K_5-Subdivisions in graphs containing K_4^-

Jie Ma*
Xingxing Yu†
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA

Abstract

Seymour and, independently, Kelmans conjectured in the 1970s that every 5-connected nonplanar graph contains a subdivision of K_5. In this paper, we prove this conjecture for graphs containing K_4^-.

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*jiema@math.gatech.edu
†yu@math.gatech.edu; Partially supported by NSA, and by NSFC Project 10628102

1 Introduction

Only finite simple graphs are considered. We adopt the notaion and terminology in [7]. Paths $P_1, \dots, P_k$ are said to be *independent* if for any $1 \le i \ne j \le k$ no end of P_i is an internal vertex of P_j. A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_i) \cup V(G_i - G_{3-i}) \ne \emptyset$ for $i \in \{1,2\}$. If $|V(G_1 \cap G_2)| = k$, then (G_1, G_2) is a *k-separation.* For a subgraph H of a graph G, an *H-bridge* of G is a subgraph of G that is induced by the edges contained in some component D of $G - V(H)$ and edges from D to H. The vertices in H that are neighbors of D are called the *attachments* of this H-bridge. For $S \subseteq V(G)$, the $G[S]$-bridges of G are also called S-bridges. Let G be a graph and $S \subseteq V(G)$, and let k be a positive integer. We say that G is (k, S)-connected if, for any cut T of G with $|T| < k$, every component of $G - T$ contains a vertex from S.

For a graph K, we follow Diestel [3] to use TK to denote a subdivision of K. The well known Kuratowski's theorem states that a graph is planar iff it contains neither TK_5 nor $TK_{3,3}$. It is known that 3-connected nonplanar graphs contain $TK_{3,3}$. Seymour [8] conjectured in 1975 that every 5-connected nonplanar graph contains a TK_5, which was posed independently by Kelmans [6] in 1979. For convenience, the vertices with degree 4 in a TK_5 are called *branch* vertices.

Clearly if G is 5-connected and contains a K_4 then G contains a TK_5; since for any vertex v there are four paths from v to the vertices of K_4 which have only v in common. It is shown in [7] that if a 5-onnected graph G contains K_4^- on vertices x_1, x_2, y_1, y_2 with $y_1y_2 \notin E(G)$, and if G contains an induced path P from x_1 to x_2 such that $G - P$ is 2-connected and $y_1, y_2 \notin P$, then G contains a TK_5 in which x_1, x_2, y_1, y_2 are branch vertices.

In this paper we prove Seymour's conjecture for those graphs that contain K_4^- as a subgraph.

Theorem 1.1 *If G is a 5-connected non-planar graph and contains K_4^- as a subgraph, then G contains a TK_5.*

Note that K_4^--free graphs have nice structural properties; for example, it is shown in [4] that if G is 5- connected and K_4^--free then G contains a contractible edge (see [5] for more results). It is our hope that by excluding K_4^- (and perhaps some other graphs) one can force usful structural properties that would lead to an eventual resolution of Seymour's conjecture.

It is shown in [7] that if G is a 5-connected nonplanar graph and has a 5-separation (G_1, G_2) such that $|G_2| \ge 7$ and G_2 has a planar drawing in a closed disc in the plane with vertices in $V(G_1 \cap G_2)$ occur on the boundary of the disc, then G has a TK_5. This result will be used to prove Theorem 1.1, and we believe that it will also be useful in an enentual resolution of Seymour's conjecture.

The proof of Theorem 1.1 can be outlined as follows. Let G be 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] = K_4^-$, with $y_1y_2 \notin E(G)$. First, we use a lemma in [7] to show that there is an induced path P in G from x_1 to x_2 such that $G - P$ is 2-connected, and $\{y_1, y_2\} \not\subseteq P$. If $y_1, y_2 \notin P$, then Theorem 1.1 follows from one of the two main results in [7]. So we may assume by symmetry that $y_1 \notin P$ and $y_2 \in P$. Now y_2 divides P to two subpaths x_1Py_2 and x_2Py_2, each has at least three vertices (since P is induced in G and $x_iy_2 \in E(G)$). By contracting $x_iPy_2 - \{x_i, y_2\}$ in $G - \{x_1, x_2\}$ we show that either the resulting graph contains disjoint paths between the new vertices and between y_1 and

y_2, or G contains a TK_5. This allows us to assume that there exist $z_i \in V(x_iPy_2 - \{x_i, y_2\})$ such that G has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively, and internally disjoint from P. Choose Y, Z so that z_1Pz_2 is maximal. We then show that either we can find a TK_5 in G or (by symmetry) there are three independent paths, A and C from z_1 to y_1 and B from y_2 to z_2 (see Figure 1). So we may assume A, B and C exist, and we choose such paths satisfying certain requirements. Then either there is a TK_5 in G, or there exist disjoint paths P, Q, with P from C to B and Q from A to B. See Figure 1. We then use this structure to show that to force a 5-separation (G_1, G_2) such that $|G_2| \geq 7$ and G_2 has a planar drawing in a closed disc in the plane with vertices in $V(G_1 \cap G_2)$ occur on the boundary of the disc. Now Theorem 1.1 follows from the second main result in [7].

Those results in [7] which we will use are stated in Section 2, along with Seymour's characterization of graphs without disjoint paths between two pairs of vertices. In Section 3, we show how to force the structure consisting of paths X, A, B, C, P, Q. In Section 4, we show how to force the desired separation (G_1, G_2).

2 Previous results

In this section we state a few results that we need to prove Theorem 1.1. The first lemma is proved in [7] which says that given an induced path X and a chain of blocks H in $G - X$, one can, with one exception, modify X to a nonseparating induced path X' such that $H \subseteq G - X'$. A graph is said to be a chain of blocks if its blocks can be labeled as $B_1, \ldots, B_k$ such that $|B_i \cap B_{i+1}| = 1$ for $i = 1, \ldots, k-1$, and $B_i \cap B_j = \emptyset$ when $1 \leq i < j - 1 \leq k - 1$. In addition, if $k = 1$ and y_1, y_2 are distinct vertices of B_1, or if $k \geq 2$ and $y_11V(B_1 - B_2)$ and $y_2 \in V(B_k - B_{k-1})$, then we say that B_1 is a chain of blocks from y_1 to y_2.

Lemma 2.1 *Let G be a graph and let x_1, x_2, y_1, y_2 be distinct vertices of G such that G is $(5, \{x_1, x_2, y_1, y_2\})$-connected. Suppose X is an induced path in G from x_1 to x_2, and H is a chain of blocks in $G - V(X)$ from y_1 to y_2. Then precisely one of the following holds:*

- (*i*) *$H = y_1y_2$ and $G - y_1y_2$ can be drawn in a closed disc in the plane without edge crossings such that x_1, y_1, x_2, y_2 occur on the boundary of the disc in this cyclic order.*
- (*ii*) *There is an induced path X' from x_1 to x_2 such that $H \subseteq G - V(X')$, and $G - V(X')$ is a chain of blocks from y_1 to y_2.*

Lemma 2.1 is used in [7] to prove the following lemma, which gives an induced path X from which we will build our structure in Figure 1.

Lemma 2.2 *Let G be a 5-connected nonplanar graph and x_1, x_2, y_1, y_2 distinct vertices of G such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Then there is an induced path X in $G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ from x_1 to x_2 such that $G - V(X)$ is 2-connected and $\{y_1, y_2\} \not\subseteq V(X)$.*

The case $\{y_1, y_2\} \cap V(X) = \emptyset$ is taken care of by the following lemma proved in [7].

Lemma 2.3 *Let G be a 5-connected nonplanar graph and let x_1, x_2, y_1, y_2 be distinct vertices of G such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Suppose there is an induced path X in $G - x_1x_2$ from x_1 to x_2 such that $G - V(X)$ is 2-connected and $\{y_1, y_2\} \cap V(X) = \emptyset$. Then G contains a TK_5 in which x_1, x_2, y_1, y_2 are branch vertices.*

We now state the result proved in [7] about TK_5 when a 5-connected graph admits a 5-separation such that one side of the separation is planar.

Theorem 2.4 *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|G_2| \geq 7$ and G_2 has a planar representation in which the vertices of $V(G_1 \cap G_2)$ are incident with a common face. Then G contains a TK_5.*

In our proof of Theorem 1.1, we need the characterizaion of graphs containing no disjoint paths between two pairs of vertices. For convenience, we introduce the following definition.

Definition 2.5 *A 3-planar graph $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \dots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that*

- (*a*) *for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,*
- (*b*) *for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and*
- (*c*) *if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc D with no edge crossings.*

If, in addition, $b_0, b_1, \dots, b_n$ are vertices in G such that $b_i \notin A_j$ for all $0 \leq i \leq n$ and $A_j \in \mathcal{A}$, $p(G, A)$ can be drawn in a closed disc D with no edge crossings, and $b_0, b_1, \dots, b_n$ occur on the boundary of D in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, \dots, b_n)$ is 3-planar. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_0, b_1, \dots, b_n)$ is 3-planar.

The following result is due to Seymour [9]; equivalent results can be found in [2, 10, 11].

Theorem 2.6 (*Seymour*) *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:*

- (*i*) *G contains disjoint paths from s_1 to t_1 and from s_2 to t_2.*
- (*ii*) *(G, s_1, t_1, s_2, t_2) is 3-planar.*

For convenience, we say that $(G, X, x_1, x_2, y_1, y_2)$ is a 6-*tuple* if the following holds:

- G is a 5-connected nonplanar graph,
- x_1, x_2, y_1, y_2 are distinct vertices of G such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$, and
- there is an induced path X in $G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ from x_1 to x_2 such that $G - V(X)$ is 2-connected, $y_1 \notin V(X)$, and $y_2 \in V(X)$.

Note that in a 6-tuple $(G, X, x_1, x_2, y_1, y_2)$, $|V(x_iXy_2)| \geq 3$.

3 Substructure

In this section, we show that in a 5-connected nonplanar graph we can find a TK_5 or a substructure (see Figure 1) satisfying a list of useful properties.

Lemma 3.1 *Let $(G, X, x_1, x_2, y_1, y_2)$ be a 6-tuple. Then G contains a TK_5, or there exist $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$ and $z_2 \in V(y_2Xx_2) - \{x_2, y_2\}$ such that $G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$ has disjoint paths Z, Y from z_1, y_1 to z_2, y_2, respectively.*

Proof. Let G' be the graph obtained from $G - \{x_1, x_2\}$ by contracting $x_iXy_2 - \{x_i, y_2\}$ to vertex u_i for $i = 1, 2$. Note that G' is 2-connected; since G is 5-connected, X is induced, and $G - X$ is 2-connected.

Suppose G' contains disjoint paths, say U, Y, from u_1, y_1 to u_2, y_2, respectively. Let v_i denote the neighbor of u_i in the path U, and let $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ be a neighbor of v_i. Let $Z := (U - \{u_1, u_2\}) \cup \{z_1, z_2, z_1v_1, z_2v_2\}$. Now Z, Y are the desired paths.

So we may assume that such disjoint paths U, Y do not exist in G'. Then by Theorem 2.6, there exists a collection $\mathcal{A}$ of subsets of $V(G') - \{u_1, u_2, y_1, y_2\}$ such that $(G', \mathcal{A}, u_1, y_1, u_2, y_2)$ is 3-planar. Since $G - V(X)$ is 2-connected, $|\{u_1, u_2\} \cap N(A)| \neq 2$ for all $A \in \mathcal{A}$. Let $\mathcal{A}' = \{A \in \mathcal{A} : |\{u_1, u_2\} \cap N(A)| = 0\}$ and $\mathcal{A}'' = \{A \in \mathcal{A} : |\{u_1, u_2\} \cap N(A)| = 1\}$. For each $A \in \mathcal{A}'$, since G is 5-connected, we have $\{x_1, x_2\} \subseteq N(A)$.

Note that in $p(G', \mathcal{A})$ (see Definition 2.5) there are edges joining the vertices in each $N(A) - \{u_1, u_2\}$. Since G is 5-connected and $G - V(X)$ is 2-connected, $p(G', \mathcal{A}) - \{u_1, u_2, y_2\}$ is a 2-connected plane graph; and the edges joining vertices of $N(A) - \{u_1, u_2\}$ (for each $A \in \mathcal{A}''$) occur on the outer cycle, say D, of $p(G', \mathcal{A}) - \{u_1, u_2, y_2\}$. Let $y_2', y_2'' \in V(D)$ be the neighbors of y_2 such that y_1, y_2', y_2'' occur on D in clockwise order and, subject to this, $y_2'Dy_2''$ is maximal. Possibly, $y_2' = y_2''$.

We may assume that $N(x_1) - X \subseteq V(y_2''Dy_1) \cup \bigcup_{\{A \in \mathcal{A}'' : u_1 \in N(A)\}} A$, and $N(x_2) - V(X) \subseteq V(y_1Dy_2') \cup \bigcup_{\{A \in \mathcal{A}'' : u_2 \in N(A)\}} A$. For, suppose x_1 has a neighbor a such that $a \notin X$, $a \notin y_2''Dy_1$, and $a \notin A$ for any $A \in \mathcal{A}''$ with $u_1 \in N(A)$. Let $w_1 \in V(D)$ such that $u_1w_1 \in E(G')$ and w_1Dy_1 is minimal, and let $z_1w_1 \in E(G)$ with $z_1 \in x_1Xy_2 - \{x_1, y_2\}$. Let $w_2 \in V(D)$ such that $u_2w_2 \in E(G')$ and y_1Dw_2 is minimal, and let $z_2w_2 \in E(G)$ with $z_2 \in y_2Xx_2 - \{x_2, y_2\}$. Since G' and H are 2-connected, there exist two independent paths P_1, P_2 from z_2 to D in $G - V(X - z_2)$ internally disjoint from $V(p(G', \mathcal{A}))$, such that P_1 ends at w_3 and P_2 ends at w_2 where y_1, w_2, w_3 occur on D in clockwise order. If there exists a path P_3' from w_3 to a in $p(G', \mathcal{A}) - \{u_1, u_2, y_2\}$ and disjoint from w_1Dw_2, then P_3', w_1Dy_1, y_1Dw_2 give three paths P_3, W_1, W_2 in G (with the same ends of P_3', w_1Dy_1, y_1Dw_2, respectively) such that $(P_1 \cup P_3 \cup ax_1) \cup (P_2 \cup W_2) \cup (z_2Xx_2) \cup (y_2Xz_2) \cup (W_1 \cup w_1z_1 \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume such a path P_3' does not exist. Then by planarity, there is a 2-cut $\{s_1, s_2\}$ in $p(G', \mathcal{A}) - \{u_1, u_2, y_2\}$ separating w_3 from a, with $s_1, s_2 \in w_1Dw_2$. This implies that $\{x_1, x_2, s_1, s_2\}$ is a 4-cut in H separating $\{a, y_1\}$ from X, contradicting the assumption that G is 5-connected.

Therefore, since G is not planar, there must exist $i \in \{1, 2\}$ and vertices $v_1, v_2 \in x_iXy_2 - y_2$ such that x_1, v_1, v_2, x_2 occur on X in this order, and one of the following holds:

(a) v_j is adajcent to $w_j \in V(D)$ in G such that $y_1, y_2', y_2'', w_1, w_2$ (if $i = 1$) or $y_1, w_1, w_2, y_2', y_2''$ (if $i = 2$) occur on D in clockwise order, and in this case we let $Q_j = v_jw_j$;

(b) there is some $A \in \mathcal{A}$ such that $G[A \cup V(v_1 X v_2)]$ has disjoint paths Q_1, Q_2 from v_1, v_2 to w_1, w_2 respectively, where w_1, w_2 are neighbors of A in G' that are not u_1 or u_2, and $y_1, y_2', y_2'', w_1, w_2$ (if $i = 1$) or $y_1, w_1, w_2, y_2', y_2''$ (if $i = 2$) occur on D in clockwise order.

Without loss of generality we may assume that the above occurs with $i = 1$. Let z be a vertex in $y_2 X x_2 - \{x_2, y_2\}$. Then by planarity of $p(G', \mathcal{A}) - \{u_1, u_2, y_2\}$ there exist neighbors z', z'' of z in $G - V(X)$ such that $G - V(X)$ contains independent paths P_1, P_2, P_3 with P_1 from y_1 to z', P_2 from z'' to w_1, and P_3 from w_2 to y_1. Now $zXx_2 \cup zXy_2 \cup (\{z, zz''\} \cup P_2 \cup Q_1 \cup x_1 X v_1) \cup (P_1 \cup \{z, zz'\}) \cup (P_3 \cup Q_2 \cup v_2 X y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z. ∎

For convenience, we say that $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ is an 8-*tuple* if

- $(G, X, x_1, x_2, y_1, y_2)$ is a 6-tuple,
- there exist $z_1 \in V(x_1 X y_2) - \{x_1, y_2\}$, $z_2 \in V(y_2 X x_2) - \{x_2, y_2\}$, and disjoint paths Z, Y in $G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$ from z_1, y_1 to z_2, y_2, respectively, and
- subject to above, $z_1 X z_2$ is maximal.

For any 8-tuple $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$, we let $H := G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$. Clearly, each z_i has at least three neighbors in $H - \{z_1, z_2, y_2\}$, and y_2 has at least one neighbor in H. So H is connected, and $H - y_2$ is 2-connected. We will derive more structural information of H.

Lemma 3.2 *Let $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ be an 8-tuple. Then G contains a TK_5, or the following holds:*

(1) *for any $i \in \{1, 2\}$, H has no path through z_i, z_{3-i}, y_1, y_2 in order, and $y_1 z_i \notin E(G)$;*

(2) *there exists $i \in \{1, 2\}$ such that H contains independent paths A, B, C, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.*

Proof. First, suppose there is a path in H from z_i (for some $i \in \{1, 2\}$) to y_2 such that z_i, z_{3-i}, y_1, y_2 occur on P in order. Then $G[\{x_1, x_2, y_1, y_2\}] \cup (X - V(z_i X y_2 - \{y_2, z_i\})) \cup P$ is a TK_5 in G with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume that such P does not exist. Hence by Lemma 3.1, we have $y_1 z_1, y_1 z_2 \notin E(G)$, and (1) holds. Thus we have shown that G has a TK_5 or (1) holds.

We now show that G has a TK_5 or (2) holds. Clearly, if (1) fails then G has a TK_5; so we may assume that (1) holds. For each $i \in \{1, 2\}$, let H_i denote the graph obtained from H by duplicating z_i and y_1, and let z_i' and y_1' denote the duplicates of z_i and y_1, respectively.

First, suppose some H_i contains three disjoint paths A', B', C' from $\{z_i, z_i', y_2\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in A', z_i' \in C'$ and $y_2 \in B'$. If $z_{3-i} \notin B'$, then after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in H from z_i to y_2 through z_{3-i}, y_1 in order, contradicting our assumption that (1) fails. Hence $z_{3-i} \in B'$, and we get the desired paths for (2) from $A' \cup B' \cup C'$, by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that for any $i \in \{1, 2\}$, H_i does not contain three disjoint paths from $\{z_i, z_i', y_2\}$ to $\{y_1, y_1', z_{3-i}\}$. Then H_i has a separation (H_i', H_i'') such that $|V(H_i' \cap H_i'')| \leq 2$, $\{z_i, z_i', y_2\} \subseteq V(H_i')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(H_i'')$.

We claim that $y_1, y_2, z_1, z_2 \notin V(H_i' \cap H_i'')$ for $i = 1, 2$. Note that $\{y_1, y_1'\} \neq V(H_i' \cap H_i'')$, since otherwise y_1 would be a cut vertex in H separating z_{3-i} from $\{y_2, z_i\}$. Now suppose one of y_1, y_1' is in $V(H_i' \cap H_i'')$; then since y_1, y_1' are duplicates (with same neighbors), the other vertex in $V(H_i' \cap H_i'')$ is a cut vertex in H separating $\{z_{3-i}, y_1\}$ from $\{z_i, y_2\}$, a contradiction. So $y_1, y_1' \notin V(H_i' \cap H_i'')$. Similar argument shows that $z_i, z_i' \notin V(H_i' \cap H_i'')$. Since $H - \{z_1, z_2, y_2\}$ is 2-connected, $z_{3-i}, y_2 \notin V(H_i' \cap H_i'')$.

For $i = 1, 2$, let $V(H_i' \cap H_i'') = \{s_i, t_i\}$, and let F_i' (respectively, F_i'') be obtained from H_i' (respectively, H_i'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') is a 2-separation of H such that $V(F_i' \cap F_i'') = \{s_i, t_i\}$, $y_2, z_i \in F_i' - \{s_i, t_i\}$, and $y_1, z_{3-i} \in F_i''$. Let Z_1, Y_2 denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, y_2, respectively; and let Z_2, Y_1 denote the $\{s_1, t_1\}$-bridges of F_1'' containing z_2, y_1, respectively.

Case 1. $Y_1 \neq Z_2$ and $Y_2 \neq Z_1$.

Then since G is 5-connected, $\{x_1, x_2, s_1, t_1\}$ cannot be a cut in G; and hence there exists $y \in V(X) - \{x_1, x_2, z_1, z_2\}$ such that $y \in N(Y_1) - \{s_1, s_2\}$.

Suppose $y \in z_1Xz_2 - \{z_1, z_2\}$. Since $H - y_2$ is 2-connected and by symmetry between s_1 and t_1, we may assume that there is a path Q_1 in $G[Y_1 + y] - s_1$ from y to t_1 and containing y_1. Now $Q_1 \cup yXy_2$ and a path in $(Y_1 \cup Y_2) - s_1$ between y_1 and y_2 form a cycle, say D. Note that the union of $(Z_1 \cup Z_2) - t_1$ and $x_1Xz_1 \cup z_2Xx_2$ contains a path from x_1 to x_2, say X', which is disjoint from D. In fact, in $(G - x_1x_2) - D$ we may choose X' to be an induced path from x_1 to x_2. Now applying Lemma 2.1 we see that there is an induced path X' in $G - x_1x_2$ from x_1 to x_2 such that $G - X'$ is 2-connected and $y_1, y_2 \notin X'$. By Lemma 2.3, G contains a TK_5, contradicting our assumption.

Thus, by symmetry between x_1Xz_1 and x_2Xz_2, we may assume that $y \in x_1Xz_1 - \{x_1, z_1\}$. Since G is 5-connected and X is induced, y has a neighbor, say y', such that $y' \notin X$, $y' \notin \{y_1, y_2\}$, and if y_2 has a unique neighbor y_2' in H then $y' \neq y_2'$.

If $y' \in Z_1 \cup Z_2$ then we may assume (by symmetry between s_1 and t_1) that $(Z_1 \cup Z_2) - t_1$ contains a path Q' from y' to z_2. Clearly, in $(Y_1 \cup Y_2) - s_1$ there is a path Y' from y_1 to y_2, which is disjoint from Q'. Now $Q' + \{y, yy'\}$ and Y' contradict the choice of Y, Z in the 8-tuple.

So we may assume $y' \in Y_1 \cup Y_2$. An easy check and symmetry between s_1 and t_1 allows us to assume that there are disjoint paths Q', Y' in $Y_1 \cup Y_2$ from y', y_1 to s_1, y_2, respectively. Let Q'' be a path in $Z_2 - t_1$ from s_1 to z_2. Now $Q' \cup Q''$ and Y' contradict the choice of Y, Z in the 8-tuple.

Case 2. $Z_1 = Y_2$ or $Z_2 = Y_1$.

We first show that $Z_1 = Y_2$ and $Z_2 = Y_1$. We only deal with the case $Z_2 = Y_1$ and $Z_1 \neq Y_2$; the other case is symmetric. So assume $Z_2 = Y_1$ and $Z_1 \neq Y_2$. Then one of $\{s_2, t_2\}$, say s_2, must be a cut vertex of $F_1' = Z_2 = Y_1$ separating y_1 from z_2. By symmetry between s_1 and t_1 and since $H - y_2$ is 2-connected, we may assume that s_2 separates $\{s_1, y_1\}$ from $\{t_1, z_2\}$. Since $\{s_2, t_2\}$ separates z_1 from y_2, $t_2 \in (Y_2 \cup Z_1) - \{s_1, t_1\}$. If $t_2 \in Y_2 - \{s_1, t_1\}$ then in $H - \{s_2, t_2\}$ there is a path from y_1 to z_1 through s_1, a contradiction. So $t_2 \in Z_1 - \{s_1, t_1\}$; then in $H - \{s_2, t_2\}$ there is a path from y_2 to z_2 through t_1, a contradiction.

Since $Z_1 = Y_2$ and $Z_2 = Y_1$, we may assume that s_2 is a cut vertex of $F_1' = Z_2 = Y_1$ separating y_1 from z_2, and t_2 is a cut vertex of $F_1'' = Z_1 = Y_2$ separating y_2 from z_1. Since $H - y_2$ is 2-connected and by symmetry between s_1 and t_1, we may assume that in Z_2, s_2 separates $\{s_1, y_1\}$ from $\{z_2, t_1\}$. Since in H, $\{s_2, t_2\}$ separates y_2 from z_1, we have $t_2 \in Z_1 - \{s_1, t_1\}$.

Moreover, since in H, $\{s_2, t_2\}$ separates y_1 from z_2, we see that t_2 separates $\{s_1, z_1\}$ from $\{t_1, y_2\}$ in Z_1. But this implies that there is no disjoint paths in H from z_1, y_1 to z_1, y_2, respectively, contradicting the existence of Y, Z in an 8-tuple. ∎

We note in passing that the structure of H satisfying (1) of Lemma 3.2 is well characterized by a result proved in [12–14]. However, we do not need the full strength of that result, and it is simpler to deal with H directly.

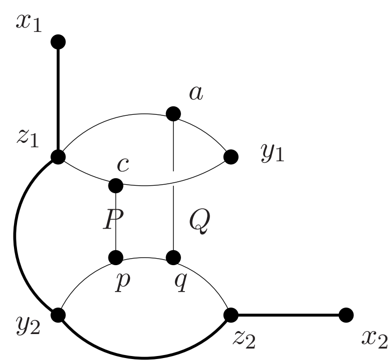

Figure 1: The substructure.

In the argument below we do not fix $i = 1$ or $i = 2$ (for the sake of symmetry). However, in the rest of this section one may view $i = 1$ as suggested by Figure 1.

Lemma 3.3 *Let $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ be an 8-tuple. Then G has a TK_5, or there exists $i \in \{1, 2\}$ such that H contains independent paths A, B, C, with A and C from z_i to y_1, and B from y_2 to z_{3-i}, and the following hold:*

(1) *there exist disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_i\}, a \in V(A) - \{y_1, z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$, and*

(2) $z_{3-i}x_{3-i} \in E(X)$.

Proof. We may assume that G has no TK_5, since otherwise the assertion of the lemma holds. First, we prove (1). By Lemma 3.2,

(i) for any $i \in \{1, 2\}$, H has no path through z_i, z_{3-i}, y_1, y_2 in order, and $y_1 z_i \notin E(G)$;

(ii) there exist $i \in \{1, 2\}$ and independent paths A, B, C in H with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

We choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,

(b) if possible the $(A \cup C)$-bridge of H containing B has attachments on both $A - \{z_i, y_1\}$ and $C - \{y_1, z_i\}$,

(c) the $(A \cup C)$-bridge of H containing B is maximal, and

(d) B', the union of B and the B-bridges of H not containing $A \cup C$, is maximal.

Since $G - V(X - z_{3-i})$ is 2-connected, there are disjoint paths P, Q from $B - y_2$ to $s, t \in V(A \cup C) - \{z_i\}$ and internally disjoint from $A \cup B \cup C$.

Claim 1. We may choose P, Q so that $s \neq y_1$ and $t \neq y_1$.
For otherwise, $H - \{z_i, y_2\}$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{y_1, v\}$ for some $v \in V(H)$, $(A \cup C) - z_i \subseteq H_1$ and $B - y_2 \subseteq H_2$. Recall that $G - V(X - \{z_1, z_2\})$ contains disjooint paths Z, Y from z_1, y_1 to z_2, y_2, respectively. If $v \notin Z$ then $Z - z_i \subseteq H_2 - \{y_1, v\}$, and hence we may choose Y so that $Y \cap A = \{y_1\}$ or $Y \cap C = \{y_1\}$; now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path that contradicts (i). So $v \in Z$. Hence $Y - y_2 \subseteq H_2 - v$, and so we may choose Z with $Z \cap A = \{z_i\}$ or $Z \cap C = \{z_i\}$. Again, $Z \cup A \cup Y$ or $Z \cup C \cup Y$ gives a path contradicting (i).

If $s \in A - y_1$ and $t \in C - y_1$ or $s \in C - y_1$ and $t \in A - y_1$, then P, Q give the desired paths for (1). So we may assume by symmetry that $s, t \in C$. We may further choose P, Q so that sCt is maximal, and assume that z_i, s, t, y_1 occur on C in order. Let $P \cap B = \{p\}, Q \cap B = \{q\}$.

Claim 2. We may assume that the $(A \cup C)$-bridge of H containing B has no attachment in $A - \{y_1, z_i\}$.
For, otherwise, there is a path R from some $r \in V(A) - \{y_1, z_i\}$ to B internally disjoint from $A \cup B \cup C$. If $R \cap (P \cup Q) \neq \emptyset$, then $P \cup Q \cup R$ contains the desired paths for (1). So we may assume $R \cap (P \cup Q) = \emptyset$. If $y_2 \notin R$, then P, R are the desired paths for (1). So we may assume $y_2 \in R$. Now consider B' defined in (d) above. If $B' - y_2$ contains independent paths P', Q' from z_{3-i} to p, q, respectively, then $z_i C s \cup P \cup P' \cup Q' \cup Q \cup tCy_1 \cup y_1 A r \cup R$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (i). So such paths P', Q' do not exist in B'. Then there is a vertex $z \in B' - y_2$ such that in $B' - y_2$, z separates z_{3-i} from p, q. Clearly, $z \in qBz_{3-i} - z_{3-i}$. Choose z so that zBz_{3-i} is minimal, and let B'' denote the z-bridge of $B' - y_2$ contaiing z_{3-i}. T Note that $z_{3-i}Bz \subseteq B''$. Recall that G is 5-connected, X is induced in G, and $H - y_2$ is 2-connected. $H - y_2$ must contain a path W from $w' \in V(B'') - z$ to $w \in V(P \cup Q \cup R \cup A \cup C) - \{z_i, y_2\}$ and internally disjoint from $P \cup Q \cup R \cup A \cup C$. By the definition of B' in (d) above, we see that any path from B' to $P \cup Q \cup R \cup A \cup C$ must intersect B. Hence we may further choose W so that $w' \in zBz_{3-i}$ and W is internally disjoint from B. Then by the choice of P, Q, we have $w = y_1$. By the minimality of zBz_{3-i}, B'' has independent paths P'', Q'' from z_{3-i} to z, w', respectively. Now $z_i C t \cup Q \cup qBz \cup P'' \cup Q'' \cup Q \cup y_1 A r \cup R$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (i).

Let J denote the union of C and the $(A \cup C)$-bridge of H containing B. Then by (i) and Theorem 2.6, there exists a collection $\mathcal{A}$ of subsets $V(J) - \{y_1, z_i, y_2, z_{3-i}\}$ such that $(J, \mathcal{A}, z_i, y_2, z_{3-i}, y_1)$ is 3-planar. We choose $\mathcal{A}$ so that for any $D \in \mathcal{A}$, if $N_H(D) = \{w_1, \ldots, w_k\}$ (where $k \in \{2, 3\}$) and $D' := H[D \cup N_H(D)]$ then $(D', w_1, \ldots, w_k)$ is not 3-planar; for otherwise there is a collection of subsets $\mathcal{A}'$ of D such that $D', \mathcal{A}'', w_1, \ldots, w_k)$ is 3-planar, and we see that with $\mathcal{A}'' = (\mathcal{A} - \{D\}) \cup \mathcal{A}'$, $(J, \mathcal{A}'', z_i, y_2, z_{3-i}, y_1)$ is 3-planar.

Let $v_1, \ldots, v_k$ denote the vertices on $C - \{z_i, y_1\}$ in order from z_i to y_1 such that each v_i is an attachment of some $(A \cup C)$-bridge of H that does not contain B but has attachments on both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$.

Claim 3. $(J, v_1, \ldots, v_k, y_1, z_{3-i}, y_2, z_i)$ is 3-planar.
For, otherwise, there exist $i \in \{1, \ldots, k\}$ and $D \in \mathcal{A}$ such that $v_i \in D$ and $|N_J(D)| = 3$ (since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_2, z_{3-i}, y_1)$ is 3-planar). Let $N_J(D) =$

$\{c_1, c_2, c\}$ such that $c_1, c_2 \in C$, $c \notin C$, and c is in the $(A \cup C)$-bridge containing B; and let $D' = H[D \cup \{c_1, c_2, c\}]$. If D' contains no disjoint paths from c_1 to c_2 and from c to v_i, then by Theorem 2.6, there is a collection of subsets $\mathcal{A}'$ of D such that $(D', \mathcal{A}', c_1, v_i, c_2, c)$ is 3-planar. This contradicts the choice of $\mathcal{A}$. So D' contains disjoint paths R from v_i to c and T from c_1 to c_2. We may assume T is induced. Let C' be obtained from C by replacing c_1Cc_2 with T. We now see that the $(A \cup C')$-bridge of H containing B has attachments on both $A - \{y_1, z_i\}$ and $C' - \{y_1, z_i\}$ (because of P, Q and T), contradicting (b).

For any $(A \cup C)$-bridge T of H not containing B, if T has attachments on A we define $a_1(T)$ and $a_2(T)$ to be the attachemnets of T on A with $a_1(T)Aa_2(T)$ maximal, and if T has attachments on C we define $c_1(T)$ and $c_2(T)$ to be the attachemnets of T on C with $c_1(T)Cc_2(T)$ maximal. We assume $z_i, a_1(T), a_2(T), y_1$ occur on A in order, and $z_i, c_1(T), c_2(T), y_1$ occur on C in order. We now further choose A, C so that subject to (a)–(d), the union of $(A \cup C)$-bridges of H with attachments on both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$ is maximal.

Claim 4. If T_1, T_2 are $(A \cup C)$-bridges of H not containing B such that T_2 has attachments on both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$, and T_1 has attachments on C (or A) only, then $c_1(T_1)Cc_2(T_1) - \{c_1(T_1), c_2(T_1)\}$ (or $a_1(T_1)Aa_2(T_1) - \{a_1(T_1), a_2(T_1)\}$) contains no attachment of T_2.
For, otherwise, we may modify C (or A) by replacing $c_1(T_1)Cc_2(T_1)$ (or $a_1(T_1)Aa_2(T_1)$) with an induced path in T_1 from $c_1(T_1)$ to $c_2(T_1)$ (or from $a_1(T_1)$ to $a_2(T_1)$). The new A and C do not affect (a)–(d) but enlarge the union of $(A \cup C)$-bridges of H with attachments in both $A - \{y_1, z_1\}$ and $C - \{y_1, z_1\}$, a contradiction.

Remark: Claim 4 basically allows us to modify A and C through the $(A \cup C)$-bridges of H not containing, without affecting (a)–(d).

Since $G - V(X)$ is 2-connected, there exists at least one $(A \cup C)$-bridge in H with attachments on both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$. Because of the disjoint paths Z and Y, $(H, z_i, y_1, z_{3-i}, y_2)$ is not 3-planar. Hence, since $(J, v_1, \ldots, v_k, y_1, z_2, y_2, z_1)$ is 3-planar and the $(A \cup C)$-bridge of H containing B has no attachment in $A - \{y_1, z_i\}$, either there exist $(A \cup C)$-bridges T_1, T_2 of H not containing B such that for any $j = 1, 2$, $z_iAa_2(T_j)$ properly contains $z_iAa_1(T_{3-j})$, or for any $j = 1, 2$, $c_1(T_j)Cy_1$ properly contains $c_2(T_{3-j})Cy_1$, or there exists an $(A \cup C)$-bridge T of H not containing B such that $T \cup a_1(T)Aa_2(T) \cup c_1(T)Cc_2(T)$ has disjoint paths from $a_1(T), a_2(T)$ to $c_2(T), c_1(T)$, respectively.

Therefore, there exist disjoint paths R_1, R_2 from $r_1, r_2 \in V(C)$ to $r_1', r_2' \in V(A)$, respectively, and internally disjoint from $A \cup C$, such that z_i, r_1, r_2, y_1 occur on C in this order and z_i, r_2', r_1', y_1 occur on A in this order.

Claim 5. We may assume that for any choice of R_1, R_2, we have $r_1, r_2 \in tCy_1$ or $r_1, r_2 \in z_iCs$.
For otherwise, there exist R_1, R_2 such that $r_1 \in z_iCs$ and $r_2 \in tCy_1$, or $r_1 \in sCt - \{s, t\}$, or $r_2 \in sCt - \{s, t\}$. Let $A' := z_iAr_2' \cup R_2 \cup r_2Cy_1$ and $C' := z_iCr_1' \cup R_1 \cup r_1Ay_1$. Note that $(A' \cup C')$-bridge of H containing B contains the $(A \cup C)$-bridge of H containing B, but we see that there are disjoint paths from $B - y_2$ so that one ends in $A' - \{z_i, y_1\}$ and one ends in $C' - \{y_1, z_i\}$, which are the desired paths.

If R_1, R_2 may be chosen so that $r_1, r_2 \in z_iCs$, then choose R_1, R_2 so that z_iAr_1' and z_iCr_2 are maximal, and let $z' := r_1'$ and $z'' = r_2$; otherwise, define $z' = z'' = z_i$. Similarly, if R_1, R_2

may be chosen so that $r_1, r_2 \in tCy_1$, then choose R_1, R_2 so that y_1Ar_2' and y_1Cr_1 are maximal, and let $y' := r_2'$ and $y'' = r_1$; otherwise, define $y' = y'' = y_1$.

By Claim 5, z_i, z', y', y_1 occur on A in order, and z_i, z'', s, t, y'', y_1 occur on C in order. Moreover, by Claim 2 and Claim 4, if $z', z'' \neq z_i$ then $\{z', z'', z_{3-i}\}$ is a cut in H, and if $y', y'' \neq y_1$ then $\{y', y'', y_1\}$ is a cut in H. So by Claim 3 and Claim 4, we see that $(H, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting (i). This completes the proof of (1).

Proof of (2). So by (1) and by the symmetry between A and C, we may assume that y_2, p, q, z_{3-i} occur on B in order. We may choose P, Q so that pBz_{3-i} is maximal, and qBz_{3-i} is minimal; and subject to these, cCy_1 is maximal, and aAy_1 is minimal.

Suppose there exist $x \in V(z_{3-i}Xx_{3-i}) - \{x_{3-i}, z_{3-i}\}$. Then by the choice of Y and Z, all neighbors of x in H must be ocntained in B'. Consider $B'' := G[(B' - z_{3-i}) + x]$.

If B'' contains disjoint paths P', Q' from y_2, x to p, q, respectively, then $P' \cup P \cup cCy_1$ and $Q' \cup Q \cup aAz_i$ contradict the choice of Y, Z. So such paths P', Q' do not exist. Then by Theorem 2.6, (B'', x, y_2, q, p) is 3-planar.

If B'' contains disjoint paths P'', Q'' from x, y_2 to p, q, respectively, then $P'' \cup P \cup cCz_1$ and $Q'' \cup Q \cup aAy_1$ contradict the choice of Y and Z. So there is a cut vertex z in B'' separating $\{x, y_2\}$ from $p, q\}$. Note that $z \in y_2Bp$.

Since x has at least three neighbors in B'' (because G is 2-connected and X is induced), we see that the component B^* of $B'' - z$ containing $\{y_2, x\}$ has other vertices. Therefore, we see from the choice of P and Q (and because $G - X$ is 2-connected), there is a path from y_1 to $B^* - z$ internally disjoint from $P \cup Q \cup A \cup C \cup (B'' - B^*)$; and so there is a path Y' from y_1 to y_2 internally disjoint from $P \cup Q \cup A \cup C \cup (B'' - B^*)$. Now $z_{3-i}Bp \cup P \cup cCz_i \cup A \cup Y'$ is a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (i). ∎

Remark. By Lemma 3.3 and its proof, we see that if G has no TK_5, then A, B, C may be chosen so that (a), (b), (c) and (d) are satisfied in the order listed, and subject to this (1) and (2) hold.

4 Proof of Theorem 1.1

Let $(G, X, x_1, x_2, y_1, y_2)$ be a 6-tuple, and assume that G contains no TK_5. Then by Lemma 3.1,

(1) there exist $z_1 \in V(x_1Xy_2) - \{x_1, y_2\}$ and $z_2 \in V(y_2Xx_2) - \{x_2, y_2\}$ such that $G - (V(X - \{z_1, z_2, y_2\}) \cup E(X))$ has disjoint paths Z, Y from z_1, y_1 to z_2, y_2, respectively.

We choose z_1, z_2, Y, Z so that

(2) z_1Xz_2 is maximal.

Then $(G, X, x_1, x_2, y_1, y_2, z_1, z_2)$ is an 8-tuple. By Lemma 3.2,

(3) for any $i \in \{1, 2\}$, H has no path through z_i, z_{3-i}, y_1, y_2 in order, and $y_1z_i \notin E(G)$;

(4) there exist $i \in \{1, 2\}$ and independent paths A, B, C in H with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

We choose A, B, C such that the following are satisfied in the listed order:

(a) A, B, C are induced paths in H,

(b) if possible the $(A \cup C)$-bridge of H containing B has attachments on both $A - \{z_i, y_1\}$ and $C - \{y_1, z_i\}$,

(c) the $(A \cup C)$-bridge of H containing B is maximal, and

(d) the union of B and the B-bridges of H not containing $A \cup C$, denoted by B', is maximal.

Note that by (d), every path in H from B' to $A \cup C$ must intersect B.
By Lemma 3.3 and the remark following its proof,

(5) there exist disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_i\}, a \in V(A) - \{y_1, z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$, and

(6) $z_{3-i}x_{3-i} \in E(X)$.

Without loss of generality we may assume $i = 1$, see Figure 1. So by (6), $z_2Xx_2 = z_2x_2$.
By symmetry between A and C, we may assume that y_2, p, q, z_2 occur on B in order. We may further choose P, Q so that

(7) pBz_2 is maximal and qBz_2 is minimal; and subject to this, cCy_1 is maximal and aAy_1 is minimal.

Suppose T is a path from $t \in V(aAy_1 - a)$ to $t' \in V(z_1Cc - c)$ internally disjoint from $A \cup B \cup C \cup P \cup Q$. Then $z_2Bq \cup Q \cup aAz_1 \cup z_1Ct' \cup T \cup tAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (3). So

(8) there is no path in H from $aAy_1 - a$ to $z_1Cc - c$ internally disjoint from $A \cup B \cup C \cup P \cup Q$.

We proceed by proving a few lemmas.

Lemma 4.1 *$B' - y_2$ has no cut vertex contained in qBz_2.*

Proof. Otherwise, let $u \in qBz_2$ be a cut vertex of $B' - y_2$, with uBz_2 minimal. Then $u \neq z_2$, since $H - y_2$ is 2-connected and B' contains no vertex in the B-bridge of H containing $A \cup C$. Since $H - y_2$ is 2-connected, there is a path S in H from $s' \in V(uBz_2 - u)$ to $s \in V(A \cup C)$ internally disjoint from $A \cup C \cup B'$. Note that S is disjoint from $(P - c) \cup (Q - a)$; otherwise we could revise the path B using $S \cup (P - c) \cup (Q - a)$ so that the new B' is larger while (a), (b) and (c) are not affected. By the choice of u, the component of $B' - (y_2Bu - u)$ which contains $uBz_2 - u$ has independent paths R_1, R_2 from z_2 to s', u, respectively. By the choice of Q in (7), $s \in C$. We choose S so that sCy_1 is minimal.

Claim 1. $s \in cCy_1 - y_1$, and there is no path in H from y_1 to B internally disjoint from $A \cup B \cup C$.

Suppose $s \in z_1Cc - c$. Then $(z_1Cs \cup S \cup R_1) \cup (R_2 \cup uBq \cup Q \cup aAy_1) \cup (y_1Cc \cup P \cup pBy_2)$ is a path through z_1, z_2, y_1, y_2 in order, contradicting (3).

If $s = y_1$, then $(R_1 \cup S) \cup (R_2 \cup uBq \cup Q \cup aAz_1 \cup z_1Xx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2, contradicting our assumption.

So $s \neq y_1$. Now assume that there is a path Y' in H from y_1 to some $y \in V(B)$ internally disjoint from $A \cup B \cup C$. By the choice of S, $y \in y_2Bu$ and Y' is disjoint from S. Hence $z_2Bs' \cup S \cup sCz_1 \cup A \cup Y' \cup yBy_2$ is a path contradicting (3). This proves Claim 1.

Since G is 5-connected, $\{a, s, x_1, x_2\}$ is not a cut in G. So there is a path T in G from $t \in V(aAy_1 \cup sCy_1) - \{a, s\}$ to $t' \in V(X - \{x_1, x_2\}) \cup V(A \cup B \cup C \cup P \cup Q \cup S)$ internally disjoint from $A \cup B \cup C \cup P \cup Q \cup S \cup X$.

Claim 2. $t' \in A \cup C \cup (x_1Xz_2 - \{x_1, x_2, z_1, z_2\})$.

By the choice of Q and S, we have $t' \notin S$. To prove $t' \notin B \cup P \cup Q$, we consider two cases.

First, assume $t \in aAy_1 - \{a\}$. Then by Claim 1 and the choice of S, we have $t' \notin uBz_2 - u$. Moreover, by Claim 1 (when $t = y_1$) or by the choice of Q in (7) (when $t \neq y_1$), we have $t' \notin Q \cup qBz_2$. If $t' \in y_2Bq - q$, then the path $(y_2Bt' \cup T \cup tAy_1) \cup C \cup (z_1Aa \cup Q \cup qBz_2)$ passes through z_2, z_1, y_1, y_2 in order, contradicting (3). So we have $t' \notin Q \cup B$. If $t' \in P - c$, then the path $(y_2Bp \cup pPt' \cup T \cup tAy_1) \cup C \cup (z_1Aa \cup Q \cup qBz_2)$ passes through z_2, z_1, y_1, y_2 in order, again contradicting (3). So $t' \notin P - c$, and in this case Claim 2 holds.

Now assume $t \in sCy_1 - s$. By the choice of S, $t' \notin uBz_2 - u$. We claim $t' \notin y_2Bu$; for, otherwise, the path $(y_2Bt' \cup T \cup tCy_1) \cup A \cup (z_1Cs \cup S \cup s'Bz_2)$ passes through z_2, z_1, y_1, y_2 in order, contradicting (3). Also, $t' \notin P - c$; as otherwise the path $(y_2Bp \cup pPt' \cup T \cup tCy_1) \cup A \cup (z_1Cs \cup S \cup s'Bz_2)$ goes through z_2, z_1, y_1, y_2 in order, contradicting (3). Finally, $t' \notin Q - \{a\}$, for otherwise the path $(y_2Bq \cup qQt' \cup T \cup tCy_1) \cup A \cup (z_1Cs \cup S \cup s'Bz_2)$ passes through z_2, z_1, y_1, y_2 in order, contradicting (3). So the assertion of Claim 2 holds.

By Claim 2, we have the following four cases.

Case 1. $\{t, t'\} \subseteq A$ or $\{t, t'\} \subseteq C$.

Suppose $\{t, t'\} \subseteq A$. $G[z_1At' \cup T \cup tAy_1]$ contains an induced path A' from z_1 to y_1 such that, with A' replacing A, (a) and (b) are not affected, but the $(A' \cup C)$-bridge of H containing B is larger, contradicting (c).

Similarly, we derive a contradiction if $\{t, t'\} \not\subseteq C$.

Case 2. $t' \in A \cup C$.

Then by Case 1, $t \in sCy_1 - s$ and $t' \in z_1Aa - a$, or $t \in aAy_1 - a$ and $t' \in z_1Cs - s$.

If $t \in sCy_1 - s$ and $t' \in z_1Aa - a$, then $(z_2Bs' \cup S \cup sCz_1 \cup z_1At' \cup T \cup tCy_1 \cup y_1Aa \cup Q \cup qBy_1$ is a path through z_2, z_1, y_1, y_2 in order, contradicting (3).

If $t \in aAy_1 - a$ and $t' \in z_1Cs - s$, then $(R_1 \cup S \cup sCy_1) \cup (R_2 \cup uBq \cup Q \cup aAz_1 \cup z_1Xx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1At \cup T \cup t'Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Case 3. $t' \in x_1Xz_1 - \{x_1, z_1\}$.

If $t \in aAy_1 - a$, then $y_1Cc \cup P \cup pBy_2$ and $T \cup tAa \cup Q \cup qBz_2$ contradict the choice of Z, Y in (1) and (2).

If $t \in sCy_1 - s$, then $y_1Aa \cup Q \cup qBy_2$ and $z_2Bs' \cup S \cup sCt \cup T$ contradict the choice of Z, Y in (1) and (2).

Case 4. $t' \in z_1Xz_2 - \{z_1, z_2\}$.

If $t \in aAy_1 - a$ then $X' := x_2z_2 \cup z_2Bq \cup Q \cup aAz_1 \cup z_1Xx_1$ is a path in G from x_1 to x_2, and in $G - V(X')$, $\{y_1, y_2\}$ is contained in the cycle $y_1At \cup T \cup t'Xy_2 \cup y_2Bp \cup P \cup cCy_1$. If $t \in sCy_1 - s$ then $X' := x_2z_2 \cup z_2Bs' \cup S \cup sCz_1 \cup z_1Xx_1$ is a path from x_1 to x_2, and in $G - V(X')$, $\{y_1, y_2\}$ is contained in the cycle $y_1Ct \cup T \cup t'Xy_2 \cup y_2Bq \cup Q \cup aAy_1$.

In either case, we may assume X' is induced (for we can simply take an induced path in $G[X']$ from x_1 to x_2). Hence by applying Lemma 2.1 we can find an induced path X'' in G from x_1 to x_2 such that $G - V(X'')$ is 2-connected and $\{y_1, y_2\} \cap V(X'') = \emptyset$. Now Lemma 2.3 shows that G contains a TK_5, a contradiction to our initial assumption. ∎

Lemma 4.2 *There is a path R in H from z_1 to $r \in V(B - y_2)$ internally disjoint from $A \cup B \cup C$.*

Proof. Suppose R does not exist. Define $a' \in V(z_1Aa - z_1)$ with z_1Aa' minimal such that there is a path Q' in H from a' to $q' \in V(B)$ internally disjoint from $A \cup B \cup C$, or there is a path Q' from a' to $a'' \in V(cCy_1 - c)$ internally disjoint from $A \cup B' \cup C$.

Define $c' \in V(z_1Cc)$ with z_1Cc' minimal such that $c' = c$ or there is a path R' from c' to $r' \in V(a'Ay_1 - a')$ internally disjoint from $A \cup B' \cup C$.

We further choose A, B, C so that, subject to (a), (b), (c) and (d), $z_1Aa' \cup z_1Cc'$ is minimal.

Claim 1. If $c' \neq c$ then Q' ends at $q' \in B$.

For, suppose $c' \neq c$ and Q' ends at $a'' \in cCy_1 - c$. Then $G[z_1Aa' \cup Q' \cup a''Cy_1]$ and $G[z_1Cc' \cup R' \cup r'Ay_1]$ contain induced paths A', C', respectively, from z_1 to y_1. Clearly, A', C' satisfy (a) and (b); but the $(A' \cup C')$-bridge of H containing B is larger than the $(A \cup C)$-bridge of G containing B, contradicting (c). Hence we have Claim 1.

Claim 2. $\{a', c'\}$ is a cut in H separating $z_1Aa' \cup z_1Cc'$ from $a'Ay_1 \cup c'Cy_1 \cup B'$.

Suppose Claim 2 is flase. Then there is a path T in H from $t_1 \in V(z_1Aa' \cup z_1Cc') - \{a', c'\}$ to $t_2 \in (B - y_2) \cup (a'Ay_1 - a') \cup (c'Cy_1 - c')$ internally disjoint from $A \cup B \cup C$. By (8) and the choice of a' and c', there are only three possibilities: $t_2 \in B - y_2$; $t_1 \in z_1Cc' - c'$ and $t_2 \in c'Cy_1 - c'$; $t_1 \in z_1Aa' - a'$ and $t_2 \in a'Ay_1 - a'$.

Suppose $t_2 \in B - y_2$. Then by the choice of a' and since R does not exist, $t_1 \in z_1Cc' - \{c', z_1\}$. Then by the choice of P, T intersects $(Q - a) \cup (pBz_2 - p)$ before it intersects P; and hence we may assume $T \cap P = \emptyset$ and $t_2 \in pBz_2 - p$. Now the path $(z_2Bt_2 \cup T \cup t_1Cz_1) \cup A \cup (y_1Cc \cup P \cup pBy_2)$ passes through z_2, z_1, y_1, y_2 in order, contradicting (3).

Now suppose $t_1 \in z_1Cc' - c'$ and $t_2 \in c'Cy_1 - c'$. First, assume that T is contained in the $(A \cup C)$-bridge of H containing B. Then since R does not exist, $t_1 \neq z_1$, and there exists a path T' from some $t' \in V(T) - \{t_1, t_2\}$ to some $t'' \in V(B)$ which is internally disjoint from $A \cup B \cup C \cup T$. By the choice of P, T' is disjoint from P, and $t'' = y_2$ or $t'' \in pBz_2 - p$. If $t'' = y_2$, then $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup A \cup (y_1Ct_2 \cup t_2Tt' \cup T') \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_1. If $t'' \in pBz_2 - p$, then $(z_2Bt'' \cup T' \cup t'Tt_1 \cup$

$t_1Cz_1) \cup A \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (3). Therefore, T is not contained in the $(A \cup C)$-bridge of H containing B. Then $c' \neq c$ and $t_2 \in c'Cc - c'$; as otherwise, let C' be an induced path in $G[(C - (t_1Ct_2 - \{t_1, t_2\})) \cup T]$ from z_1 to y_1, and we see that A and C' satisfy (a) and (b), but the $(A \cup C')$-bridge of H containing B is larger than the $(A \cup C)$-bridge of G containing B, contradicting (c). If $t_1 = z_1$ then let A' be an induced path in $G[z_1Cc' \cup R' \cup r'Ay_1]$ from z_1 to y_1 and let C' be an induced path in $G[T \cup c'Cy_1]$ from z_1 to y_1; and we see that A', C'' satisfy (a) and (b), but the $(A' \cup C')$-bridge of H containing B is larger than the $(A \cup C)$-bridge of G containing B, contradicting (c). So $t_1 \neq z_1$. Then let C' be an induced path in $G[z_1Ct_1 \cup T \cup t_2Cy_1]$ from z_1 to y_1. Now A, B, C' satisfy (a)–(d); but we see that $t_1Cc' \cup R', t_1$ become the new R', c', respectively, contradicting the choice of c'.

Hence, $t_1 \in z_1Aa' - a'$ and $t_2 \in a'Ay_1 - a'$. We claim that Q' must end at a''; otherwise, the same argument in the previous case gives a contradiction (by symmetry between A and C, the choice of Q', and the nonexistence of R). Hence by Claim 1, $c' = c$, and Q' is contained in an $(A \cup C)$-bridge of H not containing B. Suppose T is contained in an $(A \cup C)$-bridge of H not containing B. If $t_1 = z_1$ then $G[T \cup t_2Ay_1]$ has an induced path A' from z_1 to y_1 and $G[z_1Aa' \cup Q' \cup a''Cy_1]$ has an induced path C' from z_1 to y_1, such that A', C'' satisfy (a) and (b), but the $(A' \cup C')$-bridge of H containing B is larger than the $(A \cup C)$-bridge of G containing B, contradicting (c). So $t_1 \neq z_1$. Then $G[z_1At_1 \cup T \cup t_2Ay_1]$ has an induced path A' from z_1 to y_1 such that A', B, C satisfy (a)-(d), but $t_1, t_1Aa' \cup Q'$ become the new a', Q', respectively, contradicting the minimality of $z_1Aa' \cup z_1Cc'$. So T is contained in the $(A \cup C)$-bridge of H containing B. Then there is a path S from $s' \in V(T) - \{t_1, t_2\}$ to $s'' \in V(B)$ internally disjoint from $A \cup C \cup B \cup T$. Since R does not exist, $t_1 \neq z_1$. If $s'' \neq y_2$, then $t_1, t_1Ts' \cup S$ constradict the choice of c', Q'. So $s'' = y_2$. Now $z_1Xx_1 \cup z_1Xy_2 \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (z_1Aa' \cup Q' \cup a''Cy_1) \cup (S \cup s'Tt_2 \cup t_2Ay_1) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1, contradicting our assumption that G contains no TK_5. This proves Claim 2.

Let F denote the union of $z_1Aa' \cup z_1Cc'$ and the $(A \cup C)$-bridges of H whose attachments are all contained in $z_1Aa' \cup z_1Cc'$, which is not empty since R does not exist. Since $H - \{z_1, z_2, y_2\} = G - V(X)$ is 2-connected, we have

Claim 3. $F - \{z_1, a'\}$ contains a path T_1 from $z_1Aa' - \{z_1, a'\}$ to $z_1Cc - z_1$, and $F - \{z_1, c\}$ has a path T_2 from $z_1Aa' - z_1$ to $z_1Cc - \{z_1, c\}$.

Let $u \in V(x_1Xz_1), w \in V(z_1Xy_2)$ with uXw maximal such that u, w each have a neighbor in $F - \{z_1, a', c'\}$. Since $\{u, w, a', c'\}$ cannot be a cut in G (as G is 5-connected), there is a path S from $s \in V(uXw - \{u, w\})$ to $s' \in V(a'Ay_1) \cup V(cCy_1) \cup V(P \cup Q' \cup Q) \cup V(B - y_2)$ such that $s \notin \{a', c'\}$, and S is internally disjoint from $F \cup uXw \cup a'Ay_1 \cup c'Cy_1 \cup P \cup Q \cup Q' \cup B$. By Claim 2, $s' \neq z_1$.

We will cosider two cases according to the location of s. But first, we need the following which follows from Lemma 4.1 and planarity of B'.

Claim 4. (i) B' has independent paths P_1, P_2 from z_2 to q, p, respectively; and (ii) if $q' \neq p$ then either B' has independent paths from z_2 to p, q', or $q \neq q'$ and B' has independent paths from z_2 to q', q disjoint from y_2Bp.

Case 1. $s \in uXz_1 - \{u, z_1\}$.

Then $s' \notin A$; as otherwise, the paths $S \cup s'Aa \cup Q \cup qBz_2$ and $y_2Bp \cup P \cup cCy_1$ contradict the choice of Z, Y in (1) and (2). Similarly, $s' \notin Q$, $s' \notin pBz_2 - p$, and $s' \notin Q'$ when $q' \in Q'$ and $q' \neq p$.

Subcase 1.1. $s' \in y_2Bp - \{y_2, p\}$. Then by Lemma 4.1, $(B' - y_2) - qBz_2$ has a path S' from z_2 to s'. Then $(z_2Bq \cup Q \cup aAy_1) \cup (S' \cup S \cup sXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2, a contradiction.

Subcase 1.2. $s' \in P - c$. Then by Claim 4(ii), B' has independent paths P_1', P_2' from z_2 to q, s', respectively. Now $(P_1' \cup Q \cup aAy_1) \cup (P_2' \cup S \cup sXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, y_1, x_2, y_2, z_2, a contradiction.

Subcase 1.3. $s' \in Q' - a'$. If Q' ends at $q' \in B$ then we have $q' = p$, and by Claim 4(ii) there are independent paths P_1', P_2' in B' from z_2 to q, q', respectively; and hence $(P_1' \cup Q \cup aAy_1) \cup (P_2' \cup pQ's' \cup S \cup sXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. So Q' ends at $a'' \in cCy_1 - c$. Let T' be a path in $G[V(F + u)] - z_1$ from u to c' (which exists by the path T_1 in $F - \{z_1, a'\}$). Then $(P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCc' \cup T' \cup uXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Ca'' \cup a''Q's' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2, a contradiction.

Subcase 1.4. $s' \in c'Cy_1 - c'$. If $s' \in cCy_1 - c$ then we derive a contradiction as in the above paragraph by replacing $(y_1Ca'' \cup a''Q's' \cup S \cup sXy_2)$ with $(y_1Cs' \cup S \cup sXy_2)$. So $s' \in c'Cc - c'$. In particular, $c \neq c'$ and so R' ends at $r' \in a'Ay_1 - a'$. By (8), $r' \in a'Aa - a'$.

By Claim 1, Q' ends at $q' \in B$. Let T' be a path in $G[V(F + u)] - z_1$ from u to c' (which exists by the path T_1 in $F - \{z_1, a'\}$).

If $r' = a$ then $aAy_1 \cup (Q \cup qBz_2 \cup z_2x_2) \cup (aAa' \cup Q' \cup q'By_2) \cup (R' \cup T \cup uXx_1) \cup (y_1Cs' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, a, a contradiction. So $r' \neq a$. By Claim 4(ii), $B' - y_2$ contains independent paths P_1', P_2' from z_2 to q, q', respectively. $(P_1' \cup Q \cup aAy_1) \cup (P_2' \cup Q' \cup a'Ar' \cup R' \cup T' \cup uXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cs' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2, a contradiction.

Case 2. $s \in z_1Xw - \{z_1, w\}$.

If $s' \in P - c$, then $(P_2 \cup pPs' \cup S \cup sXx_1) \cup (P_1 \cup Q \cup aAy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc \cup T_1' \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. So $s' \notin P - c$.

If $s' \in B - q$ then by Lemma 4.1 and by planarity, B' contains independent paths P_1', P_2' from z_2 to q, s', respectively. Now $(P_1' \cup Q \cup aAy_1) \cup (P_2' \cup S \cup sXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc \cup T_1' \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. So $s' \notin B - q$.

Hence we have the following four cases. Note that $G[V(F + w)] - \{z_1, a'\}$ has a path T_1' from w to c (because of T_1 in $F - \{z_1, a'\}$).

Subcase 2.1. $s' \in a'Ay_1 - a'$.

If $s' \in aAy_1 - a$, then $(P_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (P_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1As' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

If $s' = a$ then $(s'Az_1 \cup z_1Xx_1) \cup s'Ay_1 \cup (S \cup sXy_2) \cup (Q \cup qBz_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup y_2Bp) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, a.

So we may assume $s' \in a'Aa - \{a, a'\}$.

If $a'' \in Q'$ then $(P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCz_1 \cup z_1Xx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Ca'' \cup Q' \cup a'As' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

So we amy assume $q' \in Q'$.

If $q = q'$, then $(P_1 \cup Q' \cup a'Aq_1 \cup z_1Xx_1) \cup (P_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cs' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

So $q \neq q'$. By Lemma 4.1, $B' - y_2$ has independent paths P'_1, P'_2 from z_2 to q, q', respectively. Now $(P'_1 \cup Q \cup aAy_1) \cup (P'_2 \cup Q' \cup a'As' \cup S \cup sXx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc \cup T'_1 \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

Subcase 2.2. $s' \in c'Cy_1 - c'$.

If $s' \in cCy_1 - c$, then $(P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCz_1 \cup z_1Xx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cs' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

So $s' \in c'Cc - c'$. In particular, $c \neq c'$ and so R' ends at $r' \in a'Ay_1 - a'$. By (8), $r' \in a'Aa - a'$. By Claim 1, Q' ends at $q' \in B$.

If $q = q'$, then $(P_1 \cup Q' \cup a'Aq_1 \cup z_1Xx_1) \cup (P_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cr' \cup R' \cup T'_1 \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

So $q \neq q'$. By Lemma 4.1, $B' - y_2$ has independent paths P'_1, P'_2 from z_2 to q, q', respectively. Now $(P'_1 \cup Q \cup aAy_1) \cup (P'_2 \cup Q' \cup a'Az_1 \cup z_1Xx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cs' \cup S \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2.

Subcase 2.3. $s' \in Q$.

Note that $G[V(F + w)] - \{z_1, c\}$ has a path T'_2 from w to a' (because of the path T_2 in $F - \{z_1, c\}$).

Then $(P_1 \cup qQs' \cup S \cup sXx_1) \cup (P_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Aa' \cup T'_2 \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Subcase 2.4. $s' \in Q'$.

We may assume Q' ends at $q' \in B$, as otherwsie, we may revise S so that $s' = a'' \in cCy_1 - c$, and we derive a contradiction as in Subcase 2.2.

If $q' = q$ then $(P_1 \cup qQ's' \cup S \cup sXx_1) \cup (P_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Ar' \cup R' \cup T'_2 \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

So $q' \neq q$. Then by Claim 4(ii), let P'_1, P'_2 be independent paths in $B' - y_2$ from z_2 to q, q', respectively. Now $(P'_2 \cup q'Q's' \cup S \cup sXx_1) \cup (P'_1 \cup Q \cup aAy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc' \cup T'_1 \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. ∎

Lemma 4.3 *There is no path in H from y_1 to B internally disjoint from $A \cup B \cup C$.*

Proof. Suppose that H has a path R' from y_1 to $r' \in V(B)$ internally disjoint from $A \cup B \cup C$, with $r'Bz_2$ minimal.

Since G is 5-connected and X is induced, x_1 has a neighbor in $G - V(X + y_1)$, say x. If $x \in A \cup B \cup C$, let $D := \{x\}$ and $x' = x$; otherwise, let D denote the $(A \cup B \cup C \cup P \cup Q \cup R \cup R')$-bridge of H containing x, and let x' be an attachment of D such that $x' \notin \{z_1, y_2, z_2, y_1\}$ (since $H - y_2$ is 2-connected). Let T be a path in D from x_1 to x' internally disjoint from $A \cup B \cup C \cup P \cup Q \cup R \cup R'$.

Case 1. For any choice of x' we have $x' \in B'$.

Then $x \in B'$. If $x = r'$, then $R' \cup r'x_1 \cup (r'Bz_2 \cup z_2x_2) \cup r'By_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, r'. So assume $x \neq r'$.

If $r' \in qBz_2$ or $x' \in qBz_2$, then by Lemma 4.1, B' has independent paths Q_1, Q_2 from z_2 to r', x', respectively. Then $(Q_1 \cup R') \cup (Q_2 \cup x'Tx_1) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

If $\{x', r'\} \subset y_2Bq - \{q\}$, then there exist two disjoint paths Q_1, Q_2 from z_2 to x', q, respectively, then $(Q_1 \cup x'Tx_1) \cup (Q_2 \cup Q \cup aAy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2.

Case 2. $x' \in A \cup Q \cup R'$ and $r' \neq q$.

Then by Lemma 4.1, B' has independent paths from z_2 to r', q, respectively, and hence $B' \cup A \cup Q \cup R'$ has independent paths P_1, P_2 from z_2 to y_1, x', respectively. So $P_1 \cup (P_2 \cup T) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Case 3. $x' \in A \cup Q \cup R'$ and $r' = q$.

If $x' \in Q \cup R'$, then there exists two independent paths Q_1, Q_2 from z_2 to x', p which are in $B' \cup Q \cup R'$. Now, $(Q_1 \cup T) \cup (Q_2 \cup P \cup cCy_1) \cup z_2x_2 \cup z_2Xy_2 \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume that $x' \in A$, then we consider R. Note $r \notin pBz_2 - \{p\}$; otherwise, there exists a path through z_2, z_1, y_1, y_2 in order: $(z_2Br \cup R) \cup A \cup (y_1Cc \cup P \cup pBy_2)$. So $r \in y_2Bp$, then there exist two disjoint paths Q_1, Q_2 form z_2 to r, q in B', then $z_1Xx_1 \cup z_1Xy_2 \cup (R \cup Q_1) \cup (C \cup y_1x_2) \cup z_2x_2 \cup z_2Xy_2 \cup (Q_2 \cup Q \cup aAx' \cup T) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_2, z_1, z_2.

Case 4. $x' \in R \cup P$.

By Lemma 4.1, B' has two independent paths from z_2 to q, r(or p), then there exist two independent paths from z_2 to q, x' in $B' \cup R \cup P$, called Q_1, Q_2 respectively. Now, $(Q_1 \cup Q \cup aAy_1) \cup (Q_2 \cup T) \cup z_2x_2 \cup z_2Xy_2 \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2.

Case 5. $x' \in C$.

We consider R. Note, $r' \notin y_2Bp - \{p\}$; otherwise, there exists path through z_2, z_1, y_1, y_2 in order: $(z_2Bp \cup P \cup cCz_1) \cup A \cup (R' \cup r'By_2)$. So assume that $r' \in pBz_2$.

If $r' \in pBq - \{q\}$, there is a path through y_2, y_1, z_1, z_2 in order: $(y_2Br' \cup R') \cup C \cup (z_1Aa \cup Q \cup qBz_2)$, contradicting to (3).

If $r' = q$, then there exist two independent paths Q_1, Q_2 from z_2 to q, p in $B' - \{y_2\}$, respectively, then $(Q_1 \cup R') \cup (Q_2 \cup P \cup cCx' \cup T) \cup x_2z_2 \cup x_2Xy_2 \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2.

If $r' \in qBz_2 - \{q\}$, then there exist two independent paths Q_1, Q_2 in $B' - pBy_2$ from z_2 to q, r' respectively, then $(Q_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (Q_2 \cup R') \cup z_2x_2 \cup z_2Xy_2 \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2. ∎

Lemma 4.4 *There is a 2-cut $\{t_1, t_2\}$ in H separating $\{y_1, z_1\}$ from $\{y_2, z_2\}$, and $\{y_1, y_2, z_1, z_2\} \cap \{t_1, t_2\} = \emptyset$.*

Proof. Suppose such a 2-cut does not exist. Then by the same argument as for (4), H has three independent paths, two from y_1 to z_2, and one from z_1 to y_2. Hence by (6), we may also assume $x_1z_1 \in E(G)$. By (3), $y_1z_1, y_1z_2 \notin E(G)$.

If all neighbors of y_1 are contained in $A \cup C \cup X$ then let $a' \in V(A), c' \in V(C)$ such that $y_1a' \in E(A)$, $y_1c' \in E(C)$, and $S = y_1a' \cup y_1c'$.

Next we define S, a', c' when y_1 has a neighbor not in $A \cup C \cup X$. So let T_1 be an $(A \cup C)$-bridge of H with y_1 as an attachment. By Claim 3, $B \not\subseteq T_1$. Recall in (6) the definition of $a_i(T)$ and $c_i(T)$ for $(A \cup C)$-bridges T not containing B.

Let $T_1, \ldots, T_k$ be a maximal sequence of $(A \cup C)$-bridges of H not containing B such that for each $i = 1, \ldots, k-1$, T_{i+1} has an attachment not in $\bigcup_{j=1}^{i}(c_1(T_j)Cy_1 \cup a_1(T_j)Ay_1)$, and an attachment not in $\bigcup_{j=1}^{i}(z_1Cc_1(T_j) \cup z_1Aa_1(T_j))$. To simplify the notation, we let $a_i \in V(A), c_i \in V(C)$ with z_1Aa_i and z_1Cc_i minimal such that a_i is an attachment of some T_j with $1 \le j \le i$, and c_i is an attachment of some T_j with $1 \le j \le i$. Let $S_i := (\bigcup_{j=1}^{i} T_j) \cup a_iAy_1 \cup c_iCy_1$.

Claim 1. For any $1 \le i \le k$ and for any $r_i \in S_i - \{a_i, c_i\}$ there exist three independent paths A_i, C_i, R_i in S_i from y_1 to a_i, c_i, r_i, respectively.

This is obvious for $i = 1$ (if $a_i = y_1$, or $c_i = y_1$, or $r_i = y_1$ then A_i or C_i or R_i is a trivial path).

Now assume it is true for some $i \le k-1$. Let $r_{i+1} \in S_{i+1} - \{a_{i+1}, c_{i+1}\}$. When $r_{i+1} \in S_i - \{a_i, c_i\}$ let $r_i := r_{i+1}$; otherwise, let $r_i \in V(a_iAy_1 - a_i) \cup V(c_iCy_1 - c_i)$ be an attachment of T_{i+1}. By induction there are three independent paths A_i, C_i, R_i in S_i from y_1 to a_i, c_i, r_i, respectively.

If $r_i = r_{i+1}$ then $A_{i+1} := A_i \cup a_iAa_{i+1}, C_{i+1} := C_i \cup c_iCc_{i+1}, R_{i+1} := R_i$ are the desired paths in S_{i+1}.

If $r_{i+1} \in T_{i+1} - (A \cup C)$ then let P_{i+1} be a path in T_{i+1} from r_i to r_{i+1} internally disjoint from $A \cup C$; we see that $A_{i+1} := A_i \cup a_iAa_{i+1}, C_{i+1} := C_i \cup c_iCc_{i+1}, R_{i+1} := R_i \cup P_{i+1}$ are the desired paths in S_{i+1}.

So we may assume by symmetry that $r_{i+1} \in a_{i+1}Aa_i - a_{i+1}$. Let Q_{i+1} be a path in T_{i+1} from r_i to a_{i+1} internally disjoint from $A \cup C$. Now $R_{i+1} := A_i \cup a_iAr_{i+1}, C_{i+1} := C_i \cup c_iCc_{i+1}, A_{i+1} := R_i \cup Q_{i+1}$ are the desired paths in S_{i+1}.

Claim 2. $a_k \in aAy_1$ and $c_k \in cCy_1$.

Otherwise, let $i \in \{1, \ldots, k\}$ be minimum such that $a_i \in z_1Aa - a$ or $c_i \in z_1Cc - c$.

Suppose $a_i = z_1$. Then $i \ge 2$ by (9), and there is a path L in T_i from z_1 to some $r_{i-1} \in S_{i-1} - \{a_{i-1}, c_{i-1}\}$ internally disjoint from $S_{i-1} \cup A \cup C$. Now $y_2Bp \cup P \cup cCc_{i-1} \cup C_{i-1} \cup R_{i-1} \cup L \cup z_1Aa \cup Q \cup qBz_2$ is a path in H contradicting (3). Therefore, $a_i \ne z_1$. Similarly, $c_i \ne z_1$.

Suppose $a_i \in z_1Aa - \{a, z_1\}$. Let $A' := A_i \cup z_1Aa_i$ and $C' := C_i \cup z_1Cc_i$. We see that the $(A' \cup C')$-bridge of H containing B is larger than the $(A \cup C)$-bridge of H, contradicting (c) of (5) (while (b) of (5) is not affected). So $a_k \in aAy_1$. Similarly, $c_k \in cCy_1$.

Define $a' := a_k$, $c' := c_k$, and $S := S_k$. Now $\{a', c', x_1, x_2\}$ cannot be a 4-cut in G (as G is 5-connected). So there is a path D in G from some vertex $r' \in V(S) - \{a', c'\}$ to $y \in V(X \cup A \cup B \cup C) - V(S)$ internally disjoint from $X \cup A \cup B \cup C \cup S$.

By Lemma 4.3, $y \notin B$. Indeed, the $(A \cup C)$-bridge of H containing S has no attachment on B (otherwise we may choose S so that $y \in B$). Thus by the definition of a' and c', $y \notin A \cup C$. So $y \in z_1Xz_2 - \{z_1, z_2\}$. By Claim 1, let Q_1, Q_2 be independent paths in $B' - y_2$ from z_2 to q, p, respectively. Let A', C', R' be independent paths in S from y_1 to a', c', r', respectively.

If $y \in z_1Xy_2 - z_1$ then $(Q_1 \cup Q \cup aAa' \cup A') \cup (Q_2 \cup P \cup cCz_1 \cup z_1x_1) \cup z_2x_2 \cup z_2Xy_2 \cup (R' \cup D \cup yXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume $y \in z_2Xy_2 - \{z_2, y_2\}$. Then $X' := x_1z_1 \cup z_1Aa \cup Q \cup qBz_2 \cup z_2x_2$ is a path in G from x_1 to x_2. In $G - V(X')$, $\{y_1, y_2\}$ is contained in the cycle $D \cup R' \cup C' \cup c'Cc \cup P \cup pBy_2 \cup y_2Xy$. Hence by Lemma 2.1, there is an induced path X'' in G form x_1 to x_2 such that $G - V(X'')$ is 2-connected and $\{y_1, y_2\} \cap V(X'') = \emptyset$. Now Lemma 2.3 finds a TK_5 in G. ∎

By Lemma 4.4, H has a separation (H', H'') such that $|V(H' \cap H'')| = 2$, $\{y_1, z_1\} \subseteq H''$ and $\{y_2, z_2\} \subseteq H'$, and $\{y_1, z_1, y_2, z_2\} \cap V(H' \cap H'') = \emptyset$. We choose (H', H'') so that H' is minimal. Then, because of the existence of R, P, Q and by (5) and (7), we see that $A \cup C \cup P \cup R \subseteq H''$. Note, if $t_1 = p$, then $t_2 \in Q$ or $t_2 \in qBz_2$; if $t_1 \neq p$, then $t_2 \in qBz_2$. Thus we may assume that $t_1 \in y_2Br$, $t_2 \in Q$ or $t_2 \in qBz_2$. Let $T' = t_2Bq \cup Q, T'' = t_2Bz_2$ if $t_2 \in B$; otherwise define $T' = t_2Qa, T'' = qBz_2 \cup qQt_2$.

Let $z \in z_1Xy_2$ with zXz_1 minimal such that $z = y_2$ or z has a neighbor in $H' - \{y_2, z_2, t_1, t_2\}$.

Lemma 4.5 *$\{x_2, y_2, z, t_1, t_2\}$ is a 5-cut in G, $G[V(H' \cup zXx_2)$ is 2-connected and $(5, \{x_2, y_2, z, t_1, t_2\})$-connected, $G[V(H' \cup zXx_2)$ has a plane representation in which x_2, y_2, z, t_1, t_2 occur on a facial cycle in this cyclic order.*

Proof. Let $H^* := G[V(H' \cup y_2Xz_2)]$. Then $(H^*, y_2, t_1, t_2, z_2)$ is 3-planar. Since otherwise by Theorem 2.6, H^* contains disjoint paths T_1, T_2 from z_2, y_2 to t_1, t_2, respectively. Now $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (R \cup rBt_1 \cup T_1 \cup z_2x_2) \cup (y_1Aa \cup T' \cup T_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 on branch vertices x_1, x_2, y_1, y_2, z_1.

We may assume that if $z \neq y_2$ and $zz' \in E(G)$ with $z' \in V(H^*) - \{t_1, t_2, y_2\}$, then $z' \in y_2Bt_1$ or H^* has a 2-cut contained in y_2Bt_1 and separating z' from $\{t_2, z_2\}$. For otherwise, by the minimality of H' (and by planarity), $H' - y_2Bt_1$ has independent paths P_1, P_2 from z_2 to z', t_2, respectively; and $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup zz' \cup zXx_1) \cup (P_2 \cup Q \cup aAy_1) \cup (y_1Cp \cup P \cup y_2Bp) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, y_1, x_2, y_2, z_2. Actually, this paragraph tells us more: for any vertex $v \in z_1Xy_2$, the conclusion holds.

If x_2 ha a neighbor $x \in H' - T''$ then, since G is 5-connected, $H' - (T'' + y_2)$ contains a path X' from x to t_1; and hence $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (R \cup rBt_1 \cup X' \cup xx_2) \cup (y_1Aa \cup T' \cup T'' \cup z_2Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1. Therefore, we may assume that all neighbors of x_2 in H' must be contained in T''.

Suppose $zXz_2 - z$ has no neighbor in $H'' - \{t_1, t_2\}$. Note, $x_1Xz_1 - \{x_1, z_1\}$ has no neighbor in $H' - \{y_2\}$; otherwise, contradicting to (2). Then $\{x_2, y_2, z, t_1, t_2\}$ is a 5-cut in G, and $G[V(H' \cup zXx_2)$ is 2-connected and $(5, \{x_2, y_2, z, t_1, t_2\})$-connected. Suppose Claim 5 fails. Then there exist $s, t \in V(zXy_2 - y_2)$ and $s', t' \in V(y_2Bt_1 - y_2)$ such that $ss', tt' \in E(G)$, y_2, s, t, z occur on X in order, and y_2, t', s', t_1 occur on B in order. Since $H - y_2$ is 2-connected and by 3-planarity of H^* and minimality of H', $(H' - y_2) - (T'' - z_2)$ has a path L from z_2 to t' disjoint from t_1Bs'. Then $z_2x_2 \cup z_2Xy_2 \cup (L \cup tt' \cup tXx_1) \cup (T'' \cup T' \cup aAy_1) \cup (y_1Cc \cup P \cup pBs' \cup ss' \cup sXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2.

Therefore, we may assume that there is $w \in V(zXz_2 - z)$ that has a neighbor in $V(H'') - \{t_1, t_2\}$. Then there is a path W in G from w to $w' \in V(A \cup C \cup P \cup t_1Bp \cup T' \cup R) - \{t_1, t_2\}$ internally disjoint from $A \cup B \cup P \cup t_1Bp \cup T' \cup R$. Note that $w \neq z_2, w \neq y_2$.

First, assume $w \in y_2Xz_2 - z_2$. If $w' \in (R \cup t_1Bp \cup P) - C$ then $(W \cup R \cup t_1Bp \cup P) - ((C - z_1) + t_1)$ has a path W' from w to z_1; and let L be a path in $H' - z_2$ from t_2 to y_2, we see that $z_1Xx_1 \cup z_1Xy_2 \cup (W' \cup wXx_2) \cup C \cup (y_1Aa \cup T' \cup L) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in

G with branch vertices x_1, x_2, y_1, y_2, z_1. If $w' \in C$, then let L be a path in $H' - y_2$ from t_1 to z_2; and $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (R \cup t_1Br \cup L \cup z_2x_2) \cup (y_2Xw \cup W \cup w'Cy_1) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1. If $w' \in A \cup T'$, then $A \cup T' \cup W$ has a path W' from w to y_1; and let L be a path in $H' - y_2$ from t_1 to z_2, and we see that $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (R \cup t_1Br \cup L \cup z_2x_2) \cup (y_2Xw \cup W') \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1. This paragraph also shows that $z \neq y_2$.

Now assume that $w \in zXy_2 - \{z, y_2\}$. Let $z' \in t_1By_2 - \{t_1, y_2\}$ such that there is a path Z from z to z' which is independent with other paths. By the choice of (H', H'') and since $H - y_2$ is 2-connected, $H' - \{y_2, t_1\}$ contains independent paths P_1, P_2 from z_2 to t_2, z', respectively; and by planarity, H' has disjoint paths Q_1, Q_2 from t_1, z' to z_2, y_2, respectively. If $w' \in R \cup P \cup C$ then $W \cup R \cup P \cup C$ contains a path W' from w to y_1; and $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup T' \cup aAy_1) \cup (P_2 \cup zZz' \cup zXx_1) \cup (W' \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume $w' \in A \cup T' \cup t_1Bp$; then $W \cup A \cup T'$ has a path W' from w to y_1 avoiding z_1 and t_2. Hence $z_1Xx_1 \cup (z_1Xz \cup zZz' \cup Q_2) \cup (R \cup t_1Br \cup Q_1 \cup z_2x_2) \cup C \cup (W' \cup wXy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 with branch vertices x_1, x_2, y_1, y_2, z_1. ∎

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