Sparse halves in K_4-free graphs

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Abstract

A conjecture of Chung and Graham states that every K_4-free graph on n vertices contains a vertex set of size $\lfloor n/2 \rfloor$ that spans at most $n^2/18$ edges. We make the first step toward this conjecture by showing that it holds for all regular graphs.

1 Introduction

Given graphs G and H, we say G is *H-free* if G does not contain H as a subgraph. The celebrated Turán theorem [14] states that for every $r \geq 3$, the maximum number of edges in an n-vertex K_r-free graph is uniquely achieved by the Turán graph $T_{r-1}(n)$, which is the complete $(r-1)$-partite graph on n vertices such that the sizes of every two parts differ by at most one. Generalizing Turán's theorem, Erdős [3] initialized the study of the following problem: Given a constant $0 \leq \alpha \leq 1$, what is the minimum valve $\beta = \beta(\alpha, r)$ such that every n-vertex K_r-free graph contains a vertex set of size $\lfloor \alpha n \rfloor$ which spans at most βn^2 edges? This is often referred as the *local density problem.*

The case $\alpha = 1/2$ is of special intersect. Erdős [4] offered \$250 for the first solution to the following long-standing conjecture on triangle-free graphs.

Conjecture 1.1 (Erdős, [3]). *Every triangle-free graph on n vertices contains a vertex set of size $\lfloor n/2 \rfloor$ that spans at most $n^2/50$ edges.*

Both of the balanced blow-ups of the 5-cycle and the Petersen graph show that the bound $n^2/50$ would be best possible if this conjecture is true. Despite extensive research [8, 7, 12, 1], Conjecture 1.1 is still open.

A similar question also has been asked for K_4-free graphs. Chung and Graham [2], and Erdős, Faudree, Rousseau and Schelp [6] posted the following conjecture.

Conjecture 1.2 (Chung et al. [2], Erdős et al. [6]). *Every K_4-free graph on n vertices contains a vertices set of size $\lfloor n/2 \rfloor$ that spans at most $n^2/18$ edges.*

The Turán graph $T_3(n)$ shows that the bound $n^2/18$ in Conjecture 1.2 would be best possible if it is true. A closely related conjecture of Erdős (see [5]), which was proved

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by Sudakov [13], states that every K_4-free graphs on n vertices can be made bipartite by deleting at most $n^2/9$ edges. An interesting interplay between these problems for *regular* graphs was observed by Krivelevich [8], where he pointed out that a bound in the local density problem can imply a bound (doubled) in the problem of making a graph bipartite; also see [13] for an illustration.

The main result of this paper is to confirm Conjecture 1.2 for all *regular* graphs. We prove it in the following form, which also characterizes the unique extremal graph.

Theorem 1.3. *Let G be a K_4-free regular graph on n vertices. If every vertex set of size $\lfloor n/2 \rfloor$ in G spans at least $n^2/18$ edges, then n is divisible by 6 and $G \cong T_3(n)$.*

We would like to remark that our proof of Theorem 1.3 actually shows that Conjecture 1.2 holds for all *almost regular* graphs, i.e. graphs whose difference of maximum degree and minimum degree is bounded by ϵn for some absolute constant $\epsilon > 0$.[1]

As a corollary, Theorem 1.3 implies the following slightly stronger version of Sudakov's theorem in the case of regular graphs.

Corollary 1.4. *Let $n \in \mathbb{N}$ be even. Then every regular K_4-free graph on n vertices can be made bipartite by removing at most $n^2/9$ edges such that each part has size exactly $n/2$.*

For odd $n \in \mathbb{N}$, one could easily obtain a similar result as in Corollary 1.4.

We now introduce a crucial tool in our proof of Theorem 1.3, which also can be viewed as a strengthening of the local density problem. Erdős, Faudree, Rousseau and Schelp conjectured in [6] that for every $\alpha \in [17/30, 1]$, every triangle-free graph on n vertices contains a vertex set of size $\lfloor \alpha n \rfloor$ that spans at most $(2\alpha - 1)n^2/4$ edges. This was confirmed by Krivelevich [8] for all $\alpha \in [3/5, 1]$. The coming result shows that the bound $(2\alpha - 1)n^2/4$ can be improved in the range where α is relatively large.

Theorem 1.5. *Let $\alpha, c \in [0, 1]$ satisfy $\alpha + c \geq 1$. Then the following hold:*

(1). Every n-vertex triangle-free graph with cn^2 edges contains a vertex set of size $\lfloor \alpha n \rfloor$ that spans at most $(2\alpha - 1)cn^2$ edges.

(2). Assume that $\alpha n \in \mathbb{N}$ and G is an n-vertex triangle-free graph. If every vertex set of size αn in G spans at least $(2\alpha - 1)cn^2$ edges, then G is regular, and vice versa.

Note that by Mantel's theorem [11], we have $(2\alpha - 1)cn^2 \leq (2\alpha - 1)n^2/4$.

The rest of the paper is organized as follows. In Section 2 we present some preliminary results. In Section 3 we prove Theorem 1.5. In Section 4 we complete the proof of Theorem 1.3, by dividing it into three parts according to the edge density. In Section 5 we conclude this paper by mentioning some related problems.

2 Preliminaries

We first introduce our notation (which is conventional). Given a graph G, we will use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. Let $e(G) = |E(G)|$.

[1] Our calculations indicate that ϵ can be chosen as $\epsilon = 1/500$.

We use $d(G), \Delta(G), \delta(G)$ to denote the average degree, maximum degree, and minimum degree of G, respectively. For $v \in V(G)$, let $N_G(v)$ be the set of the neighbors of v in G and let $d_G(v) = |N_G(v)|$. For $S \subset V(G)$ we use $G[S]$ to express the induced subgraph of G on S and let $e_G(S)$ be the number of edges in $G[S]$. For two disjoint vertex sets $S, T \subset V(G)$, let $G[S,T]$ be the induced bipartite subgraph of G with two parts S, T and let $e_G(S,T)$ be the number of edges in $G[S,T]$. If it is clear from the context we omit the subscript G. We also omit floors and ceilings when they are not essential in our proofs.

The following propositions can be found in literatures (e.g. [7]). For completeness we include their proofs here.

Proposition 2.1. *Let $0 \le \alpha \le 1$. Then every n-vertex graph G with e edges contains a vertex set of size αn that spans at most $\alpha^2 e$ edges.*

Proof. Choose $S \subset V(G)$ with $|S| = \alpha n$ uniformly at random. Then for every edge e, the probability that e is contained in S is $\frac{\alpha n}{n} \cdot \frac{\alpha n - 1}{n-1} \le \alpha^2$. So, the expected value of $e(S)$ is at most $\alpha^2 e$. Hence there exists a vertex set of size αn in G that spans at most $\alpha^2 e$ edges. ∎

Proposition 2.2. *Let G be an n-vertex graph with e edges. Let $A \cup B = V(G)$ be a partition with $|A| = \alpha n \le n/2$. Then there exists $S \subset B$ with $|S| = (1/2 - \alpha)\, n$ such that*

$$\begin{aligned} e(A \cup S) &\le e(A) + \frac{1/2 - \alpha}{1 - \alpha} e(A,B) + \left(\frac{1/2 - \alpha}{1 - \alpha} \right)^2 e(B) \\ &= e(G) - \frac{1}{2(1-\alpha)} e(A,B) - \frac{3/2 - 2\alpha}{2(1-\alpha)^2} e(B). \end{aligned}$$

Proof. Choose $S \subset B$ with $|S| = (1/2 - \alpha)\, n$ uniformly at random. Then, for every $e \in E(G[A,B])$ the probability that e is contained in $A \cup S$ is $\frac{1/2-\alpha}{1-\alpha}$. Similar to the proof of Proposition 2.1, for every $e' \in E(G[B])$ the probability that e' is contained in S is at most $\left(\frac{1/2-\alpha}{1-\alpha}\right)^2$. So, the expected value of $e(A \cup S)$ is at most $e(A) + \frac{1/2-\alpha}{1-\alpha} e(A,B) + \left(\frac{1/2-\alpha}{1-\alpha}\right)^2 e(B)$. Therefore, there exists $S \subset B$ with $|S| = (1/2 - \alpha)\, n$ such that the desired inequality holds. ∎

3 Local densities in triangle-free graphs

In this section we prove Theorem 1.5. First we show the following proposition for the "vice versa" part of Theorem 1.5 (2).

Proposition 3.1. *Let $\alpha, c \in [0,1]$, $n \in \mathbb{N}$ such that $\alpha n \in \mathbb{N}$. Suppose that G is a triangle-free regular graph on n vertices with cn^2 edges. Then every $S \subseteq V(G)$ with $|S| = \alpha n$ spans at least $(2\alpha - 1)cn^2$ edges.*

Proof. Let $S \subset V(G)$ be a set with size αn let $T = V(G) \setminus S$. Since G is regular, every vertex has degree $2cn$, which shows that

$$2e(S) + e(S,T) = \sum_{v \in S} d(v) = 2\alpha cn^2 \quad \text{and} \quad e(S,T) \le \sum_{v \in T} d(v) = 2(1-\alpha)cn^2.$$

Therefore,

$$e(S) = \frac{1}{2}\left(2e(S) + e(S,T) - e(S,T)\right) \ge \frac{1}{2}(2\alpha cn^2 - 2(1-\alpha)cn^2) = (2\alpha - 1)cn^2,$$

which completes the proof of Proposition 3.1. ∎

Now we prove Theorem 1.5. The core of the proof is a probabilistic argument. For convenience we will assume $\alpha n \in \mathbb{N}$ in the coming presentation, while the proof for the case $\alpha n \notin \mathbb{N}$ holds analogously.

Proof of Theorem 1.5. Let $\alpha + c \geq 1$ and G be an n-vertex triangle-free graph with cn^2 edges. Our goal is to find a subset $S \subseteq V(G)$ with $|S| = \alpha n$ that spans at most $(2\alpha - 1)cn^2$ edges. It is clear that we may assume $\alpha < 1$. We divide the proof into two cases by considering the value of $\delta(G)$.

First suppose that $\delta(G) \geq (1-\alpha)n$.[2] Suppose for the contrary that every subset of size αn spans more than $(2\alpha - 1)cn^2$ edges. For every $v \in V(G)$, let $B_v = N(v)$ and $A_v = V(G) \setminus B_v$. Since G is triangle-free, B_v is an independent set and hence $e(A_v) + e(A_v, B_v) = cn^2$. Let $d_v = d(v)/n$. By the similar argument as in Proposition 2.2, there exists $S \subseteq B_v$ with $|S| = (\alpha + d_v - 1)n$ such that

$$e(A_v \cup S) \leq e(A_v) + \frac{\alpha + d_v - 1}{d_v} e(A_v, B_v).$$

Since $|A_v \cup S| = \alpha n$, by assumption, we have

$$e(A_v) + \frac{\alpha + d_v - 1}{d_v} e(A_v, B_v) \geq e(A_v \cup S) > (2\alpha - 1)cn^2,$$

which together with $e(A_v) + e(A_v, B_v) = cn^2$ gives

$$cn^2 - \frac{1-\alpha}{d_v} e(A_v, B_v) > (2\alpha - 1)cn^2.$$

Therefore,

$$\sum_{v \in V(G)} \left(cn^2 - \frac{1-\alpha}{d_v} e(A_v, B_v) \right) d_v > \sum_{v \in V(G)} (2\alpha - 1)cn^2 d_v,$$

which implies

$$(1-\alpha) \sum_{v \in V(G)} e(A_v, B_v) < 2(1-\alpha)cn^2 \sum_{v \in V(G)} d_v.$$

Since $\sum_{v \in V(G)} d_v = 2cn$ and $\alpha < 1$, this gives

$$\sum_{v \in V(G)} e(A_v, B_v) < 4c^2 n^3.$$

On the other hand, since B_v is independent for each v, by the Cauchy-Schwarz inequality

$$\sum_{v \in V(G)} e(A_v, B_v) = \sum_{v \in V(G)} \sum_{u \in N(v)} d(u) = \sum_{u \in V(G)} (d(u))^2 \geq \frac{1}{n} \left(\sum_{u \in V(G)} d(u) \right)^2 = 4c^2 n^3,$$

which is a contradiction. Therefore, if $\delta(G) \geq (1-\alpha)n$, then there exists a vertex set of size αn that spans at most $(2\alpha - 1)cn^2$ edges. Note that if every vertex set of size αn spans at least $(2\alpha - 1)cn^2$ edges, then by the above arguments, we see that $d(v)$ must be the same for all $v \in V(G)$, that is, G is regular.

[2] We point out that this case holds even without requiring $\alpha + c \geq 1$.

Now suppose that $\delta(G) < (1-\alpha)n$, where $\alpha + c \geq 1$. Choose $v \in V(G)$ such that $d(v) = \delta(G) < (1-\alpha)n$ and remove v from G. We iteratively remove a vertex with the minimum degree in the remaining graph until there is no vertex left or the remaining graph G' satisfies $\delta(G') \geq (1-\alpha)n$. Let A denote the set of vertices we removed in this process and let $k = |A|/n$. If $|A| = n$, then $e(G) < (1-\alpha)n^2 \leq cn^2$, a contradiction. So $|A| < n$, which implies that $G' \neq \emptyset$. Since $\delta(G') \geq (1-\alpha)n$, we have $|V(G')| > (1-\alpha)n$. Therefore, $k = |A|/n = (n - |V(G')|)/n < \alpha$. Let $B = V(G) \setminus A$ and let $G' = G[B]$. Also let $\tilde{n} = (1-k)n$ and $\tilde{\alpha} = \frac{\alpha - k}{1-k}$. Since $\delta(G') \geq (1-\alpha)n = (1-\tilde{\alpha})\tilde{n}$, by the previous case, there exists $S \subseteq B$ with $|S| = \tilde{\alpha}\tilde{n}$ such that $e(S) \leq (2\tilde{\alpha}-1)e(B)$. Now we obtain a desired subset $A \cup S$ in G with size $|A \cup S| = kn + \tilde{\alpha}\tilde{n} = \alpha n$ and

$$\begin{aligned} e(A \cup S) = e(A) + e(A,S) + e(S) &\leq e(A) + e(A,B) + (2\tilde{\alpha}-1)e(B) \\ &= (2\tilde{\alpha}-1)\left(e(A) + e(A,B) + e(B)\right) + 2(1-\tilde{\alpha})\left(e(A) + e(A,B)\right) \\ &< (2\tilde{\alpha}-1)cn^2 + 2(1-\tilde{\alpha})k(1-\alpha)n^2 \leq (2\alpha-1)cn^2, \end{aligned}$$

where the second last inequality is strict since $e(A) + e(A,B) < |A|(1-\alpha)n = k(1-\alpha)n^2$, and the last inequality follows from

$$(2\tilde{\alpha}-1)c + 2(1-\tilde{\alpha})(1-\alpha)k - (2\alpha-1)c = \frac{2k(1-\alpha)(\alpha+c-1)}{k-1} \leq 0.$$

Therefore in case of $\delta(G) < (1-\alpha)n$, there always exists a subset of size αn spanning strictly less than $(2\alpha-1)cn^2$ edges. Together with Proposition 3.1, we have finished the proofs of Theorem 1.5 for both (1) and (2). ∎

4 Sparse halves

In this section we prove Theorem 1.3. Let G be a K_4-free graph on n vertices. For a vertex set $S \subset V(G)$ with $|S| = \lfloor n/2 \rfloor$, we call it a *sparse half* of G if $e(S) \leq n^2/18$.

We will consider three cases regarding the edge density of G and use quite different techniques in each case. If G is sparse, then we will use some probabilistic arguments to show that it contains a sparse half. If G is dense, then a result of Lyle [9] gives a nice structure on G and this enables us to find a sparse half. The most intricate case is when the edge density of G is intermediate. In this case, assuming G does not contain a sparse half, we will first find three large disjoint independent sets in G (by using Theorem 1.5), and then building on these sets, use probabilistic arguments (in a complicated way) to derive a contradiction. Finally, we infer Theorem 1.3 from these cases in Section 4.4.

In the rest of this section we will state our results without assuming the parities of integers n. However for convenience, in the proofs we will always view n as even in order to avoid the floors (while the same arguments also work for odd n). For Theorem 1.3, we will see in Section 4.4 that it suffices to only consider when n is divisible by 6.

4.1 Sparse range

In this section we will prove the following for graphs with few edges.

Theorem 4.1. *Suppose that G is a K_4-free graph on n vertices with at most $0.26n^2$ edges. Then G contains a vertex set of size $\lfloor n/2 \rfloor$ that spans strictly less than $n^2/18$ edges.*

We need the following two lemmas from [13] which are proved by probabilistic arguments. Let $t(G)$ denote the number of triangles in G.

Lemma 4.2 (Sudakov, [13]). *Every graph G on n vertices contains a bipartite subgraph G' such that*

$$e(G') \geq \frac{1}{n} \sum_{v \in V(G)} (d(v))^2 - \frac{2}{n} \sum_{v \in V(G)} e(N(v)) \geq \frac{4(e(G))^2}{n^2} - \frac{6t(G)}{n}.$$

Lemma 4.3 (Sudakov, [13]). *Every K_4-free graph on n vertices contains a bipartite subgraph G' such that*

$$e(G') \geq \frac{e(G)}{2} + \frac{1}{n} \sum_{v \in V(G)} \left(\frac{4(e(N(v)))^2}{(d(v))^2} - \frac{e(N(v))}{2} \right).$$

The next lemma shows that if a K_4-free graph G contains a large enough bipartite subgraph, then it contains a sparse half.

Lemma 4.4. *Let G be a K_4-free graph on n vertices with cn^2 edges. Suppose that there is a partition $A \cup B = V(G)$ such that $e(A, B) > 9c^2n^2/4$. Then G contains a vertex set of size $\lfloor n/2 \rfloor$ that spans strictly less than $n^2/18$ edges.*

Proof. Suppose for the contrary that every vertex set of size $n/2$ in G spans at least $n^2/18$ edges. By Proposition 2.1, we may assume that $c \geq \frac{2}{9}$. Assume that $a := |A|/n \leq 1/2$. Applying Proposition 2.1 to $G[B]$, we obtain a vertex set $S \subset B$ with $|S| = n/2$ such that $e(S) \leq \left(\frac{1/2}{1-a}\right)^2 e(B)$. By assumption we have $\left(\frac{1/2}{1-a}\right)^2 e(B) \geq n^2/18$, which implies

$$e(B) \geq \frac{2(1-a)^2}{9} n^2.$$

Now applying Proposition 2.2 to $A \cup B$, we see that there exists $T \subset V(G)$ with $|T| = n/2$ such that $A \subset T$ and $e(T) \leq cn^2 - \frac{1}{2(1-a)} e(A, B) - \frac{3/2-2a}{2(1-a)^2} e(B)$. By assumption, we have

$$\frac{n^2}{18} \leq cn^2 - \frac{1}{2(1-a)} e(A, B) - \frac{3/2 - 2a}{2(1-a)^2} e(B) \leq cn^2 - \frac{1}{2(1-a)} e(A, B) - \frac{3/2 - 2a}{9} n^2,$$

which implies

$$e(A, B) \leq 2(1-a) \left(c - \frac{3/2 - 2a}{9} - \frac{1}{18} \right) n^2 = \left(2(1-a)c - \frac{4}{9}(1-a)^2 \right) n^2.$$

Since the maximum of $2(1-a)c - 4(1-a)^2/9$ is attained when $a = 1 - 9c/4$ (note that $a = 1 - 9c/4 \leq 1/2$ as $c \geq 2/9$), we obtain

$$e(A, B) \leq \left(2\left(1 - \left(1 - \frac{9c}{4}\right)\right) c - \frac{4}{9} \left(1 - \left(1 - \frac{9c}{4}\right)\right)^2 \right) n^2 = \frac{9}{4} c^2 n^2,$$

a contradiction. ∎

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $c = e(G)/n^2$ and let $\lambda = 8/13$. By Lemmas 4.2 and 4.3, there exists a partition $A \cup B = V(G)$ such that

$$\begin{aligned} e(A,B) &\geq (1-\lambda)\left(\frac{1}{n}\sum_{v\in V(G)}(d(v))^2 - \frac{2}{n}\sum_{v\in V(G)} e\left(N(v)\right)\right) \\ &\quad + \lambda\left(\frac{e(G)}{2} + \frac{1}{n}\sum_{v\in V(G)}\left(\frac{4\left(e(N(v))\right)^2}{(d(v))^2} - \frac{e(N(v))}{2}\right)\right) \\ &= \frac{\lambda}{2}e(G) + \frac{4\lambda}{n}\sum_{v\in V(G)}(d(v))^2\left(\left(\frac{e\left(N(v)\right)}{(d(v))^2}\right)^2 - \frac{2-3\lambda/2}{4\lambda}\frac{e\left(N(v)\right)}{(d(v))^2} + \frac{1-\lambda}{4\lambda}\right). \end{aligned}$$

Since

$$x^2 - \frac{2-3\lambda/2}{4\lambda}x + \frac{1-\lambda}{4\lambda} \geq \frac{88\lambda - 73\lambda^2 - 16}{256\lambda^2},$$

we obtain

$$\begin{aligned} e(A,B) &\geq \frac{\lambda}{2}e(G) + \frac{88\lambda - 73\lambda^2 - 16}{64\lambda}\sum_{v\in V(G)}\frac{d(v)^2}{n} \\ &\geq \frac{\lambda}{2}e(G) + \frac{88\lambda - 73\lambda^2 - 16}{64\lambda}\sum_{v\in V(G)}\left(\frac{\sum_{v\in V(G)} d(v)}{n}\right)^2 \\ &= \left(\frac{\lambda}{2}c + \frac{88\lambda - 73\lambda^2 - 16}{16\lambda}c^2\right)n^2 = \left(\frac{4}{13}c + \frac{111}{104}c^2\right)n^2. \end{aligned}$$

Since $\frac{4}{13}c + \frac{111}{104}c^2 > \frac{9}{4}c^2$ holds for all $c \in (0, \frac{32}{123})$ and $\frac{32}{123} > 0.26$, we derive that $e(A,B) > \frac{9}{4}c^2n^2$ whenever $c \leq 0.26$. Therefore, by Lemma 4.4, G contains a vertex set of size $n/2$ that spans strictly less than $n^2/18$ edges. ∎

4.2 Dense range

In this section we prove the following for graphs with high minimum degree.

Theorem 4.5. *Suppose that G is a K_4-free graph on n vertices with $\delta(G) \geq 0.59n$. Then G contains a vertex set of size $\lfloor n/2 \rfloor$ that spans at most $n^2/18$ edges. Moreover, if every vertex set of size $\lfloor n/2 \rfloor$ in G spans at least $n^2/18$ edges, then $G \cong T_3(n)$.*

To show this, we need a structural result on dense K_4-free graphs. A K_r-free graph G is *maximal* if adding any new edge to G will result in a copy of K_r. Let G_1 and G_2 be two vertex disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

Theorem 4.6 (Lyle, [9])**.** *Let G be a maximal K_4-free on n vertices with $\delta(G) \geq 4n/7$. Then either G contains an independent set of size at least $4\delta(G) - 2n$ or G is the join of an independent set and a triangle-free graph.*

Our next lemma shows that if a K_4-free graph G contains a large induced triangle-free graph, then G contains a sparse half.

Lemma 4.7. *Let G be a K_4-free graph on n vertices. Suppose that G contains an induced triangle-free subgraph Γ with at least $2n/3$ vertices. Then G contains a vertex set of size $n/2$ that spans at most $n^2/18$ edges. Moreover, if $|V(\Gamma)| > 2n/3$, then G contains a vertex set of size $n/2$ which spans strictly less than $n^2/18$ edges.*

Proof. Let $A \subset V(G)$ such that $\Gamma = G[A]$ and let $x = |A|/n$. We may assume that $x \le 5/6$ since otherwise we could choose $A' \subset A$ with $|A'| = 5n/6$ and consider $G[A']$ instead. Let $\alpha = 1/(2x)$. Then $\alpha \ge 3/5$. By a result of Krivelevich on triangle-free graphs [8], there exists $T \subset A$ with $|T| = \alpha|A| = n/2$ such that

$$e(T) \le \frac{2 \times \frac{1}{2x} - 1}{4}|A|^2 = \frac{(1-x)x}{4}n^2 \le \frac{n^2}{18},$$

where in the last inequality we used the assumption that $x \ge 2/3$. Notice that if $x > 2/3$, then the inequality above is strict. This proves the lemma. ∎

We also need the following slightly stronger version of Krivelevich's theorem on local densities of triangle-free graphs. A proof is included in the appendix, which follows from a detailed analysis of Krivelevich's proof in [8] as well as the proof of Erdős et al. in [6].

Theorem 4.8 (Krivelevich, [8])**.** *Let $3/5 < \alpha \le 1$, $n \in \mathbb{N}$ and $\alpha n \in \mathbb{N}$. Let G be a triangle-free graph on n vertices. If every vertex set of size αn in G spans at least $\frac{2\alpha-1}{4}n^2$ edges, then $G \cong T_2(n)$.*

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. It is clear that to prove Theorem 4.5, it suffices to consider maximal K_4-free graphs. Let G be a maximal K_4-free graph on n vertices with $\delta(G) \ge 0.59n > 4n/7$. Then by Theorem 4.6, either G is the join of an independent set and a triangle-free graph or G contains an independent set of size at least $4\delta(G) - 2n$.

First, suppose that the former case occurs, that is, G is the join of an independent set I and a triangle-free graph Γ. Let $\alpha = |V(\Gamma)|/n$. So $|I| = (1-\alpha)n$. We may assume that $\alpha > 1/2$ since otherwise we can simply choose a subset of I with size $n/2$ which spans none of edges. On the other hand, if $\alpha > 2/3$, then by Lemma 4.7, we are done. So we may assume that $1/2 < \alpha \le 2/3$.

Let $c = e(\Gamma)/(\alpha n)^2$. If $c < 2/9$, then by Proposition 2.1, there exists $S \subset V(\Gamma)$ with $|S| = n/2$ such that

$$e(S) \le \left(\frac{1/2}{\alpha}\right)^2 c(\alpha n)^2 = \frac{1}{4}cn^2 < \frac{n^2}{18}.$$

So we may assume that $c \ge 2/9$. Since Γ has at least $2\alpha^2 n^2/9$ edges, there exists some $v \in V(\Gamma)$ such that $d_\Gamma(v) \ge 4\alpha^2 n/9 \ge (\alpha - 1/2)n$, where the last inequality holds as $1/2 < \alpha \le 2/3$. Let $T \subset N_\Gamma(v)$ be any subset with $|T| = (\alpha - 1/2)n$. Since Γ is triangle-free, T is an independent set. Therefore, $I \cup T$ has size $n/2$ and satisfies

$$e(I \cup T) \le (1-\alpha)\left(\alpha - \frac{1}{2}\right)n^2 \le \frac{n^2}{18},$$

where the last inequality uses the assumption that $\alpha \le 2/3$. Notice that if $\alpha < 2/3$, then the inequality above is strict.

Now we may assume that G contains an independent set A whose size is at least $4\delta(G) - 2n \geq 9n/25$. We may just take A such that $|A| = 9n/25$. Let $B = V(G) \setminus A$. By Proposition 2.2, there exists $U \subset B$ with $|U| = 7n/50$ such that

$$\begin{aligned} e(A \cup U) &\leq \frac{7/50}{16/25} e(A,B) + \left(\frac{7/50}{16/25}\right)^2 e(B) = \frac{175}{1024} e(A,B) + \frac{49}{1024}(e(A,B) + e(B)) \\ &\leq \frac{175}{1024}\left(\frac{9}{25}n \times \frac{16}{25}n\right) + \frac{49}{1024} \times \frac{n^2}{3} = \frac{4249}{76800}n^2 < \frac{n^2}{18}. \end{aligned}$$

Therefore, $A \cup U$ is a sparse half with $e(A \cup U) < n^2/18$.

From the arguments above, one could see that if every vertex set of size $n/2$ in G spans at least $n^2/18$ edges, then G must be the join of a triangle-free graph Γ and an independent set I with $|V(\Gamma)| = 2n/3$. Since every vertex set of size $n/2 = \frac{3}{4}|V(\Gamma)|$ in Γ spans at least $n^2/18 = (2\cdot 3/4 - 1)|V(\Gamma)|^2/4$ edges, by Theorem 4.8 we have $\Gamma \cong T_2(2n/3)$, which implies $G \cong T_3(n)$. This finishes the proof of Theorem 4.5. ∎

4.3 Intermediate range

In this section we will prove the following result for regular graphs.

Theorem 4.9. *Every K_4-free regular graph G on n vertices with $e(G) \leq 0.297n^2$ contains a vertex set of size $\lfloor n/2 \rfloor$ that spans strictly less than $n^2/18$ edges.*

We would like to remind the reader that the assumption that G is regular in Theorem 4.9 can be replaced by $\Delta(G) - \delta(G) \leq \epsilon n$ for some absolute (but small) constant $\epsilon > 0$. However, in order to keep the proof simple we shall only consider regular graphs.

The proof ideas are as follows. First, under the assumption that all subsets of size $n/2$ span at least $n^2/18$ edges, we show that G must contain many triangles. Then we show that there exists a partition $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that V_1, V_2, V_3 are independent sets and $|V_1| + |V_2| + |V_2|$ is relatively large. Finally, utilizing this partition, we employ some ad hoc probabilistic arguments to find a sparse half and thus reach a contradiction.

Recall that $t(G)$ denotes the number of triangles in G.

Lemma 4.10. *Let G be an n-vertex K_4-free graph with cn^2 edges and $n/2 \leq \delta(G) \leq \Delta(G) \leq 9n/14$. Suppose that every vertex set of size $\lfloor n/2 \rfloor$ in G spans at least $n^2/18$ edges. Then we have $t(G) \geq \frac{c}{27(1-2c)}n^3$.*

Proof. For every $v \in V(G)$ let $\alpha_v = \frac{n}{2d(v)}$ and $c_v = e(N(v))/\left(d(v)\right)^2$. First notice that $c_v \geq 2/9$ for all $v \in V(G)$, since otherwise by Proposition 2.1, there would be a set $S \subset N(v)$ with $|S| = n/2$ such that $e(S) \leq \alpha_v^2 \cdot e(N(v)) < n^2/18$, a contradiction.

Fix $v \in V(G)$. Since $d(v) \leq 9n/14$, we see $\alpha_v \geq 7/9 \geq 1 - c_v$. By Theorem 1.5 and our assumption, there exists a vertex set $T \subset N(v)$ with $|T| = n/2$ such that $\frac{n^2}{18} \leq e(T) \leq (2\alpha_v - 1)e(N(v))$. This implies that for all $v \in V(G)$,

$$e(N(v)) \geq \frac{n^2}{18}\frac{1}{2\alpha_v - 1} = \frac{n^2}{18}\frac{d(v)}{n - d(v)}.$$

Summing over all $v \in V(G)$, we obtain that $t(G) = \frac{1}{3}\sum_{v\in V(G)} e(N(v))$ is at least

$$\frac{n^2}{54}\sum_{v\in V(G)}\frac{d(v)}{n-d(v)} \geq \frac{n^2}{54}\frac{\sum_{v\in V(G)} d(v)}{\sum_{v\in V(G)}(n-d(v))} = \frac{n^2}{54}\frac{2e(G)}{n^2-2e(G)} = \frac{c}{27(1-2c)}n^3.$$

Here we used Jensen's inequality and the fact that $\frac{x}{n-x}$ is concave up for $x \in (0,n)$. ∎

We also need the following lemma in [13]. For distinct $u, v \in V(G)$, let $N(uv)$ denote the set of common neighbors of u and v and let $d(uv) = |N(uv)|$.

Lemma 4.11 (Sudakov, [13]). *Every graph G with e edges and m triangles contains a triangle uvw such that $d(uv) + d(vw) + d(wu) \geq \frac{9m}{e}$.*

Notice that if G is K_4-free, then $N(uv)$ is independent for all $uv \in E(G)$ and $N(uv) \cap N(vw) = \emptyset$ for all triangles uvw in G. The following lemma is an immediate consequence of Lemmas 4.10 and 4.11.

Lemma 4.12. *Let G be an n-vertex K_4-free graph with cn^2 edges and $n/2 \leq \delta(G) \leq \Delta(G) \leq 9n/14$. Suppose every vertex set of size $\lfloor n/2 \rfloor$ in G spans at least $n^2/18$ edges. Then there exist three disjoint independent sets V_1, V_2, V_3 in G such that*

$$|V_1| + |V_2| + |V_3| \geq \frac{n}{3(1-2c)}.$$

Now we are ready to prove Theorem 4.9.

Proof of Theorem 4.9. Let G be a K_4-free regular graph on n vertices with cn^2 edges, where $c \leq 0.297$. Suppose that every vertex set of size $n/2$ in G spans at least $n^2/18$ edges. By Theorem 4.1, we may assume that $c \in [1/4, 0.297]$. So every vertex has degree $2cn$ with $n/2 \leq 2cn \leq 0.594n < 9n/14$. Then by Lemma 4.12, there exist three disjoint independent sets V_1, V_2, V_3 in G such that

$$|V_1| + |V_2| + |V_3| = g(c)n, \quad \text{where} \quad g(c) = \frac{1}{3(1-2c)}.$$

Let $V_4 = V(G) \setminus \left(\bigcup_{i=1}^3 V_i\right)$ and let $x_i = |V_i|/n$ for $i \in [4]$. Without loss of generality we may assume that $1/2 > x_1 \geq x_2 \geq x_3$. Let $e_{ij} = e(V_i, V_j)$ for all $\{i, j\} \subset [4]$ (so $e_{ij} = e_{ji}$) and let $e_4 = e(V_4)$. We will consider four cases depending on the values of x_1, x_2 and x_3.

Case 1: $x_1 + x_2 \geq x_1 + x_3 \geq x_2 + x_3 \geq \frac{1}{2}$.

V_1	V_2	V_3	V_4
x_1	$1/2 - x_1$	0	0
x_1	0	$1/2 - x_1$	0
0	x_2	$1/2 - x_2$	0
$1/2 - x_2$	x_2	0	0
$1/2 - x_3$	0	x_3	0
0	$1/2 - x_3$	x_3	0

Table 1: different schemes for choosing $n/2$ vertices from G.

Now we choose different $n/2$ vertices from G according to Table 1. For example, the second row in Table 1 means to choose all vertices of V_1 and choose a set $S \subset V_2$ with

$|S| = (1/2 - x_1)\, n$ uniformly at random. Then the expected value of $e(V_1 \cup S)$ is $\frac{1/2 - x_2}{x_2} e_{12}$. So there exists $S \subset V_2$ with $|S| = (1/2 - x_1)\, n$ such that $e(V_1 \cup S) \leq \frac{1/2 - x_2}{x_2} e_{12}$. By assumption, we have

$$\frac{1/2 - x_1}{x_2} e_{12} \geq \frac{n^2}{18} \quad \Rightarrow \quad e_{12} \geq \frac{1}{36} \frac{x_2}{1 - 2x_1} n^2.$$

Similarly, one can get from Table 1 that for all $(i, j) \in [3] \times [3]$ with $i \neq j$,

$$\frac{1/2 - x_i}{x_j} e_{ij} \geq \frac{n^2}{18} \quad \Rightarrow \quad e_{ij} \geq \frac{1}{36} \frac{x_j}{1 - 2x_i} n^2.$$

Adding them up, we obtain that $e_{12} + e_{13} + e_{23} = \frac{1}{2} \sum_{i \neq j} e_{ij}$ is at least

$$\frac{1}{18} \left(\frac{x_2 + x_3}{1 - 2x_1} + \frac{x_1 + x_3}{1 - 2x_2} + \frac{x_2 + x_1}{1 - 2x_3} \right) n^2 = \frac{1}{18} \left(\frac{g(c) - x_1}{1 - 2x_1} + \frac{g(c) - x_2}{1 - 2x_2} + \frac{g(c) - x_3}{1 - 2x_3} \right) n^2.$$

Since $\frac{g(c) - x}{1 - 2x}$ is concave up, by Jensen's inequality we see that

$$e_{12} + e_{13} + e_{23} \geq \frac{1}{6} \cdot \frac{g(c) - (x_1 + x_2 + x_3)/3}{1 - 2(x_1 + x_2 + x_3)/3} n^2 = \frac{g(c)}{3(3 - 2g(c))} n^2. \tag{1.1}$$

On the other hand, since G is regular,[3] we have

$$e_{14} + e_{24} + e_{34} + 2e_4 = \sum_{v \in V_4} d(v) = 2cn \times |V_4| = 2c(1 - g(c))n^2. \tag{1.2}$$

Since $G[V_4]$ is K_4-free, by Turán's Theorem we get

$$e_4 \leq \frac{1}{3} |V_4|^2 = \frac{(1 - g(c))^2}{3} n^2. \tag{1.3}$$

Therefore, it follows from (1.1), (1.2) and (1.3) that (recall that V_1, V_2, V_3 are independent)

$$cn^2 + \frac{(1 - g(c))^2}{3} n^2 \geq e(G) + e_4 \geq \frac{g(c)}{3(3 - 2g(c))} n^2 + 2c(1 - g(c))n^2,$$

which is a contradiction because

$$h(c) := \frac{g(c)}{3(3 - 2g(c))} + 2c\,(1 - g(c)) - \left(c + \frac{(1 - g(c))^2}{3} \right)$$

is decreasing in c for $c \in [1/4, 0.297]$ and $h(0.297) > 0$ (see [10]). This proves Case 1.

Case 2: $x_2 + x_3 \leq x_1 + x_3 \leq x_1 + x_2 < \frac{1}{2}$.

Note that this case can exist only when $g(c) < 3/4$, which implies $c < 5/18$.

Now we choose $n/2$ vertices according to Table 2. Then similar to Case 1, we obtain that for every $k \in [3]$ and $\{i, j\} = [3] \backslash \{k\}$,

$$e_{ij} + \frac{1/2 - x_i - x_j}{x_k} (e_{ik} + e_{jk}) \geq \frac{n^2}{18}, \tag{2.1}$$

[3] We point out that throughout the proof of Theorem 4.9, this is the only place where we need the restriction that G is regular.

V_1	V_2	V_3	V_4
x_1	x_2	$1/2 - x_1 - x_2$	0
x_1	$1/2 - x_1 - x_3$	x_3	0
$1/2 - x_2 - x_3$	x_2	x_3	0
x_1	0	0	$1/2 - x_1$
0	x_2	0	$1/2 - x_2$
0	0	x_3	$1/2 - x_3$

Table 2: different schemes for choosing $n/2$ vertices from G.

and for all $i \in [3]$

$$\frac{1/2 - x_i}{x_4} e_{i4} + \left(\frac{1/2 - x_i}{x_4}\right)^2 e_4 \geq \frac{n^2}{18}. \tag{2.2}$$

By simplifying the linear combination of

$$\sum_{k \in [3]} \left(\frac{x_k}{x_4} \times (2.1)\right) + \sum_{i \in [3]} \left(\frac{x_4}{1/2 - x_i} \times (2.2)\right),$$

we can derive that

$$\begin{aligned} e(G) + \frac{e_4}{2x_4} \geq \frac{x_1 + x_2 + x_3}{18x_4} n^2 + \frac{x_4}{18}\left(\frac{1}{1/2 - x_1} + \frac{1}{1/2 - x_2} + \frac{1}{1/2 - x_3}\right) n^2 \\ \geq \frac{x_1 + x_2 + x_3}{18x_4} n^2 + \frac{x_4}{18} \times \frac{3}{1/2 - (x_1 + x_2 + x_3)/3} n^2 = \frac{1 - x_4}{18x_4} n^2 + \frac{x_4}{3 - 2(1 - x_4)} n^2. \end{aligned}$$

Since $e_4 \leq |V_4|^2/3 = x_4^2 n^2/3$ and $x_4 = 1 - g(c)$, the inequality above implies

$$c + \frac{1 - g(c)}{6} \geq \frac{1 - (1 - g(c))}{18(1 - g(c))} + \frac{1 - g(c)}{3 - 2(1 - (1 - g(c)))} = \frac{g(c)}{18(1 - g(c))} + \frac{1 - g(c)}{3 - 2g(c)},$$

which is a contradiction because

$$k(c) := c + \frac{1 - g(c)}{6} - \left(\frac{g(c)}{18(1 - g(c))} + \frac{1 - g(c)}{3 - 2g(c)}\right)$$

is strictly smaller than 0 for $c \in [1/4, 5/18)$ (see [10]). This proves Case 2.

Case 3: $x_2 + x_3 < \frac{1}{2} \leq x_1 + x_3 \leq x_1 + x_2$.

V_1	V_2	V_3	V_4
x_1	$1/2 - x_1$	0	0
x_1	0	$1/2 - x_1$	0
$1/2 - x_2 - x_3$	x_2	x_3	0
x_1	0	0	$1/2 - x_1$
$1/2 - x_2 - x_4$	x_2	0	x_4
$1/2 - x_3 - x_4$	0	x_3	x_4

Table 3: different schemes for choosing $n/2$ vertices from G.

We choose $n/2$ vertices according to Table 3. Similar as above, we can obtain that

$$\frac{1/2 - x_1}{x_i} e_{1i} \geq \frac{n^2}{18} \quad \text{for} \quad i \in \{2,3\}, \tag{3.1}$$

$$e_{23} + \frac{1/2 - x_2 - x_3}{x_1}(e_{12} + e_{13}) \geq \frac{n^2}{18}, \tag{3.2}$$

$$\frac{1/2 - x_1}{x_4} e_{14} + \left(\frac{1/2 - x_1}{x_4}\right)^2 e_4 \geq \frac{n^2}{18}, \quad \text{and} \tag{3.3}$$

$$e_{j4} + e_4 + \frac{1/2 - x_j - x_4}{x_1}(e_{1j} + e_{14}) \geq \frac{n^2}{18} \quad \text{for} \quad j \in \{2,3\}. \tag{3.4}$$

By simplifying the linear combination of

$$\sum_{i=2,3}\left(\frac{x_i^2}{(1/2 - x_1)x_1} \times (3.1)\right) + (3.2) + \frac{x_4^2}{(1/2 - x_1)x_1} \times (3.3) + \sum_{j=2,3}(3.4),$$

we derive that

$$e(G) + \frac{e_4}{2x_1} \geq \left(\frac{1}{6} + \frac{x_2^2 + x_3^2 + x_4^2}{9x_1(1 - 2x_1)}\right) n^2.$$

Since $e_4 \leq |V_4|^2/3 = x_4^2 n^2/3$, the inequality above implies that

$$c + \frac{x_4^2}{6x_1} \geq \frac{1}{6} + \frac{x_2^2 + x_3^2 + x_4^2}{9x_1(1 - 2x_1)}.$$

This is a contradiction due to the following claim whose proof can be found in the appendix.

Claim 4.13. *Under the conditions of Case 3, we have*

$$\frac{1}{6} + \frac{x_2^2 + x_3^2 + x_4^2}{9x_1(1 - 2x_1)} - \frac{x_4^2}{6x_1} - c > 0.$$

This contradiction completes the proof of Case 3.

Case 4: $x_2 + x_3 \leq x_1 + x_3 < \frac{1}{2} \leq x_1 + x_2$.

V_1	V_2	V_3	V_4
x_1	$1/2 - x_1$	0	0
x_1	0	$1/2 - x_1$	0
x_1	$1/2 - x_1 - x_3$	x_3	0
$1/2 - x_2 - x_3$	x_2	x_3	0
x_1	0	0	$1/2 - x_1$
0	x_2	0	$1/2 - x_2$
$1/2 - x_3 - x_4$	0	x_3	x_4
0	$1/2 - x_3 - x_4$	x_3	x_4

Table 4: different schemes for choosing $n/2$ vertices from G.

Choosing $n/2$ vertices according to Table 4, we obtain that

$$\frac{1/2-x_i}{x_{3-i}}e_{i,3-i} \geq \frac{n^2}{18} \quad \Rightarrow \quad e_{i,3-i} \geq \frac{x_{3-i}}{1/2-x_i}\frac{n^2}{18} \quad \text{for each } i \in [2], \tag{4.1}$$

$$e_{j3} + \frac{1/2-x_j-x_3}{x_2}(e_{12}+e_{j3}) \geq \frac{n^2}{18} \quad \text{for each } j \in [2], \tag{4.2}$$

$$\frac{1/2-x_k}{x_4}e_{k4} + \left(\frac{1/2-x_k}{x_4}\right)^2 e_4 \geq \frac{n^2}{18} \quad \text{for each } k \in [2], \quad \text{and} \tag{4.3}$$

$$e_{34} + e_4 + \frac{1/2-x_3-x_4}{x_\ell}(e_{\ell 3}+e_{\ell 4}) \geq \frac{n^2}{18} \quad \text{for each } \ell \in [2]. \tag{4.4}$$

By simplifying the linear combination of

$$\frac{1}{2}\left(1+\frac{1}{1-2x_3}-\frac{1}{x_1+x_2}\right)\sum_{i\in[2]}(4.1) + \frac{1}{1-2x_3}\sum_{j\in[2]}\left(\frac{x_{3-j}}{x_1+x_2}\times(4.2)\right)$$
$$+\frac{1}{2(x_1+x_2)}\sum_{k\in[2]}\left(\frac{x_4}{1/2-x_k}\times(4.3)\right) + \frac{x_1+x_2}{1/2-x_3-x_4}\sum_{\ell\in[2]}\left(\frac{x_\ell}{1/2-x_3-x_4}\times(4.4)\right),$$

it yields that

$$e(G) + \frac{1-x_1-x_2}{2(x_1+x_2)x_4}e_4 \geq \left(\frac{1}{2}\left(1+\frac{1}{1-2x_3}-\frac{1}{x_1+x_2}\right)\left(\frac{x_2}{1/2-x_1}+\frac{x_1}{1/2-x2}\right)\right.$$
$$\left.+\frac{1}{1-2x_3}+\frac{1}{2(x_1+x_2)}\left(\frac{x_4}{1/2-x_1}+\frac{x_4}{1/2-x_2}\right)+1\right)\frac{n^2}{18}.$$

Since $e_4 \leq |V_4|^2/3 = x_4^2 n^2/3$, the inequality above implies that

$$c + \frac{(1-x_1-x_2)x_4}{6(x_1+x_2)} \geq \frac{1}{18}\left(\frac{1}{2}\left(1+\frac{1}{1-2x_3}-\frac{1}{x_1+x_2}\right)\left(\frac{x_2}{1/2-x_1}+\frac{x_1}{1/2-x2}\right)\right.$$
$$\left.+\frac{1}{1-2x_3}+\frac{1}{2(x_1+x_2)}\left(\frac{x_4}{1/2-x_1}+\frac{x_4}{1/2-x_2}\right)+1\right).$$

Again, this is a contradiction because of the following claim, whose proof is included in the appendix.

Claim 4.14. *Under the conditions of Case 4, we have*

$$c + \frac{(1-x_1-x_2)x_4}{6(x_1+x_2)} < \frac{1}{18}\left(\frac{1}{2}\left(1+\frac{1}{1-2x_3}-\frac{1}{x_1+x_2}\right)\left(\frac{x_2}{1/2-x_1}+\frac{x_1}{1/2-x2}\right)\right.$$
$$\left.+\frac{1}{1-2x_3}+\frac{1}{2(x_1+x_2)}\left(\frac{x_4}{1/2-x_1}+\frac{x_4}{1/2-x_2}\right)+1\right).$$

This completes the proof of Theorem 4.9. ∎

4.4 Proof of Theorem 1.3

Let G be a K_4-free regular graph on n vertices such that every vertex set of size $\lfloor n/2 \rfloor$ in G spans at least $n^2/18$ edges. Our goal is to show that n is divisible by 6 and $G \cong T_3(n)$

First we show that it suffices to consider the case that n is divisible by 6. Assume that we have proved for all n that are divisible by 6, and now consider the case that n is not divisible by 6. Let H be the blow-up of G obtained by replacing every vertex $i \in V(G)$ by a set V_i of size 6 and replacing every edge $ij \in E(G)$ by a complete bipartite graph with parts V_i and V_j. Then H contains $N := 6n$ vertices and is K_4-free and regular, hence by our assumption, if let $S \subset V(H)$ be a subset of size $N/2 = 3n$ spanning the minimum number of edges, then we have $e(S) \leq N^2/18$. We may assume that S either contains V_i or is disjoint from V_i for all but at most one i, since if there are two indices i, j satisfying $1 \leq |S \cap V_\ell| \leq 5$ for $\ell \in \{i, j\}$, then we could increase one of the intersections and decreasing the other until $|S \cap V_\ell| \in \{0, 6\}$ for some ℓ, without increasing $e(S)$. So, S contains $\lfloor n/2 \rfloor$ sets V_i. By our assumption on G, $e(S) \geq 36 \lceil n^2/18 \rceil > N^2/18$, a contradiction.

Now we assume that n is divisible by 6. Let $e(G) = cn^2$ for some $c \in (0, 1/3]$. Then every vertex in G has degree $2cn$. If $c \leq 0.26$, then by Theorem 4.1, there exists a vertex set of size $n/2$ that spans strictly less than $n^2/18$ edges, a contradiction. If $c \geq 0.295$, then by Theorem 4.5, we can derive that $G \cong T_3(n)$. So it remains to consider $0.26 < c < 0.295$. In this case, by Theorem 4.9, G contains a vertex set of size $n/2$ that spans strictly less than $n^2/18$ edges, again a contradiction. We have completed the proof of Theorem 1.3. ∎

5 Concluding remarks

In this paper we consider the local density problem, and prove Conjecture 1.2 for all K_4-free regular graphs. To fully resolve Conjecture 1.2 there are two barriers in our proofs. The first one is the minimum degree condition: the proof of Theorem 4.5 requires the minimum degree to be at least $4n/7$ for the structure from Theorem 4.6, while the proof of Theorem 4.9 requires the minimum degree to be at least $n/2$ for a good lower bound on the number of triangles. The other barrier is the regular condition: the proof of Case 1 of Theorem 4.9 (i.e., the footnote 3) requires G to be regular for obtaining a lower bound on the number of edges that contains at least one vertex of V_4.

A closely related problem is the problem of making a graph bipartite. A famous conjecture of Erdős [3] states that every triangle-free graph on n vertices can be made bipartite by deleting at most $n^2/25$ edges. This is still open, with the extremal graphs to be the balanced blow-ups of the 5-cycle. Following from Krivelevich's observation [8], we see that for regular graphs, Conjecture 1.1 would imply the above conjecture of Erdős. So it seems interesting (but perhaps still difficult) to attack Conjecture 1.1 for regular graphs.

For analogous problems on K_r-free graphs and other related problems, we direct interested readers to [2, 5, 6, 13].

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Appendices

A Proof of Theorem 4.8

In this section we prove Theorem 4.8. We need the following lemmas.

Lemma A.1 ([6]). *Let $0 < \alpha \leq 1$ and let G be a triangle-free graph on αn vertices with at least $(2\alpha - 1)n^2/4$ edges. Then G contains a matching with at least $(2\alpha - 1)n/2$ edges.*

Lemma A.2. *Let $1/2 \leq \alpha \leq 1$, $n \in \mathbb{N}$ and $\alpha n \in \mathbb{N}$. Let G be bipartite graph on n vertices. If every vertex set of size αn in G spans at least $(2\alpha - 1)n^2/4$ edges, then $G \cong T_2(n)$.*

Proof. Let $V_1 \cup V_2 = V(G)$ be a partition such that G is a bipartite graph with parts V_1 and V_2. Let $x = |V_1|/n$ and we may assume that $x \geq 1/2$. By assumption, $x < \alpha$, since otherwise there would be a subset of A of size αn that spans zero edges. Now choose a random set $S \subset B$ with $|S| = (\alpha - x)n$. Then $|A \cup B| = \alpha n$ and $e(A \cup S) \leq x(\alpha - x)n^2 \leq (2\alpha - 1)n^2/4$. By assumption the inequality above must be tight, which means $x = 1/2$ and $G[A, S]$ is a complete bipartite graph. Since S was chosen randomly, G must be a complete bipartite graph with $|V_1| = n/2$. Therefore, $G \cong T_2(n)$. ∎

Now we prove Theorem 4.8.

Proof of Theorem 4.8. First one could see from Kriveleich's proof (i.e. the proof of Theorem 4) in [8] that if G does not contain an independent set of size $(1 - \alpha)n$, then there exists a vertex set of size αn in G that spans strictly less than $\frac{2\alpha-1}{4}n^2$ edges. So by assumption there exists an independent set in G whose size is $(1 - \alpha)n$. Next, we use the argument of Erdős et al. [6] to show that $G \cong T_2(n)$.

Let $A \subset V(G)$ be an independent set of size $(1 - \alpha)n$. By Lemma A.1, there exists a matching M in $G[V(G) \setminus A]$ with $(2\alpha - 1)n/2$ edges. Let $C = V(M)$ and let $B = V(G) \setminus (A \cup C)$. Note that $|C| = (2\alpha - 1)n$.

Since G is triangle-free and M is a matching, every vertex in A is adjacent to at most half of the vertices in C. Therefore, $e(A, C) \leq (1 - \alpha)(2\alpha - 1)n^2/2$ and hence

$$e(A \cup C) = e(A, C) + e(C) \leq \frac{(1-\alpha)(2\alpha-1)n^2}{2} + \frac{(2\alpha-1)^2 n^2}{4} = \frac{2\alpha-1}{4}n^2.$$

Since $|A \cup C| = \alpha n$, by assumption, $e(A \cup C) \geq (2\alpha - 1)n^2/4$. So all inequalities above must be tight, which means $G[C]$ is a balanced complete bipartite graph. and every vertex in A is adjacent to exactly half of the vertices in C.

Let $C_1 \cup C_2 = C$ be a partition such that $G[C] = G[C_1, C_2]$ and note that $|C_1| = |C_2| = |C|/2 = (2\alpha - 1)n/2$. For $i \in \{1, 2\}$ let

$$A_i = \{u \in A : \exists v \in C_i, uv \in E(G)\} \quad \text{and} \quad B_i = \{u \in B : \exists v \in C_i, uv \in E(G)\},$$

and let $B_3 = B \setminus (B_1 \cup B_2)$. Since $G[C_1, C_2]$ is a complete bipartite graph and every $v \in A$ is adjacent to at least half vertices in $C_1 \cup C_2$, we have $uw \in E(G)$ for all $u \in A_i$ and $w \in C_i$ for $i \in \{1, 2\}$. Notice that $A_1 \cup A_2$ is a partition of A, and for $i \in \{1, 2\}$ we have $uv \notin E(G)$ for all $u \in B_i, v \in A_i$, since otherwise there exists $w \in C_1$ such that u, v, w induces a copy of K_3 in G, a contradiction. Therefore, if $B_3 = \emptyset$, then G is bipartite with two parts $V_1 = C_1 \cup A_2 \cup B_2$ and $V_2 = C_2 \cup A_1 \cup B_1$, and by Lemma A.2, $G \cong T_2(n)$. So we may assume that $B_3 \neq \emptyset$.

Let $\widehat{C}_1 = C_1 \cup B_2$ and $\widehat{C}_2 = C_2 \cup B_1$. Let $x_i = |A_i|/n$, $y_i = |\widehat{C}_i|/n$ for $i \in \{1, 2\}$, and $z = |B_3|/n$. Since $|\widehat{C}_1 \cup \widehat{C}_2 \cup A_1 \cup B_3| = n - |A_2| \geq \alpha n$, there exists $U_1 \subset \widehat{C}_1$ with $|U_1| = \alpha n - |B_3 \cup A_1 \cup \widehat{C}_2| = (\alpha - z - x_1 - y_2)n$. Since $|B_3 \cup A_1 \cup \widehat{C}_2 \cup U_1| = \alpha n$, by assumption

$$\frac{2\alpha-1}{4}n^2 \leq e(B_3 \cup A_1 \cup \widehat{C}_2 \cup U_1) \leq zx_1n^2 + (x_1 + y_2)(\alpha - z - x_1 - y_2)n^2.$$

Similarly, there exists $U_2 \subset \widehat{C}_2$ with $|U_2| = (\alpha - z - x_2 - y_2)n$, and

$$\frac{2\alpha-1}{4}n^2 \leq e(B_3 \cup A_2 \cup \widehat{C}_1 \cup U_2) \leq zx_2n^2 + (x_2 + y_1)(\alpha - z - x_2 - y_1)n^2.$$

Adding up these two inequalities we obtain

$$\begin{aligned}\frac{2\alpha-1}{2} &\le zx_1+(x_1+y_2)(\alpha-z-x_1-y_2)+zx_2+(x_2+y_1)(\alpha-z-x_2-y_1)\\ &= \alpha(x_1+x_2+y_1+y_2)-z(y_1+y_2)-\big((x_1+y_2)^2+(x_2+y_1)^2\big)\\ &\le \alpha(1-z)-z(\alpha-z)-\frac{(x_1+x_2+y_1+y_2)^2}{2}\\ &= \alpha(1-z)-z(\alpha-z)-\frac{(1-z)^2}{2}=\frac{z^2}{2}-(2a-1)z+\frac{2\alpha-1}{2},\end{aligned}$$

which implies that $z^2/2-(2a-1)z\ge 0$. However, since $0<z\le 1-\alpha<4\alpha-2$ (here we used $\alpha>3/5$ and $B_3\ne\emptyset$),

$$\frac{z^2}{2}-(2a-1)z=\frac{z}{2}(z-(4\alpha-2))<0,$$

a contradiction. ∎

B Proofs of Claims 4.13 and 4.14

In this section we prove Claims 4.13 and 4.14.

Proof of Claim 4.13. Since $x_2^2+x_3^2\ge(x_2+x_3)^2/2$, it suffices to show

$$\frac{(x_2+x_3)^2/2+x_4^2}{9x_1(1-2x_1)}-\frac{x_4^2}{6x_1}+\frac{1}{6}-c>0.$$

Plugging $x_4=1-g(c)$ and $x_2+x_3=g(c)-x_1$ into the inequality above, it becomes

$$\ell(c,x_1):=\frac{(g(c)-x_1)^2+2(1-g(c))^2}{18x_1(1-2x_1)}-\frac{(1-g(c))^2}{6x_1}+\frac{1}{6}-c>0. \tag{3.5}$$

Then with the aid of computer [10] one can see that

$$\min\{\ell(c,x):x\in(0,1/2),c\in[1/4,1/3]\}>0.003.$$

Therefore, (3.5) is true. ∎

Proof of Claim 4.14. First since $1/(1/2-x)$ is concave up, by Jensen's inequality

$$\frac{x_4}{1/2-x_1}+\frac{x_4}{1/2-x_2}\ge\frac{4x_4}{1-(x_1+x_2)}.$$

Since $x_1+x_2\ge 1/2$ and $x_2\le x_1<1/2$,

$$\frac{x_2}{1/2-x_1}+\frac{x_1}{1/2-x_2}-\frac{2(x_1+x_2)}{1-(x_1+x_2)}=\frac{2(x_1-x_2)^2(2x_1+2x_2-1)}{(1-2x_1)(1-2x_2)(1-x_1-x_2)}\ge 0.$$

It suffice to show that

$$\begin{aligned}c+\frac{(1-x_1-x_2)x_4}{6(x_1+x_2)}<\frac{1}{18}\Bigg(&\left(1+\frac{1}{1-2x_3}-\frac{1}{x_1+x_2}\right)\cdot\frac{x_1+x_2}{1-(x_1+x_2)}\\ &+\frac{1}{1-2x_3}+\frac{1}{2(x_1+x_2)}\cdot\frac{4x_4}{1-(x_1+x_2)}+1\Bigg),\end{aligned}$$

Let $x = x_1 + x_2$. Then $x_3 = g(c) - x$ and the inequality above can be simplified as

$$\begin{aligned} m(x,c) := & \left(1 + \frac{1}{1-2(g(c)-x)} - \frac{1}{x}\right) \cdot \frac{x}{1-x} \\ & + \frac{1}{1-2(g(c)-x)} + \frac{2(1-g(c))}{x(1-x)} + 1 - \frac{3(1-x)(1-g(c))}{x} - 18c > 0. \end{aligned} \tag{4.5}$$

Then with the aid of computer [10] one can see that

$$\min\{m(x,c) : x \in [1/2,1], c \in [1/4,1/3]\} > 0.099.$$

Therefore, (4.5) is true. ∎