_2, a_3, a_4$ and containing the edge $v_2v_3$. Let $U_1'$ be obtained from $(U_1 - a_1) + v_2v_3$ by duplicating $u_1$ twice, as $u_1', u_1''$. We wish to see if $U_1'$ has three disjoint paths from $\{u_1, u_1', u_1''\}$ to $\{a_2, a_3, a_4\}$ and containing $v_2v_3$. So we apply Lemma 3.4.

If Lemma 3.4(i) holds then $U_1'$ has three disjoint paths from $\{u_1, u_1', u_1''\}$ to $\{a_2, a_3, a_4\}$ and containing $v_2v_3$. So $(U_1 - a_1) + v_2v_3$ has three independent paths from $u_1$ to $a_2, a_3, a_4$ and containing the edge $v_2v_3$. Hence, $G$ has a topological $H$ rooted at $u_1, u_2, A$, and (i) holds.

Suppose Lemma 3.4(ii) holds. Then $U_1'$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $\{u_1, u_1', u_1''\} \subseteq U_{11}$, and $\{a_2, a_3, a_4\} \subseteq U_{12}$. If $v_2v_3 \in U_{12}$ then the separation
Let \( G \) shows that (ii) holds. If \( v_2v_3 \in U_{11} \) then the separation \( (G[U_{11} - \{u_1', u_2''\}], U_{12}) \) shows that (iii) holds. If Lemma 3.4(iii) holds then \( U'_{10} \) has a separation \( (U_{11}, U_{12}) \) such that \( |V(U_{11} \cap U_{12})| \leq 1 \), \( \{u_1, u_1', u_2''\} \subseteq U_{11} \) and \( v_1, v_2, v_3 \in U_{12} \). Now the separation \( (G[U_{11} - \{u_1', u_2''\}], G[V(U_{12})] \cup U_2) \) shows that then (ii) holds.

Suppose Lemma 3.4(iv) holds. Then \( U'_{10} \) has a separation \( (U_{11}, U_{12}) \) such that \( |V(U_{11} \cap U_{12})| = 3 \), \( \{u_1, u_1', u_2''\} \subseteq U_{11} \) and \( \{a_2, a_3, a_4\} \subseteq U_{12} \). If \( v_2v_3 \in U_{11} \) then \( G[V(U_{11}) - \{u_1', u_2''\}] + \{a_1\}, U_2, G[U_{12} + a_1] \) show that \( (G, u_1, u_2, A) \) is an obstruction of type I, and (iv) holds. If \( v_2v_3 \in U_{12} \) then the separation \( (G[V(U_{12} + a_1)], G[U_{11} + a_1] \cup U_2) \) shows that (iii) holds.

Since \( u_1' \) and \( u_2'' \) are duplicates of \( u_1 \), Lemma 3.4(v) cannot occur. So we may assume Lemma 3.4(vi) holds. Again, since \( u_1' \) and \( u_2'' \) are duplicates of \( u_1 \), \( U'_{10} \) is the edge disjoint union of graphs \( G_i \), \( 1 \leq i \leq 5 \), such that \( |V(G_i \cap G_j)| = 1 \) for \( i \in \{1, 2\} \) and \( j \in \{3, 4, 5\} \), \( G_1 \cap G_2 \subseteq G_3 \cup G_4 \cup G_5 \), \( G_i \cap G_j \subseteq G_1 \cup G_2 \) for \( 3 \leq i \neq j \leq 5 \), \( v_2v_3 \subseteq G_1 \), \( \{u_1, u_1', u_2''\} \subseteq G_2 \), and \( |V(G_j) \cap \{a_2, a_3, a_4\}| = 1 \) for \( j \in \{3, 4, 5\} \). Then \( G[G_2 - \{u_1', u_1''\} + a_1], U_2 \cup G[V(G_1 + a_1)], \{a_1\}, G_3, G_4, G_5 \) show that \( (G, u_1, u_2, A) \) is an obstruction of type IV, so (iv) holds.

As an easy corollary of Lemma 3.5, we can deal with obstructions of type VI.

**Corollary 3.6.** Let \( (G, u_1, u_2, A) \) be a quadruple, and let \( A := \{a_1, a_2, a_3, a_4\} \). Suppose there exist \( xy \in E(G) \) such that \( x, y \in V(G) - A - \{u_1, u_2\} \) and \((G/xy, u_1, u_2, A)\) is of type VI. Then one of the following holds.

(i) \((G, u_1, u_2, A)\) is feasible.

(ii) \(G\) has a separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| \leq 2\), and for some \( i \in \{1, 2\}\), \( u_i \in G_1 - G_2\), and \( A \cup \{v_3-i\} \subseteq L\).

(iii) \(G\) has a separation \((G_1, G_2)\) such that \( |V(G_1 \cap G_2)| \leq 4\), \( A \subseteq G_1\) and \( \{u_1, u_2\} \subseteq G_2 - G_1\).

(iv) \((G, u_1, u_2, A)\) is an obstruction of types I, IV, or VI.

**Proof.** Let \( G/xy \) be the edge-disjoint union of subgraphs \( U_1, U_2, A_1 \) such that \( |V(U_1 \cap A_1)| = 3\), \( |V(U_2 \cap A_1)| = 3\), \( |V(U_1 \cup U_2)| \subseteq A \cup V(A_1)|, |V(U_1 \cup U_2)| = 1\), \( A \subseteq A_1\), and \( u_1 \in U_1 - A_1\) and \( u_2 \in U_2 - A_1\). Let \( v \) denote the vertex of \( G/xy \) resulting from the contraction of \( xy \).

If \( v \notin V(U_i \cap A_1) \) for \( i = 1, 2 \) then we see that \((G, u_1, u_2, A)\) is an obstruction of type VI. Otherwise, we may assume by symmetry that \( v \in U_2 \cap A_1\). Now \((U_1, A_1 \cup U_2)\) is a separation which allows us use Lemma 3.5. So the assertion of the lemma holds.

## 4 Contraction critical quadruples

In this section we prove lemmas to be used to deal with contraction critical quadruples \((G, u_1, u_2, A)\): those such that for any \( xy \in E(G - (A \cup \{u_1, u_2\}))\), \((G/xy, u_1, u_2, A)\) is an obstruction.

**Lemma 4.1.** Let \( (G, u_1, u_2, A) \) be a quadruple, and let \( A := \{a_1, a_2, a_3, a_4\} \). Suppose there exist \( xy \in E(G - A - \{u_1, u_2\})\) such that \((G/xy, u_1, u_2, A)\) is of type I. Then one of the following holds.

\(\)
(i) \((G, u_1, u_2, A)\) is feasible.

(ii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 2\), \(u_1 \in G_1 - G_2\), and \(A \cup \{u_2\} \subseteq G_2\).

(iii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 4\), \(\{u_1, u_2\} \subseteq G_1 - G_2\), and \(A \subseteq G_2\).

(iv) \((G, u_1, u_2, A)\) is an obstruction of types I, II or IV.

Proof. Let \(G/xy\) be the edge disjoint union of \(U_1 \cup U_2 \cup A_1\) such that \(V(U_1 \cap U_2) = \{a_1, a_2\}\), \(V(U_1 \cap A_1) = \{a_1, a_2, v_1\}\), \(V(U_2 \cap A_1) = \{a_1, a_2, v_2, v_3\}\), \(V(U_1 \cap U_2) = \{a_1, a_2\}, u_1 \in U_1 - A_1\), and \(u_2 \in U_2 - A_1\). Let \(v\) denote the vertex resulting from the contraction of \(x, y\).

We may assume \(v = v_1\). For, suppose \(v \neq v_1\). Then \((U_1, G - (U_1 - \{a_1, a_2, v_1\}))\) is a separation in \(G\) which allows us to apply Lemma 3.5; so (i) or (ii) or (iii) or (iv) holds.

Let \(U_1', A_1'\) be obtained from \(U_1, A_1\), respectively, by uncontracting \(v\) to \(xy\). Note the symmetry between \(U_1'\) and \(U_2\). We choose \(U_1', U_2, A_1\) so that, subject to \(a_1, a_2 \in U_1' \cap U_2, U_1' \cup U_2\) is maximal. Then \(xy, v_2v_3 \notin A_1'\). Moreover we may assume \(a_3a_4 \notin A_1'\); otherwise, \((G - a_3a_4, G[\{a_3, a_4\}])\) shows that (iii) holds.

We may assume that for some permutation \(ij\) of \(\{1, 2\}\), \(U_1' - a_j\) has three independent paths from \(u_1\) to \(a_i, x, y\), respectively, and \(U_2 - a_i\) has three independent paths from \(u_2\) to \(a_j, v_2, v_3\), respectively. To see this, let \(H\) be obtained from \(U_1' \cup U_2\) by duplicating each \(u_i\) twice with \(u_i', u_i''\). If \(H\) contains six disjoint paths from \(\{u_i, u_i', u_i'' : i = 1, 2\}\) to \(\{a_1, a_2, v_2, v_3, x, y\}\) then the desired permutation and six paths exist. So we may assume by Menger’s theorem that \(H\) has a separation \((H_1, H_2)\) such that \(|V(H_1 \cap H_2)| \leq 5\), \(\{u_i, u_i', u_i'' : i = 1, 2\} \subseteq V(H_1)\) and \(\{a_1, a_2, v_2, v_3, x, y\} \subseteq V(H_2)\). It is easy to see that \(|V(H_1 \cap H_2) \cap V(U_1')| \leq 2\), or \(|V(H_1 \cap H_2) \cap V(U_2)| \leq 2\), or \(|V(H_1 \cap H_2) \cap V(\{u_i\})| = 3\) and \(V(H_1 \cap H_2) \cap V(U_1') \cap \{a_1, a_2\} \neq \emptyset\), or \(|V(H_1 \cap H_2) \cap V(U_2)| = 3\) and \(V(H_1 \cap H_2) \cap V(U_2) \cap \{a_1, a_2\} \neq \emptyset\). If the first two cases occur, \(V(H_1 \cap H_2) \cap V(U_1') \leq 2\) or \(|V(H_1 \cap H_2) \cap V(U_2)| \leq 2\) then (ii) holds. If the next two cases occur, then by Lemma 3.5 the assertion of the lemma holds.

Let \(J\) denote the union of the six paths in \(U_1' - a_j\) and \(U_2 - a_i\). If \(A_1^* := (A_1' - \{a_1, a_2\}) + \{a_3a_4, v_2v_3, xy\}\) contains a cycle \(C\) through \(\{a_3a_4, v_2v_3, xy\}\) then \(C - \{a_3a_4, v_2v_3, xy\}\) and \(J\) form a topological \(H\) rooted at \(u_1, u_2, A\), and (i) holds. So we may assume that such a cycle \(C\) does not exist in \(A_1^*\). Then by Lemma 3.3, we have three cases to consider.

In the first case, \(A_1^*\) has a separation \((A_{11}, A_{12})\) such that \(|V(A_{11} \cap A_{12})| \leq 1\) and \(|E(A_{11}) \cap \{a_3a_4, v_2v_3, xy\}| = 1\). If \(xy \in A_{11}\), then \(U_1' \cup G[V(A_{11}) + \{a_1, a_2\}] + U_2\) and \(G[V(A_{12}) + \{a_1, a_2\}]\) show that \((G, u_1, u_2, A)\) is an obstruction of type I. If \(v_2v_3 \in A_{11}\) then \(U_1' \cup G[V(A_{11}) + \{a_1, a_2\}]\) show that \((G, u_1, u_2, A)\) is an obstruction of type I. If \(a_3a_4 \in A_{11}\) then \(G[V(A_{11}) + \{a_1, a_2\}] + U_1' \cup U_2 \cup G[V(A_{12}) + \{a_1, a_2\}]\) show that (iii) holds.

In the second case, \(A_1^*\) has a separation \((A_{11}, A_{12})\) such that \(|V(A_{11} \cap A_{12})| = 2\) and \(|E(A_{11}) \cap \{a_3a_4, v_2v_3, xy\}| = 1\). If \(xy \in A_{11}\) or \(v_2v_3 \in A_{11}\), then \(U_1' \cup G[V(A_{11}) + \{a_1, a_2\}] \cup U_2\) contradicts the maximality of \(U_1' \cup U_2\). So \(a_3a_4 \in A_{11}\). Then \(G[V(A_{11}) + \{a_1, a_2\}] + U_1' \cup U_2 \cup G[V(A_{12}) + \{a_1, a_2\}]\) shows that (iii) holds.

Therefore, we may assume that \(A_1^* - \{a_3a_4, v_2v_3, xy\}\) is not connected. Since \(a_3a_4, v_2v_3, xy \notin A_1'\), \(A_1'\) consists of disjoint subgraphs \(A_{11}, A_{12}\) such that each of \(a_3a_4, v_2v_3, xy\) has one end in \(A_{11}\) and the other in \(A_{12}\). Now \(U_1', U_2, A_{11}, A_{12}, \{a_1\}, \{a_2\}\) show that \((G, u_1, u_2, A)\) is an obstruction of type II.
Lemma 4.2. Let \((G, u_1, u_2, A)\) be a quadruple with \(A = \{a_1, a_2, a_3, a_4\}\). Suppose there exist \(xy \in E(G - A - \{u_1, u_2\})\) such that \((G/xy, u_1, u_2, A)\) is of type II. Then one of the following holds.

(i) \((G, u_1, u_2, A)\) is feasible.

(ii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 2\), \(u_1 \in G_1 - G_2\), and \(A \cup \{u_2\} \subseteq G_2\).

(iii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 4\), \(\{u_1, u_2\} \subseteq G_1 - G_2\), and \(A \subseteq G_2\).

(iv) \((G, u_1, u_2, A)\) is an obstruction of types I, II, III, IV.

Proof. Let \(G/xy\) be the edge-disjoint union of \(U_1, U_2, A_1, A_2, A_3\) such that \(V(U_1 \cap A_i) = \{v_i\}\) for \(i = 1, 2\) and \(V(U_1 \cap A_3) = \{v_3, v_4\}\). \(V(U_2 \cap A_i) = \{w_i\}\) for \(1 \leq i \leq 3\), \(V(U_1 \cap U_2) \subseteq \{v_1, v_2, v_3, v_4\} \cap \{w_1, w_2, w_3\}\). \(V(A_i \cap A_j) \subseteq V(U_1 \cap U_2)\) for \(1 \leq i \neq j \leq 3\), \(u_i \in U_1 - (A_1 \cup A_2 \cup A_3 \cup A_4)\) for \(i = 1, 2\), \(a_i \in A_1\) for \(i = 1, 2\) and \(a_3, a_4 \in A_3\), if \(|V(A_i)| = 1\) then \(A_i \subseteq A_j\) for all \(j \neq i\), and if \(w_3 \in A\) then \(w_3 \in U_2 \cap A_i\) for \(i = 1, 2, 3\).

Let \(v\) denote the vertex resulting from the contraction of \(xy\). If \(v \notin \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\}\), then \((G, u_1, u_2, A)\) is also an obstruction of type II. So we may assume that \(v \in \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\}\). By symmetry, it suffices to consider four cases: \(v = v_1\), \(v = v_4\), \(v = w_1\), and \(v = w_3\).

Case 1. \(v = v_1\).

Then by Lemma 3.5 we may assume that \(\{w_1, w_2, w_3, v_2\} \cap A = \emptyset\). Let \(U'_1, A'_1\) be obtained from \(U_1, A_1\), respectively, by uncontracting \(v\) to \(xy\).

We may assume that \(U_2\) has three independent paths from \(u_2\) to \(w_1, w_2, w_3\), respectively. Otherwise, \(U_2\) has a separation \((U_{21}, U_{22})\) such that \(|V(U_{21} \cap U_{22})| \leq 2\), \(u_2 \in U_{21} - U_{22}\), and \(\{w_1, w_2, w_3\} \subseteq U_{22}\). Now the separation \((U_{21}, U_{22} \cup U'_1 \cup A'_1 \cup A_2 \cup A_3)\) in \(G\) shows that (ii) holds.

We may also assume that \(A'_1\) has disjoint paths from \(\{x, y\}\) to \(\{a_1, w_1\}\). For, otherwise, \(A'_1\) has a separation \((A_{11}, A_{12})\) such that \(|V(A_{11} \cap A_{12})| \leq 1\), \(\{x, y\} \subseteq A_{11}\), and \(\{a_1, w_1\} \subseteq A_{12}\). Now \(U_1 \cup A_{11}, U_2, A_{12}, A_2, A_3\) show that (ii) holds, or \((G, u_1, u_2, A)\) is also an obstruction of type II.

We may assume that for each \(i \in \{3, 4\}\), \(A_3\) has disjoint paths from \(\{w_3, v_i\}\) to \(\{a_3, a_4\}\), which avoids \(v_{7-i}\) if \(v_{7-i} \notin A\). For, suppose no such disjoint paths exist. Then \(A_3\) has a separation \((A_{31}, A_{32})\) such that \(|V(A_{31} \cap A_{32})| \leq 1\) (if \(v_{7-i} \in A\)), \(|V(A_{31} \cap A_{32})| \leq 2\) and \(v_{7-i} \in A_{31} \cap A_{32}\) (when \(v_{7-i} \notin A\)), \(\{w_3, v_i\} \subseteq A_{31}\), and \(\{a_3, a_4\} \subseteq A_{32}\). Now the separation \((G[V(A_{32} \cup \{a_1, a_2\}] \cup U_1 \cup U'_2 \cup A'_1 \cup A_2 \cup G[V(A_{31} \cup \{a_1, a_2\})])\) shows that (iii) holds.

We may assume that \(A_2\) has a path from \(w_2\) to \(a_2\) which avoids \(v_2\) when \(v_2 \neq a_2\). Otherwise, \(A_2\) has a separation \((A_{21}, A_{22})\) such that \(A_{21} \cap A_{22} = \emptyset\) (when \(v_2 = a_2\) or \(A_{21} \cap A_{22} = \{v_2\}\) (when \(v_2 \neq a_2\), \(a_2 \in A_{21}\), and \(w_2 \in A_{22}\). Now the separation \((U_2 \cup A_{22} \cup U'_2 \cup A'_1 \cup A_{22} \cup A_3)\) shows that (iii) holds.

We may assume that if \(\{v_3, v_4\} \neq \{a_3, a_4\}\) then \(v_4 \notin \{a_3, a_4\}\).

Now if \(U'_1 - (A - \{v_3\})\) contains disjoint paths from \(u_1\) to \(x, y, v_3\), respectively, then (i) holds. Thus we may assume that \(U'_1 - (A - \{v_3\})\) has a separation \((U_{11}, U_{12})\) such that
$|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{x, y, v_3\} \subseteq U_{12}$. Choose this separation to minimize $U_{12}$.

We may assume $|V(U_{11} \cap U_{12})| = 2$. For, otherwise, we may assume $v_2, v_4 \in N(U_{11} - U_{12})$ (or else (ii) holds). Recall that $v_2 \notin A$. By Lemma 3.5 we may also assume $v_4 \notin A$; so $v_2, v_4 \in U_{11} - U_{12}$. Then $G[U_{11} + v_4], U_2, A_2, G[U_{12} + v_4] \cup A_1 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type III. So let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$.

By the minimality of $U_{12}$, $U_{12} - A$ contains disjoint paths from $\{s_1, s_2\}$ to $\{x, y\}$. For, otherwise, $U_{12} - A$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 1$, $\{s_1, s_2\} \subseteq K$, and $\{x, y\} \subseteq L$. Then $(U_{11} \cup G[K + v_3], G[L + v_3])$ is a separation in $U_{11} - (A - \{v_3\})$, contradicting the minimality of $U_{12}$.

Suppose $v_3 \notin N(U_{11} - U_{12})$. If $v_2 \notin U_{11} - U_{12}$, then (ii) holds. So we may assume that $v_2 \notin U_{11} - U_{12}$. Then $U_{11}, U_2, A_2, G[U_{12} + v_3] \cup A_1 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type III. So we may assume $v_4 \in N(U_{11} - U_{12})$.

We may assume that $G[U_{11} + v_4]$ has three independent paths from $u_1$ to $s_1, s_2, v_4$, respectively. Otherwise, $G[U_{11} + v_4]$ has a separation $(K, L)$ such that $|V(K \cap L)| \leq 2$, $u_1 \in K - L$ and $\{s_1, s_2, v_4\} \subseteq L$. If $v_2 \notin K - L$ or $|V(K \cap L)| \leq 1$ then (ii) holds. So assume $v_2 \in K - L$ and $|V(K \cap L)| = 2$. Then $K, U_2, L \cup G[U_{12} + v_4] \cup A_1 \cup A_2 \cup A_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type III.

We may assume $v_4 \notin A$. For, otherwise, we have $v_3, v_4 \in A$. If $v_2 \notin U_{11} - U_{12}$ then $(G[U_{11} + v_4], G[U_{12} + v_3] \cup U_2 \cup A_1 \cup A_2 \cup A_3)$ allows us to apply Lemma 3.5; so the assertion of the lemma holds. So we may assume $v_2 \in U_{11} - U_{12}$. Then $U_1$ has three independent paths from $u_1$ to $x, y, v_4$, respectively; and (i) holds.

Thus we may assume $v_4 \notin A$, and hence $v_4 \in U_{11} - U_{12}$. So $U_{11}$ has three independent paths from $u_1$ to $s_1, s_2, v_4$, respectively; thus $U_1 - A$ has three independent paths from $u_1$ to $x, y, v_4$, respectively.

If $A_3' := A_3 - (\{v_3\} - A)$ has disjoint paths from $\{v_3, v_4\}$ to $\{a_3, a_4\}$, then (i) holds. So we may assume that $A_3'$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 1$, $\{v_3, v_4\} \subseteq A_{31}$ and $\{a_3, a_4\} \subseteq A_{32}$. Now $V(A_{32}) = \{v_3\} \subseteq \{a_3, a_4\}$; otherwise (iii) holds. If $v_2 \notin U_{11} - U_{12}$ then $U_{11}, A_3, U_2, G[U_{12} + v_3] \cup A_1 \cup A_2$ show that $(G, u_1, u_2, A)$ is an obstruction of type III. So assume $v_2 \in U_{11} - U_{12}$. Then $U_{11}, U_2, A_1 \cup G[U_{12} + v_3], A_2, A_3 - v_3$ show that $(G, u_1, u_2, A)$ is an obstruction of type II.

**Case 2.** $v = v_4$.

Let $U_1', A_3'$ be obtained from $U_1, A_3$, respectively, by uncontracting $v$ to $xy$. By Lemma 3.5, we may assume $\{v_1, v_2, w_1, w_2, w_3\} \cap A = \emptyset$.

We may assume that $A_3'$ has three disjoint paths from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For, if such paths do not exist, then $A_3'$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 2$, $\{v_3, x, y\} \subseteq A_{31}$, and $\{a_3, a_4, w_3\} \subseteq A_{32}$. Now $U_1' \cup A_{31}, U_2, A_1, A_2, A_{32}$ show that $(G, u_1, u_2, A)$ is an obstruction of type II.

We may assume that $U_2$ has three independent paths from $v_2$ to $w_1, w_2, w_3$, respectively; or else (ii) holds. Also we may assume that, for $i = 1, 2, A_i$ has a path from $w_i$ to $a_i$; otherwise (ii) holds.

Thus if $U_1'$ has three independent paths from $u_1$ to $v_3, x, y$, respectively, then (i) holds. So
we may assume that \( U'_1 \) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11}\cap U_{12})| \leq 2\), \( u_1 \in U_{11} - U_{12} \), and \( \{v_3, x, y\} \subseteq U_{12} \).

If \( v_1, v_2 \notin U_{11} - U_{12} \) then (ii) holds. So we may assume that \( v_1 \in U_{11} - U_{12} \). If \( v_2 \notin U_{11} - U_{12} \) then \( U_{11}, U_2, A_1, U_{12} \cup A_2 \cup A'_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type III. So we assume that \( v_2 \in U_{11} - U_{12} \). Then \( U_{11}, U_2, A_1, A_2, A'_3 \cup U_{12} \) show that \((G, u_1, u_2, A)\) is an obstruction of type II.

Case 3. \( v = w_3 \).

Let \( U'_2, A'_3 \) be obtained from \( U_2, A_3 \), respectively, by uncontracting \( v \) to \( xy \). Note the symmetry between \( U_1 \) and \( U'_2 \). We choose \( U_1, U'_2, A_1, A_2, A_3 \) to maximize \( U_1 \cup U'_2 \).

We may assume that \( A'_3 \) contains three disjoint paths: one from \( \{x, y\} \) to \( \{v_3, v_4\} \) and the other two from \( \{a_3, a_4\} \) to \( \{v_3, v_4, x, y\} \). For, suppose not. Then \( A''_3 := A'_3 + \{a_3a_4, v_3v_4, xy\} \) contains no cycle through \( S := \{a_3a_4, v_3v_4, xy\} \). So we may apply Lemma 3.3. First, suppose \( A''_3 \) has a separation \((A_{31}, A_{32})\) such that \(|V(A_{31} \cap A_{32})| \leq 1\) and and \(|E(A_{32}) \cap S| = 1\). If \( xy \in A_{32} \) or \( v_3v_4 \in A_{32} \) then we see that \((G, u_1, u_2, A)\) is an obstruction of type II; and if \( a_3a_4 \in A_{32} \) then we see that (iii) holds. Now, suppose \( A''_3 \) has a separation \((A_{31}, A_{32})\) such that \(|V(A_{31} \cap A_{32})| = 2\), \(|E(A_{32}) \cap S| = 1\), and \(|V(A_{32})| \geq 3\). Then by the maximality of \( U_1 \cup U'_2 \), we see that \( a_3a_4 \in S \), which shows (iii) holds. We may thus assume that \( S \) is an edge cut of \( A''_3 \). In this case, \((G, u_1, u_2, A)\) is an obstruction of type IV.

We may assume that for any \( i \in \{1, 2\} \), \( U'_2 - (A - \{w_i\}) \) contains three independent paths from \( u_2 \) to \( w_i, x, y \), respectively. For, suppose not. Then \( U'_2 - (A - \{w_i\}) \) has separation \((U_{21}, U_{22})\) such that \(|V(U_{21} \cap U_{22})| \leq 2\), \( u_2 \in U_{21} - U_{22} \), and \( \{w_i, x, y\} \subseteq U_{22} \). Choose this separation to minimize \( U_{22} \). We may assume \( w_{3-i} \in N(U_{21} - U_{22}) \) and \(|V(U_{21} \cap U_{22})| = 2\); or else (ii) holds. Then by Lemma 3.5, we may assume \( w_{3-i} \notin A \) (and hence, we may also assume that \( v_{3-i} \notin A \)). So \( w_{3-i} \in U_{21} - U_{22} \). By the minimality of \( U_{22} \) there are disjoint paths in \( U_{22} - A \) from \( V(U_{21} \cap U_{22}) \) to \( \{x, y\} \). We may further assume that \( U_{21} \) has three independent paths from \( u_1 \) to \( V(U_{21} \cap U_{22}) \); for otherwise \( U_{21} \) has a separation \((K, L)\) such that \(|V(K \cap L)| \leq 2\), \( U_{11} \cap U_{12} \subseteq L \), and \( u_2 \in K - L \), which gives the separation \((L, G - (L - K))\) in \( G \) showing that (ii) holds. Thus \( U'_2 - (A - \{w_{3-i}\}) \) has three independent paths from \( u_2 \) to \( w_{3-i}, x, y \), respectively. If \( U_1 \) contains three independent paths from \( u_1 \) to \( v_1, v_3, v_4 \), respectively, then (i) holds. So we may assume that \( U_1 \) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 2\), \( u_1 \in U_{11} - U_{12} \), and \( \{v_1, v_3, v_4\} \subseteq U_{12} \). we may assume \(|V(U_{11} \cap U_{12})| = 2\) and \( v_i \in U_{11} - U_{12} \); or else (ii) holds. Then \( U_{11}, U_2, A_1, U_{12} \cup U_{22} \cup A_1 \cup A'_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type III.

Similarly, we may assume that for any \( i \in \{1, 2\} \), \( U_i - (A - \{v_i\}) \) contains three independent paths from \( u_1 \) to \( v_1, v_3, v_4 \), respectively. Now it is easy to see that (i) holds.

Case 4. \( v = w_1 \).

Let \( U'_2, A'_1 \) be obtained from \( U_2, A_1 \), respectively, by uncontracting \( v \) to \( xy \).

We may assume that \( A'_1 \) has disjoint paths from \( \{v_1, a_1\} \) to \( \{x, y\} \). For otherwise, \( A'_1 \) has a separation \((A_{11}, A_{12})\) such that \(|V(A_{11} \cap A_{12})| \leq 1\), \( \{v_1, a_1\} \subseteq A_{11} \) and \( \{x, y\} \subseteq A_{12} \). Now \( U_{11}, U_2 \cup A_{12}, A_{11}, A_2, A_3 \) show that \((G, u_1, u_2, A)\) is an obstruction of type II.

Subcase 4.1. \( U_1 - \{v_2\} \cap A \) has three independent paths from \( u_1 \) to \( v_1, v_3, v_4 \), respectively. We may assume that \( A'_3 := A_3 - \{w_3\} - A \) has disjoint paths from \( \{v_3, v_4\} \) to \( \{a_3, a_4\} \).
For, otherwise, $A_3$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 2$, $w_3 \in A_{31} \cap A_{32}$ if $w_3 \notin A$, $\{v_3, v_4\} \subseteq A_{31}$, and $\{a_3, a_4\} \subseteq A_{32}$. Then the separation $(G[A_{32} + \{a_1, a_2\}], A_{31} \cup U_1 \cup U_2 \cup A'_1 \cup A_2)$ show that (iii) holds.

If $U_2 - (A - \{w_2\})$ has three independent paths from $u_{02}$ to $w_{02}, x, y$, respectively, then (i) holds. So we may assume that $U_2 - (A - \{w_2\})$ has a separation $(U_{21}, U_{22})$ such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_{2} \in U_{21} - U_{22}$, and $\{w_2, x, y\} \subseteq U_{22}$. We choose this separation to minimize $U_{22}$.

We may assume $w_3 \in N(U_{21} - U_{22})$, or else (ii) holds. Thus we may assume by Lemma 3.5 that $w_3 \notin A$ and $V(U_{21} \cap U_{22}) \cap A = \emptyset$; so $w_3 \in U_{21} - U_{22}$. By the minimality of $U_{22}$, $U_{22} - \{w_2\} \cap A$ contains disjoint paths from $V(U_{21} \cap U_{22})$ to $\{x, y\}$. Thus, $U_2 - \{w_2\} \cap A$ has three independent paths from $u_{02}$ to $w_3, x, y$, respectively.

Suppose for some $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has three independent paths from $u_1$ to $v_1, v_2, v_i$, respectively. If $A''_3 := A_3 - (\{v_{3-i}\} - A)$ has disjoint paths from $\{v_i, w_3\}$ to $\{a_3, a_4\}$, then (i) holds. So we may assume that $A_3$ has a separation $(A_{31}, A_{32})$ such that $|V(A_{31} \cap A_{32})| \leq 1$ (or $|V(A_{31} \cap A_{32})| \leq 2$ and $v_{3-i} \in A_{31} \cap A_{32}$), $\{v_i, w_3\} \subseteq A_{31}$, and $\{a_3, a_4\} \subseteq A_{32}$. Now the separation $(G[A_{32} + \{a_1, a_2\}], U_1 \cup U_2 \cup A'_1 \cup A_2 \cup G[A_{31} + \{a_1, a_2\}])$ show that (iii) holds.

Thus may assume that for any $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has no three independent paths from $u_1$ to $v_1, v_2, v_i$, respectively. Then for any $i \in \{3, 4\}$, $U_1 - (A - \{v_i\})$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_2, v_i\} \subseteq U_{12}$. If $v_{7-i} \in A$ then the separation $(G[U_{11} + v_{7-i}], G[U_{12} + v_{7-i}] \cup U_2 \cup A'_1 \cup A_2 \cup U_3)$ and Lemma 3.5 imply the assertion. So we may assume $v_3, v_4 \notin A$.

Clearly, $U_1 + \{v, v_3, v_4\}$ has no three independent paths from $u_1$ to $v_1, v_2, v$, respectively. So $U_1 + \{v, v_3, v_4\}$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_2, v\} \subseteq U_{12}$. If $v \notin U_{11} \cup U_{12}$ then $(U_{11}, G - (U_{11} - U_{12}))$ shows that (ii) holds. If $v \in U_{11} \cup U_{12}$ then $U_{11} - v, U_{21}, A_3, (U_{12} - v) \cup A'_1 \cup A_2$ show that $(G, u_1, w_2, A)$ is an obstruction of type IV.

Subcase 4.2. $U_1 - \{v_2\} \cap A$ has no three independent paths from $u_1$ to $v_1, v_3, v_4$, respectively.

Then $U_1 - \{v_2\} \cap A$ has a separation $(U_{11}, U_{12})$ such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in U_{11} - U_{12}$, and $\{v_1, v_3, v_4\} \subseteq U_{12}$. Choose this separation so that $U_{12}$ is minimal.

We may assume $|V(U_{11} \cap U_{12})| = 2$ and $v_2 \in N(U_{11} - U_{12})$; otherwise (ii) holds. Let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$. By Lemma 3.5, we may assume $\{s_1, s_2, v, w_2\} \cap A = \emptyset$. Thus $v_2 \in U_{11} - U_{12}$.

We may further assume that $U_{11}$ has three independent paths from $u_1$ to $s_1, s_2, v_2$, respectively; otherwise we have (ii). By the minimality of $U_{12}$, for any $i \in \{3, 4\}$, $U_{12} - (A - v_i)$ has disjoint paths from $\{s_1, s_2\}$ to $\{v_1, v_i\}$. So for any $i \in \{3, 4\}$, $U_1 - (A - v_i)$ has three independent paths from $u_1$ to $v_1, v_2, v_i$, respectively.

We may also assume that $U_2$ has three independent paths from $u_2$ to $x, y, w_3$, respectively. For, suppose not. Then $U_2$ has a separation $(U_{21}, U_{22})$ such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in U_{21} - U_{22}$ and $\{x, y, w_3\} \subseteq U_{22}$. If $u_2 \notin U_{21} - U_{22}$ then (ii) holds. So assume $u_2 \in U_{21} - U_{22}$. Then $U_{11}, U_{21}, A_2, U_{12} \cup U_{22} \cup A'_1 \cup A_3$ show that $(G, u_1, w_2, A)$ is an obstruction of type III.

Suppose $\{v_3, v_4\} = \{a_3, a_4\}$. If $A_3$ has a path from $w_3$ to $v_4$ then (i) holds. So we may assume that $A_3$ has a separation $(A_{31}, A_{32})$ such that $A_{31} \cap A_{32} = \{v_3\}$, $w_3 \in A_{32}$, and
So we may assume that \( v_4 \notin A \). If \( A_4 - v_4 \) has disjoint paths from \( \{v_3, w_3\} \) to \( \{a_3, a_4\} \) then (i) holds. So we may assume that \( A_4 \) has a separation \( (A_{31}, A_{32}) \) such that \(|V(A_{31} \cap A_{32})| \leq 2\), \( v_4 \in A_{31} \cap A_{32}, \{v_3, w_3\} \subseteq A_{31} \), and \( a_3, a_4 \) \( \subseteq A_{32} \). Now the separation \( (G[A_{32} + \{a_1, a_2\}], U_1 \cup U_2 \cup A_1' \cap A_2 \cup A_{31}) \) shows that (iii) holds.

**Lemma 4.3.** Let \( (G, u_1, u_2, A) \) be a quadruple, and let \( A := \{a_1, a_2, a_3, a_4\} \). Suppose there exists \( xy \in E(G - A - \{u_1, u_2\}) \) such that \( (G/xy, u_1, u_2, A) \) is of type III. Then one of the following holds.

- (i) \((G, u_1, u_2, A)\) is feasible.
- (ii) \( G \) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 2\), \( u_1 \in G_1 - G_2 \), and \( A \cup \{u_2\} \subseteq G_2 \).
- (iii) \( G \) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 4\), \( \{u_1, u_2\} \subseteq G_1 - G_2 \), and \( A \subseteq G_2 \).
- (iv) \((G, u_1, u_2, A)\) is an obstruction of types I, II, III, IV, \( V \).

*Proof.* Let \( G/xy \) be the edge disjoint union of \( U_1, U_2, A_1, A_2 \) such that \( V(U_1 \cap A_1) = \{v_1\} \)
and \( V(U_2 \cap A_1) = \{w_1\} \), \( V(U_1 \cap A_2) = \{v_2, v_3\} \)
and \( V(U_2 \cap A_2) = \{w_2, w_3\} \), \( V(U_1 \cup U_2) \subseteq \{(v_1 \cap w_1) \cup \{v_2, v_3\} \cap \{w_2, w_3\}\} \), \( a_1 \in A_1, a_2, a_3, a_4 \in A_2 \), and \( u_i \in U_i - (A_1 \cup A_2) \) for \( i = 1, 2 \).

Let \( v \) denote the vertex resulting from the contraction of \( xy \). If \( v \notin \{v_1, v_2, v_3, w_1, w_2, w_3\} \)
then \((G, u_1, u_2, A)\) is an obstruction of type III. So we may assume by symmetry that \( v = v_1 \) or \( v = v_2 \). By Lemma 3.5 we may assume that \( \{w_1, w_2, w_3\} \cap A = \emptyset \).

We may assume that \( U_2 \) has three independent paths from \( u_2 \) to \( w_1, w_2, w_3 \), respectively; for otherwise (ii) holds.

**Case 1.** \( v = v_1 \).

Let \( U'_1, A'_1 \) be obtained from \( U_1, A_1 \), respectively, by uncontracting \( v \) to \( xy \). We may assume that \( A'_1 \) has disjoint paths from \( \{x, y\} \) to \( \{a_1, w_1\} \). Otherwise, \( A'_1 \) has a separation \((A_{11}, A_{12})\)
such that \(|V(A_{11} \cap A_{12})| \leq 1\), \( \{x, y\} \subseteq A_{11} \), and \( \{a_1, w_1\} \subseteq A_{12} \). Now \( U'_1 \cup A_{11}, U_2, A_{12}, A_2 \)
show that \((G, u_1, u_2, A)\) is an obstruction of type III.

We may assume that for some \( i \in \{2, 3\} \), \( U'_i - (A - v_i) \) has three independent paths from \( u_1 \) to \( x, y, v_1 \), respectively. For, suppose not. Then \( U'_i - (A - \{v_2\}) \) has a separation \((U_{11}, U_{12})\)
such that \(|V(U_{11} \cap U_{12})| \leq 2\), \( u_1 \in U_{11} - U_{12} \), and \( \{x, y, v_2\} \subseteq U_{12} \). Choose \( U_1 \) to minimize \( U_{12} \). Then \( v_3 \in N(U_{11} - U_{12}) \); otherwise (ii) holds. So we may assume \( v_3 \notin A \) by Lemma 3.5; hence \( v_3 \notin U_{11} - U_{12} \). Moreover, we may assume \( U_{11} \) has three independent paths from \( u_1 \) to \( V(U_{11} \cup U_{12}) \cup \{v_3\} \); otherwise (ii) holds. Also by Lemma 3.5 we may assume \( v_2 \notin A \) if \( v_2 \in U_{11} \cup U_{12} \). So by the minimality of \( U_{12} \), \( U_{12} - A \) contains disjoint paths from \( V(U_{11} \cup U_{12}) \) to \( \{x, y\} \). So \( U'_1 - (A - \{v_3\}) \) has three independent paths from \( u_1 \) to \( x, y, v_3 \), respectively.

Thus we may assume that \( U'_1 - (A - v_2) \) has three independent paths from \( u_1 \) to \( x, y, v_2 \), respectively. If \( A_2 - \{v_3\} \) \( \notin A \) has three disjoint paths from \( \{a_2, a_3, a_4\} \) to \( \{v_2, w_2, w_3\} \) then (i) holds. So we may assume that \( A_2 \) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| \leq 2\),
\{a_2, a_3, a_4\} \subseteq A_{21}, \text{ and } \{v_2, w_2, w_3\} \subseteq A_{22}, \text{ or } |V(A_{21} \cap A_{22})| \leq 3, \ v_3 \in A_{21} \cap A_{22} - A, \ \{a_2, a_3, a_4\} \subseteq A_{22}, \text{ and } \{v_2, w_2, w_3\} \subseteq A_{21}. \text{ Then the separation } (G[A_{22} + a_1], A_{21} \cup U'_1 \cup A'_1 \cup U_2) \text{ shows that } (iii) \text{ holds.}

**Case 2.** \(v = v_3\).

Let \(U'_1, A'_2\) be obtained from \(U_1, A_2\), respectively, by uncontracting \(v\) to \(xy\). We choose such \(U'_1, U_2, A_1, A'_2\) to maximize \(U'_1 \cup U_2\). We may assume \(v_1 \notin A\) by Lemma 3.5.

We may assume that \(U'_1\) has three independent paths from \(u_1\) to \(v_2, x, y\), respectively. For, otherwise, \(U'_1\) has a separation \((U_1, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 2, \ u_1 \in U_{11} - U_{12}, \text{ and } \{x, y, v_3\} \subseteq U_{12}\). Then \(v_1 \in U_{11} - U_{12};\) otherwise (ii) holds. So \(U_{11}, U_2, A_1, U_{12} \cup A'_2\) show that \((G, u_1, u_2, A)\) is of type III.

If \(A''_2 := A'_2 + w_2 w_3\) has three disjoint paths from \(\{v_2, x, y\}\) to \(\{a_2, a_3, a_4\}\) and through \(w_2 w_3\), then (i) holds. So we may assume that such paths do not exist, and apply Lemma 3.4.

First, suppose Lemma 3.4(ii) holds. Then \(A''_2\) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| \leq 2, \ \{v_2, x, y\} \subseteq A_{21}, \ \{a_2, a_3, a_4\} \subseteq A_{22}\). If \(w_2 w_3 \in A_{21}\) then \(U'_1, U_2 \cup A_{22}, A_1, G[A_{21} - w_2 w_3]\) contradict the choice of \(U'_1, U_2, A_1, A'_2\) \text{(maximality of } U'_1 \cup U_2\). So \(w_2 w_3 \in A_{22}\). Then \(U'_1 \cup A_{21}, U_2, A_1, G[A_{22} - w_2 w_3]\) show that \((G, u_1, u_2, A)\) is an obstruction of type III.

Now suppose Lemma 3.4(iii) holds. Then \(A''_2\) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| \leq 1, \ \{x, y, v_3\} \subseteq A_{21}, \text{ and } \{w_2, w_3\} \subseteq A_{22}\). So the separation \((U'_1 \cup A_{21} \cup A_{22}, U_2 \cup G[A_{22} - w_2 w_3])\) shows that (ii) holds.

Suppose Lemma 3.4(iv) holds. Then \(A''_2\) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| = 3, \ {x, y, v_3}\} \subseteq A_{21}, \text{ and } \{a_2, a_3, a_4\} \subseteq A_{22}\). If \(w_2 w_3 \in A_{22}\) then \(U'_1 \cup A_{21}, U_2, A_1, G[A_{22} - w_2 w_3]\) contradict the choice of \(U'_1, U_2, A_1, A'_2\) \text{(maximality of } U'_1 \cup U_2\). So \(w_2 w_3 \in A_{22}\). Now the separation \((G[A_{22} + a_1], U'_1 \cup U_2 \cup A_1 \cup G[A_{21} - w_2 w_3])\) shows that \((iii)\) holds.

Suppose Lemma 3.4(v) holds. Then \(A''_2 = G_1 \cup G_2 \cup G_3\) such that \(G_1 \cap G_2 = \emptyset, w_2 w_3 \in G_2, |V(G_1 \cap G_2)| \leq 1, |V(G_2 \cap G_3)| \leq 1, |V(G_1) \cap \{a_2, a_3, a_4\}| = 1 = |V(G_1) \cap \{v_2, x, y\}|, \text{ and } |V(G_3) \cap \{a_2, a_3, a_4\}| = |V(G_3) \cap \{v_2, x, y\}| = 2. \text{ Then } U'_1, U_2 \cup G[G_2 - w_2 w_3], A_1, G_1, G_3 \text{ show that } (G, u_1, u_2, A) \text{ is an obstruction of type IV.}

Finally, assume that Lemma 3.4(vi) holds. Then \(A''_2 = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5\) such that \(|V(G_i \cap G_j)| = 1 \text{ for } i \in \{1, 2\} \text{ and } j \in \{3, 4, 5\}, |V(G_1 \cap G_2)| \leq G_3 \cup G_4 \cup G_5, G_i \cap G_j \subseteq G_1 \cap G_2 \text{ for } 3 \leq i \neq j \leq 5, w_2 w_3 \in G_1, \text{ and either } \{a_2, a_3, a_4\} \subseteq G_2 \text{ and } |V(G_i) \cap \{v_2, x, y\}| = 1 \text{ for } j \in \{3, 4, 5\} \text{ or } \{v_2, x, y\} \subseteq G_2 \text{ and } |V(G_j) \cap \{a_2, a_3, a_4\}| = 1 \text{ for } j \in \{3, 4, 5\}. \text{ In the former case, } (G[G_2 + a_1], G[G_1 - w_2 w_3] \cup U'_1 \cup U_2 \cup A_1 \cup G_3 \cup G_4 \cup G_5) \text{ shows that } (iii) \text{ holds. Thus, we may assume the latter case. Then } U'_1 \cup G_2, U_2 \cup G[G_1 - w_2 w_3], A_1, G_3, G_4, G_5 \text{ show that } (G, u_1, u_2, A) \text{ is an obstruction of type IV.} \quad \blacksquare

**Lemma 4.4.** Let \((G, u_1, u_2, A)\) be a quadruple, and let \(A := \{a_1, a_2, a_3, a_4\}\). Suppose there exists \(xy \in E(G - A - \{u_1, u_2\})\) such that \((G/xy, u_1, u_2, A)\) is of type IV. Then one of the following holds.

(i) \((G, u_1, u_2, A)\) is feasible.

(ii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 2, u_1 \in G_1 - G_2, \text{ and } A \cup \{u_2\} \subseteq G_2\).

(iii) \(G\) has a separation \((G_1, G_2)\) such that \(|V(G_1 \cap G_2)| \leq 4, \ {u_1, u_2}\} \subseteq G_1 - G_2, \text{ and } A \subseteq G_2\).
(iv) \((G, u_1, u_2, A)\) is an obstruction of types I, II, IV.

**Proof.** Let \(G/xy\) be the edge disjoint union of \(U_1, U_2, A_1, A_2, A_3, A_4\) such that \(V(U_1 \cap A_i) = \{v_i\}\) and \(V(U_2 \cap A_i) = \{w_i\}\) for \(1 \leq i \leq 4\), \(V(U_1 \cap U_2) \subseteq \bigcup_{i=1}^{4} (\{v_i\} \cap \{w_i\})\), \(a_i \in A_i\) for \(i = 1, 2, 3, 4\), and \(u_i \in U_i - (A_1 \cup A_2 \cup A_3 \cup A_4)\) for \(i = 1, 2\).

Let \(v\) denote the vertex resulting from the contraction of \(xy\). If \(v \notin \{v_i, w_i : 1 \leq i \leq 4\}\) then \((G, u_1, u_2, A)\) is an obstruction of type IV, and (iv) holds. So by symmetry we may assume that \(v = v_1\). Let \(U_1'\), \(A_1'\) be obtained from \(U_1, A_1\), respectively, by uncontracting \(v\) to \(xy\).

We may assume that \(A_1'\) contains disjoint paths from \(\{x, y\}\) to \(\{a_1, w_1\}\). For, if such paths do not exist, then \(A_1'\) has a separation \((A_{11}, A_{12})\) such that \(|V(A_{11} \cap A_{12})| \leq 1\), \(\{x, y\} \subseteq A_{11}\), and \(\{a_1, w_1\} \subseteq A_{12}\). Now \(U_1' \cup A_{11}, U_2, A_{12}, A_3, A_4\) show that \((G, u_1, u_2, A)\) is an obstruction of type IV, and (iv) holds.

Moreover, for each \(i \in \{2, 3, 4\}\), if \(A_i \neq \{a_i\}\) then we may assume \(a_i \notin \{v_i, w_i\}\). \(A_i - v_i\) (respectively, \(A_i - w_i\)) has a path between \(v_i\) (respectively, \(v_i\)) and \(a_i\). Otherwise, we can enlarge \(U_1'\) or \(U_2\).

**Case 1.** There exist two \(i \in \{2, 3, 4\}\) such that \(J_i := U_1' - (A - \{v_i\})\) has no three independent paths from \(u_1\) to \(x, y, v_i\), respectively.

First, suppose \(J_2\) contains no three independent paths from \(u_1\) to \(x, y, v_2\), respectively. Then \(J_2\) has a separation \((J_{21}, J_{22})\) such that \(|V(J_{21} \cap J_{22})| \leq 2\), \(u_1 \in J_{21} - J_{22}\), and \(\{x, yv_2\} \subseteq J_{22}\). We choose \((J_{21}, J_{22})\) so that \(|V(J_{21} \cap J_{22})|\) is minimum and then \(J_{21}\) is minimal.

If \(\{v_3, v_4\} \cap N(J_{21} - J_{22}) = \emptyset\) then the separation \((J_{21}, G - (J_{21} - J_{22}))\) shows that (ii) holds. So we may assume by symmetry that \(v_3 \in N(J_{21} - J_{22})\). We may also assume \(|V(J_{21} \cap J_{22})| = 0\); otherwise, the separation \((G[J_{21} + \{v_3, v_4\}], G - (J_{21} - J_{22} - \{v_3, v_4\}))\) shows that (ii) holds.

Suppose \(|V(J_{21} \cap J_{22})| = 1\). Then we may assume that \(v_4 \in N(J_{21} - J_{22})\); otherwise, the separation \((G[J_{21} + \{v_3\}], G - (J_{21} - J_{22} - \{v_3, v_4\}))\) shows that (ii) holds. Moreover, the separation \((G[J_{21} + \{v_3, v_4\}], G - (J_{21} - J_{22} - \{v_3, v_4\}))\) allows us to use Lemma 3.5 to assume \(v_3, v_4 \notin A\). Hence, \(v_3, v_4 \in J_{21}\). Then \(J_{21}, U_2, A_3, A_4, J_{22} \cup A_1' \cup A_2\) show that \((G, u_1, u_2, A)\) is an obstruction of type II.

So we may assume that \(|V(J_{21} \cap J_{22})| = 2\). Let \(V(J_{21} \cap J_{22}) = \{s_1, s_2\}\). So by the minimality of \(|V(J_{21} \cap J_{22})|\), \(J_{22} - (A - \{v_2\})\) contains disjoint paths from \(\{s_1, s_2\}\) to \(\{x, y\}\).

By the minimality of \(J_{21}\), we see that \(G[J_{21} + v_3]\) has three independent paths from \(u_1\) to \(s_1, s_2, v_3\), respectively. So \(J_4\) has three independent paths from \(u_1\) to \(x, y, v_3\), respectively. Similarly, if \(v_4 \in N(J_{21} - J_{22})\) then \(J_4\) has three independent paths from \(u_1\) to \(x, y, v_4\), respectively. Thus we may assume that \(v_4 \notin N(J_{21} - J_{22})\). Then by Lemma 3.5 we may assume \(v_3 \notin A\); and hence we may assume \(v_3 \notin A\).

If \(U_2\) has three independent paths from \(u_2\) to \(w_1, w_2, w_4\), respectively, then we see that (i) holds. So we may assume that \(U_2\) has a separation \((U_2, U_2')\) such that \(|V(U_2 \cap U_2')| \leq 2\), \(u_2 \in U_2 - U_2'\), and \(\{w_1, w_2, w_4\} \subseteq U_2'.\) Then we may assume that \(|V(U_2' \cap U_2)| = 2\) and \(w_3 \in U_2' - U_2\) as otherwise (ii) holds. Now \(J_{21}, U_2, A_3, J_{22} \cup U_2' \cup A_1' \cup A_2 \cup A_4\) show that \((G, u_1, U_2, A)\) is an obstruction of type II.

**Case 2.** There exist two \(i \in \{2, 3, 4\}\) such that \(J_i := U_1' - (A - \{v_i\})\) has three independent paths from \(u_1\) to \(x, y, v_i\), respectively.

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Without loss of generality, we may assume that for $i = 2, 3, J$, has three independent paths from $u_1$ to $x, y, v_i$, respectively.

If $U'_2 := U_2 - \{w_2\} \cap A$ has three independent paths from $w_2$ to $w_1, w_3, w_4$, respectively, then (i) holds. So we may assume that $U'_2$ has a separation $(U_21, U_22)$ such that $|V(U_21 \cap U_22)| \leq 2$, $u_2 \in U_21 - U_22$, and $\{w_1, w_3, w_4\} \subseteq U_22$. Choose $(U_21, U_22)$ so that $|V(U_21 \cap U_22)|$ is minimum and then $U_22$ is minimal. Thus, $U_22 - (A \cap \{w_3\})$ has disjoint paths from $V(U_21 \cap U_22)$ to $\{w_1, w_4\}$.

We may assume $w_2 \in N(U_21 - U_22)$ and $|V(U_21 \cap U_22)| = 2$, as otherwise (ii) holds. Thus by Lemma 3.5 we may assume that $w_2 \notin A$ and $V(U_21 \cap U_22) \cap A = \emptyset$. So $w_2 \in U_21 - U_22$. Hence, $U_21$ has three independent paths from $w_2$ to $V(U_21 \cap U_22) \cup \{w_2\}$. Therefore, $U_2 - (A \cap \{w_3\})$ has three independent paths from $w_2$ to $w_1, w_2, w_4$, respectively. Again, $(G, u_1, u_2, A)$ is feasible, and (i) holds.

**Lemma 4.5.** Let $G$ be a graph, let $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of $G$, and let $A := \{a_1, a_2, a_3, a_4\}$. Suppose there exist $xy \in E(G - A - \{u_1, u_2\})$ such that $(G/xy, u_1, u_2, A)$ is of type $V$. Then one of the following holds.

(i) $(G, u_1, u_2, A)$ is feasible.

(ii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 2$, $u_1 \in G_1 - G_2$, and $A \cup \{w_2\} \subseteq G_2$.

(iii) $G$ has a separation $(G_1, G_2)$ such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq G_1 - G_2$, and $A \subseteq G_2$.

(iv) $(G, u_1, u_2, A)$ is an obstruction of types I, II, III, IV or V.

**Proof.** Let $G/xy$ be the edge-disjoint union of $U_1, U_2, A_1, A_2$ such that $V(U_1 \cap A_1) = \{v_1\}$, $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_1) = \{w_1, w_2\}$, $V(U_2 \cap A_2) = \{w_3\}$, $V(U_1 \cap U_2) \subseteq (\{v_1\} \cap \{w_1, w_2\}) \cup (\{v_2, v_3\} \cap \{w_3\})$, $a_1, a_2 \in A_1$, $a_3, a_4 \in A_2$, and $u_i \in U_i - (A_1 \cap A_2)$ for $i = 1, 2$.

Let $v$ denote the vertex resulting from the contraction of $xy$. If $v \notin \{v_i, w_i : 1 \leq i \leq 3\}$ then it is easy to see that $(G, u_1, u_2, A)$ is also an obstruction of type $V$, and (iv) holds. Thus, we may assume $v \in \{v_i, w_i : 1 \leq i \leq 3\}$. By symmetry, we need to consider only two cases: $v = v_1$ or $v = v_2$. By Lemma 3.5 we may assume that $\{w_1, w_2, w_3\} \cap A = \emptyset$.

We may assume that $U_2$ contains three independent paths from $w_2$ to $w_1, w_2, w_3$, respectively; for otherwise Menger’s theorem shows that (ii) holds.

**Case 1.** $v = v_2$.

Let $U'_1, A'_2$ be obtained from $U_1, A_2$ by uncontracting $v$ to $xy$. We may assume that $A'_2$ contains three disjoint paths from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For if such three paths do not exist then $A'_2$ has a separation $(A_{21}, A_{22})$ such that $|V(A_{21} \cap A_{22})| \leq 2$, $\{a_3, a_4, w_3\} \subseteq A_{22}$ and $\{v_3, x, y\} \subseteq A_{21}$. Then $U'_1 \cup A_{21}, U_2, A_1, A_{22}$ show that $(G, u_1, u_2, A)$ is an obstruction of type $V$.

We may assume $v_1 \notin A$. For, suppose $v_1 \notin A$, then $(A_1 \cup U_2, A'_2 \cup U'_1 + a_2)$ is a separation in $G$, and hence by Lemma 3.5, the assertion of the lemma holds.

We may assume that $A_1 - v_1$ contains disjoint paths from $\{w_1, w_2\}$ to $\{a_1, a_2\}$. For, otherwise, $A_1$ has a separation $(A_{11}, A_{12})$ such that $|V(A_{11} \cap A_{12})| \leq 2$, $v_1 \in A_{11} \cap A_{12}$,
\{w_1, w_2\} \subseteq A_{12} and \{a_1, a_2\} \subseteq A_{12}. Then the separation \((G[A_{12} + \{a_3, a_4\}], A_{11} \cup A_2' \cup U_1' \cup U_2)\) shows that (iii) holds.

If \(U_1'\) contains three independent paths from \(u_1\) to \(v_3, x, y\), then (i) holds. So we may assume that \(U_1'\) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 3\), \(u_1 \in U_{11} - U_{12}\), and \(\{v_3, x, y\} \subseteq U_{12}\). If \(v_1 \notin U_{11} - U_{12}\) then (ii) holds. So assume \(v_1 \in U_{11} - U_{12}\). Then \(U_{11}, U_2, A_1, A_2' \cup U_12\) show that \((G, u_1, u_2, A)\) is an obstruction of type \(V\).

**Case 2.** \(v = v_1\).

Let \(U_1', A_1'\) be obtained from \(U_1, A_1\), respectively, by uncontracting \(v\) to \(xy\). We choose \(U_1', U_2, A_1', A_2\) to maximize \(U_1' \cup U_2\).

We may assume that \(A_1^* := A_1' + \{a_1 a_2, w_1 w_2, xy\}\) contains a cycle through \(a_1 a_2, w_1 w_2, xy\). For, suppose not. Then by Lemma 3.3 there are three possibilities. First, suppose \(A_1^*\) has a separation \((K, L)\) such that \(|V(K \cap L)| \leq 1\) and \(|E(K) \cap \{a_1 a_2, w_1 w_2, xy\}| = 1\). If \(w_1 w_2 \in K\), then the separation \((U_1' \cup L \cup A_2, K \cup U_2)\) shows that (ii) holds. If \(xy \in K\) then \(U_1' \cup K, U_2, L, A_2\) show that \((G, u_1, u_2, A)\) is an obstruction of type \(V\). If \(a_1 a_2 \in K\) then \((G[K + \{a_3, a_4\}], L \cup U_1' \cup U_2 \cup A_2)\) shows that (iii) holds. Now, suppose \(A_1^*\) has a separation \((K, L)\) such that \(|V(K \cap L)| = 2\), \(|E(K) \cap \{a_1 a_2, w_1 w_2, xy\}| = 1\), and \(|V(K)| \geq 3\). If \(w_1 w_2 \in K\) or \(xy \in K\) then \(U_1' \cup K, U_2, L, A_2\) or \(U_1', U_2 \cup L, K, A_2\) contradicts the choice of \(U_1', U_2, A_1', A_2\) (maximality of \(U_1' \cup U_2\)). If \(a_1 a_2 \in K\) then the separation \((G[K + \{a_3, a_4\}], U_1' \cup U_2 \cup L \cup A_2)\) shows that (iii) holds. Finally, \(a_1 a_2, w_1 w_2, xy\) is an edge cut in \(A_1^*\). Then it is easy to check that \((G, u_1, u_2, A)\) is an obstruction of type II, and (iv) holds.

We may assume that for any \(i \in \{2, 3\}\), \(A_2 - \{v_{5-i}\} - A\) contains disjoint paths from \(\{w_3, v_i\}\) to \(\{a_3, a_4\}\). For suppose the contrary. Then by symmetry we may assume that \(A_2 - \{v_3\} - A\) contains no disjoint paths from \(w_3, v_2\) to \(a_3, a_4\). So Menger’s theorem implies that \(A_2\) has a separation \((A_{21}, A_{22})\) such that \(|V(A_{21} \cap A_{22})| \leq 1\) (when \(v_3 \notin A\), \(|V(A_{21} \cap A_{22})| \leq 2\) and \(v_3 \in A_{21} \cap A_{22}\) (when \(v_3 \notin A\)), \(\{a_3, a_4\} \subseteq A_21\) and \(\{w_3, v_2\} \subseteq A_{22}\). We may assume that \(V(A_{21}) = V(A_{21} \cap A_{22}) \cup \{v_3\} = \{a_3, a_4\}\), or else the separation \((G[A_{21} + \{a_1, a_2\}], A_{22} \cup U_1' \cup U_2 \cup A_1')\) shows that (iii) holds. As \(w_1, w_2, v_2\) separates \(v_2\) from \(A \cup \{u_1\}\) in \(G\), we may assume by Lemma 3.5 that \(v_2 \notin A\). If \(U_1'\) has three independent paths from \(u_1\) to \(x, y, v_3\), respectively, then we see that (i) holds. So we may assume that \(U_1'\) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 2\), \(u_1 \in U_{11} - U_{12}\) and \(\{x, y, v_3\} \subseteq U_{12}\). If \(v_2 \notin U_{11} - U_{12}\) then (ii) holds. So assume \(v_2 \in U_{11} - U_{12}\). Then \(U_{11}, U_2, U_{12} \cup A_1', A_2 - v_3\) show that \((G, u_1, u_2, A)\) is an obstruction of type III, and (iv) holds.

We may assume that \(U_1' - (A - \{v_3\})\) has no three independent paths from \(u_1\) to \(x, y, v_3\), respectively. For, such paths together with disjoint paths in \(A_2\) from \(\{w_3, v_3\}\) to \(\{a_3, a_4\}\), three paths in \(U_2\) from \(u_1, w_1, w_2, v_3\), and \(C - \{a_1 a_2, w_1 w_2, xy\}\), give a topological \(H\) in \(G\) rooted at \(u_1, u_2, A\); so (i) holds.

Thus, \(U_1' - (A - \{v_3\})\) has a separation \((U_{11}, U_{12})\) such that \(|V(U_{11} \cap U_{12})| \leq 2\), \(u_1 \in U_{11} - U_{12}\), and \(\{x, y, v_3\} \subseteq U_{12}\). We choose \(U_{11}, U_{12}\) so that \(|V(U_{11} \cap U_{12})|\) is minimum and then \(U_{12}\) is minimum.

We may assume that \(v_2 \in N(U_{11} - U_{12})\) and \(|V(U_{11} \cap U_{12})| = 2\); or else (ii) holds. So by Lemma 3.5 we may assume \(v_2 \notin A\). So \(v_2 \in U_{11} - U_{12}\). By the minimality of \(|V(U_{11} \cap U_{12})|\), \(U_{11}\) has three independent paths from \(u_1\) to \(x, y, v_2\), respectively. By the minimality of \(U_{12}\), \(U_{12} - \{v_3\} \cap A\) has disjoint paths from \(V(U_{11} \cap U_{12})\) to \(\{x, y\}\), respectively. Thus, \(U_1' - (A - \{v_3\})\) has no three independent paths from \(u_1\) to \(x, y, v_3\), respectively.
\{v_2\}) has three independent paths from \(u_1\) to \(x, y, v_2\), respectively. So these paths, disjoint paths in \(A_2 - \{v_3\} - A\) from \(\{v_2, v_3\}\) to \(\{v_3, a_2\}\), three paths in \(U_2\) from \(w_2 \to u_1, w_2, w_3\), and \(C - \{a_1, a_2, w_1, w_2, xy\}\), give a topological \(H\) in \(G\) rooted at \(u_1, u_2, A\); so (i) holds.

5 Proof of main theorem

Proof. Suppose this is not true. Let \((G, u_1, u_2, A)\) be a counterexample with \(|V(G)|\) minimum.

We claim that no cut of size at most 4 in \(G\) is disjoint from \(\{u_1, u_2\}\), and \(A\) from \(\{u_1, u_2\}\) from \(A\). For, suppose \(G\) has a cut \(S\) such that \(|S| \leq 4\) and \(S \cap \{u_1, u_2\} = \emptyset\), and \(S\) separates \(\{u_1, u_2\}\) from \(A\). Then \(|S| = 4\) for any such choice of \(S\); otherwise, (iii) holds. But this shows that \(G\) admits a good 4-separation, a contradiction.

We also claim that \(u_1\) is not adjacent to \(u_2\). For, suppose \(u_1u_2 \in E(G)\). Then let \(G'\) be obtained from \(G\) by duplicating \(u_1\) and \(u_2\), and let \(u_i', i = 1, 2\), denote the duplicate of \(u_i\). Now by (2), \(G'\) contains four disjoint paths from \(\{u_1, u_1', u_2, u_2'\}\) to \(A\). These paths and \(u_1u_2\) form a topological \(H\) in \(G\) rooted at \(u_1, u_2, A\), a contradiction.

We further claim that \(N(u_1) \cap N(u_2) \subseteq A\). Now let \(u \in N(u_1) \cap N(u_2) - A\). Let \(G'\) be obtained from \(G - u\) by duplicating \(u\) (with duplicate \(u_i')\) for \(i = 1, 2\). By (2), \(G'\) contains four disjoint paths from \(\{u_1, u', u_2, u_2'\}\) to \(A\). These paths together with \(u_1u_2\) form a topological \(H\) in \(G\) rooted at \(u_1, u_2, A\), a contradiction.

We now show that there exists an edge \(xy \in E(G)\) such that \(x, y \notin A \cup \{u_1, u_2\}\), and if \(d(u_i) = 3\) then \(\{x, y\} \notin N(u_i)\). If \(V(G) = A \cup \{u_1, u_2\}\) then, since \(u_1u_2 \notin E(G), u_1\) and \(u_2\) are the components of \(G - A\), so \((G, u_1, u_2, A)\) may be viewed as an obstruction of type IV. Thus, we may assume \(V := V(G) - (A \cup \{u_1, u_2\}) \neq \emptyset\). We may assume that \(G[V]\) contains no edge, as any edge in \(G[V]\) gives the desired edge. Therefore, since \(N(u_1) \cap N(u_2) \subseteq A\), \(V(G) - A\) can be partitioned into two sets \(V_1, V_2\), such that \(u_i \in V_i\) for \(i = 1, 2\). Now \(G[V_1], G[V_2]\) can be partitioned into two sets \(V_1, V_2\), such that \(u_i \in V_i\) for \(i = 1, 2\). Now \((G[V_1], G[V_2], a_1, a_2, a_3, a_4\) show that \((G, u_1, u_2, A)\) is an obstruction of type IV.

By the choice of \(G\), \((G/xy, u_1, u_2, A)\) satisfies (i) or (ii) or (iii) or (iv). If \((G/xy, u_1, u_2, A)\) satisfies (i) then \((G, u_1, u_2, A)\) also satisfies (i).

Suppose \((G/xy, u_1, u_2, A)\) satisfies (ii). Let \((G_1, G_2)\) be a separation in \(G\) such that \(V(G_1 \cap G_2) \leq 2, u_i \in G_1 - G_2,\) and \(A \cup \{v_{3-i}\} \subseteq G_2\). By the minimality of \(G\), \(G_1 - G_2 = \{u_i\}\). Thus \(x, y \in N(u_i)\), a contradiction. So \((G/xy, u_1, u_2, A)\) cannot satisfy (ii).

Suppose \((G/xy, u_1, u_2, A)\) satisfies (iv). Then \((G, u_1, u_2, A)\) satisfies (i)–(iv) by Lemmas 3.6, 4.1, 4.2, 4.3, 4.4, and 4.5.

So we may assume that \((G/xy, u_1, u_2, A)\) satisfies (iii). Let \((G_1, G_2)\) be a separation in \(G\) such that \(V(G_1 \cap G_2) = 4, \{u_1, u_2\} \subseteq G_1 - G_2,\) and \(A \subseteq G_2\). Let \(v\) denote the vertex resulting from the contraction of \(xy\). If \(v \notin G_1 \cap G_2\) for one such separation, then (iii) also holds for \((G, u_1, u_2, A)\). Thus we may assume that \(v \in G_1 \cap G_2\) for all such separations. So \(G_2\) has four disjoint paths from \(A' := V(G_1 \cap G_2)\) to \(A\). We choose \((G_1, G_2)\) to minimize \(G_1\).

Let \(A' = \{a'_1, a'_2, a'_3, v\}\). By the minimality of \((G, u_1, u_2, A)\), \((G, u_1, u_2, A')\) is not a counterexample. Thus, \((G_1, u_1, u_2, A')\) satisfies (i)–(iv). If \((G_1, u_1, u_2, A')\) satisfies (i) then \((G, u_1, u_2, A)\) also satisfies (i).

If \((G_1, u_1, u_2, A')\) satisfies (ii) then \(G_1\) has a separation \((K, L)\) such that \(|V(K \cap L)| \leq 2, 20\).
\(u_i \in K - L\) and \(A' \cup \{u_{3-i}\} \subseteq L\). If \(v \notin K \cap L\) or \(|V(K \cap L)| \leq 1\) then (ii) holds for \((G, u_1, u_2, A)\). If \(v \in K \cap L\) and \(|V(K \cap L)| = 2\) then by the minimality of \(G\), \(V(K - L) = \{u_i\}\). This shows that \(x, y \in N(u_i)\), a contradiction.

Now suppose \((G_1, u_1, u_2, A')\) satisfies (iii) then \(G_1\) has a separation \((K, L)\) such that \(|V(K \cap L)| = 4\), \(\{u_1, u_2\} \subseteq K - L\) and \(A' \subseteq L\). So \(v \in K \cap L\). But this contradicts the minimality of \(G_1\).

Therefore, \((G_1, u_1, u_2, A')\) satisfies (iv).

(4) \(G\) contains no 5-cut \(S\) such that \(u_1, u_2\) belong to different components of \(G - S\), and the components of \(G - S\) containing \(u_1\) or \(u_2\) are disjoint from \(A\).

Otherwise, let \(S\) be a 5-cut in \(G\) and \(U_1\) and \(U_2\) be components of \(G - S\) such that for \(i = 1, 2\), \(u_i \in U_i\) and \(U_i \cap A = \emptyset\).

We now apply Lemma ??(ii) cannot occur; otherwise \(G\) would satisfy (ii). By (2), Lemma ??(i) cannot occur. So Lemma ??(iii) occurs. Thus for any \(v \in N(U_1) \cap N(U_2)\) with \(v \notin A\) and for \(i = 1, 2\), \(G[U_i \cup N(U_i)]\) contains three paths \(P_{1i}, P_{2i}, P_{3i}\) from \(u_i\) to \(N(U_i) \cap S\) such that \(P_{1j} \cap P_{2i} = \{u_i\}\) whenever \(j \neq k\), \(v \in P_{1i} \cap P_{2k}\), and each vertex in \(S - \{v\}\) belongs to precisely one of these paths.

If \(G - (U_1 \cup U_2 \cup \{v\})\) has four disjoint paths from \(S - \{v\}\) to \(A\), then these paths and \(P_{j1}\), \(i = 1, 2\) and \(j = 1, 2, 3\), form a topological \(H\) in \(G\) rooted at \(u_1, u_2, a_1, a_2, a_3, a_4\), a contradiction. Thus such paths do not exist. So \(G - (U_1 \cup U_2 \cup \{v\})\) has a cut \(T\) with \(|T| \leq 3\) separating \(S - \{v\}\) from \(A\). Hence \(T \cup \{v\}\) is a cut in \(G\) separating \(A\) from \(\{u_1, u_2\}\), contradicting (2).

Thus for any \(v \in N(U_1) \cap N(U_2) - A\), \(G - (U_1 \cup U_2 \cup \{v\})\) has a cut \(T\) with \(|T| \leq 3\) and separating \(S - \{v\}\) from \(A\). If \(|T| \leq 2\) then \(T \cup \{v\}\) shows that (iii) holds, a contradiction. So \(|T| = 3\), which shows that (v) holds, a contradiction.

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References


