SHARP VERTEX CONNECTIVITY OF THE MARKOFF GRAPHS MODULO p

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Abstract. The Markoff graph G_p modulo a prime p is an undirected graph whose vertices are the nonzero solutions over the finite field $\mathbb{F}_p$ of the normalized Markoff equation

$$x_1^2 + x_2^2 + x_3^2 = x_1x_2x_3,$$

where two vertices are adjacent if they differ by a Vieta involution. A major breakthrough of Bourgain, Gamburd, and Sarnak established that G_p contains a giant connected component. Combined with Chen's remarkable divisibility theorem, this implies that G_p is connected for all sufficiently large primes p.

In the same paper, Bourgain, Gamburd, and Sarnak further asked whether the family $\{G_p : p \geq 5\}$ forms an expander family. This motivates us to investigate the robustness of connectivity in the Markoff graphs. In this short note, we show that if the Markoff graph G_p is connected, then it is in fact 2-connected. Consequently, the Markoff graph G_p is 2-connected for all sufficiently large primes p. This is sharp in the sense that G_p is not 3-connected for any prime $p \geq 7$.

1. Introduction

The classical *Markoff equation*

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3 \tag{1}$$

was introduced by Markoff [10] in his study of indefinite binary quadratic forms and Diophantine approximation. A fundamental feature of the Markoff equation is that the surface defined by it admits three non-commuting *Vieta involutions*

$$(x_1, x_2, x_3) \longmapsto (3x_2x_3 - x_1, x_2, x_3),$$
$$(x_1, x_2, x_3) \longmapsto (x_1, 3x_1x_3 - x_2, x_3),$$
$$(x_1, x_2, x_3) \longmapsto (x_1, x_2, 3x_1x_2 - x_3).$$

Markoff [10] proved that every positive integral solution to (1) can be obtained by repeatedly applying three Vieta involutions to the fundamental solution $(x_1, x_2, x_3) = (1, 1, 1)$. From the graph-theoretic perspective, one may form an undirected graph whose vertices are the positive integral solutions of (1), with two solutions connected by an edge if they differ by a Vieta involution. In this sense, Markoff's theorem asserts that the graph is connected. Indeed, it is an infinite tree, known as the *Markoff tree*.

Baragar [1] considered a local analogue of the connectivity theorem. To be precise, let $p \geq 5$ be a prime, and consider the normalized Markoff equation over the finite field $\mathbb{F}_p$

$$x_1^2 + x_2^2 + x_3^2 = x_1x_2x_3. \tag{2}$$

Note that the scaling

$$(x_1, x_2, x_3) \longmapsto (3x_1, 3x_2, 3x_3)$$

identifies the solutions of (1) with those of (2).

Let X_p be the set of solutions $(x_1, x_2, x_3) \neq (0, 0, 0)$ of the normalized Markoff equation. The associated Vieta involutions on X_p are given by

$$m_1(x_1, x_2, x_3) := (x_2x_3 - x_1, x_2, x_3),$$
$$m_2(x_1, x_2, x_3) := (x_1, x_1x_3 - x_2, x_3),$$
$$m_3(x_1, x_2, x_3) := (x_1, x_2, x_1x_2 - x_3).$$

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The *Markoff graph* G_p is an undirected graph whose vertex set is X_p and whose edges connect two vertices $x, y \in X_p$ if $m_i(x) = y$ for some $i \in \{1, 2, 3\}$. For every prime $p \geq 5$, Baragar [1] conjectured that the Markoff graph G_p is connected.

Bourgain, Gamburd, and Sarnak [2, 3] made the first major progress toward this conjecture. They proved that, for every $\varepsilon > 0$ and all sufficiently large primes p depending on ε, the graph G_p has a giant connected component $\mathcal{C}_p$ in the sense that

$$\left|X_p \setminus \mathcal{C}_p\right| < p^{\varepsilon}.$$

Chen [7] then proved that the cardinality of every connected component of G_p is divisible by p. Combined with the theorem of Bourgain, Gamburd, and Sarnak, this implies that G_p is connected for every sufficiently large prime p. Recently, Martin [11] gave a short alternative proof to Chen's divisibility theorem.

Eddy, Fuchs, Litman, Martin, and Tripeny [9] further developed the approach of Bourgain, Gamburd, and Sarnak to show that Baragar's conjecture is true for all primes greater than $3.449 \cdot 10^{392}$. On the other hand, Brown [4] developed a fast connectivity test and verified the conjecture for all primes in $[5, 10^6)$. Nevertheless, Baragar's full conjecture remains open.

Beyond connectivity, Bourgain, Gamburd, and Sarnak [3] raised a stronger question of whether the family of Markoff graphs $\{G_p : p \geq 5 \text{ prime}\}$ forms an expander family. The expansion problem naturally motivates us to study the robustness of connectivity in the Markoff graphs. The main goal of this paper is to investigate the 2-connectivity of the Markoff graphs. Here and throughout, by 2-connectivity, we mean 2-vertex-connectivity: the graph has at least three vertices and remains connected after deleting any one vertex.[1] Our first main result is that every edge of the Markoff graph is contained in a cycle.

Theorem 1.1. Let $p \geq 5$ be a prime. Every edge of the Markoff graph G_p is contained in a cycle of G_p.

An edge of a graph is called a *bridge* if the deletion of this edge increases the number of connected components of this graph. In other words, Theorem 1.1 can be rephrased as the Markoff graph G_p has no bridge for $p \geq 5$. The next result follows from Theorem 1.1 immediately.

Corollary 1.2. Let $p \geq 5$ be a prime. If the Markoff graph G_p is connected, then it is 2-connected.

Combining Corollary 1.2 with the results of Eddy, Fuchs, Litman, Martin, and Tripeny [9] and the results of Brown [4] gives the 2-connectivity of the Markoff graph G_p for an explicit range of primes.

Corollary 1.3. The Markoff graph G_p is 2-connected for every prime p in the range $[5, 10^6) \cup (3.449 \cdot 10^{392}, +\infty)$.

Remark 1.4. One can easily verify that G_5 is 3-connected (see [8, Figure 13.1] for an illustration of G_5). For every prime $p \geq 7$, however, the conclusion of Corollary 1.3 is best possible, as G_p is not 3-connected. Indeed, [6, Lemma 2.3] shows that G_p has $3(p-3)$ vertices with a loop when $p \equiv 3 \pmod 4$, and $3(p-5)$ such vertices when $p \equiv 1 \pmod 4$. Moreover, each vertex of G_p has at most one loop. Hence every vertex has either two or three distinct neighbors, and for every prime $p \geq 7$ there exists a vertex with exactly two distinct neighbors. Since every 3-connected graph has minimum degree at least 3, it follows that G_p is not 3-connected for any prime $p \geq 7$.

The remainder of the paper is organized as follows. In Section 2, we pass to the quotient graph $\overline{G}_p$ that captures the double sign-change symmetries of the Markoff graph G_p and identify a distinguished class of quotient vertices, called *cusps*. Their locally connecting property and lifting property play central roles in our proof, with Lemma 2.3 providing the key ingredient. In Section 3, we apply the results of Section 2 to prove our main results.

2. The quotient graph

We define the *double sign-change automorphisms* $\sigma_1, \sigma_2, \sigma_3 \colon X_p \to X_p$ by

$$\begin{aligned}
\sigma_1(x_1, x_2, x_3) &:= (x_1, -x_2, -x_3),\\
\sigma_2(x_1, x_2, x_3) &:= (-x_1, x_2, -x_3),\\
\sigma_3(x_1, x_2, x_3) &:= (-x_1, -x_2, x_3).
\end{aligned}$$

[1] By the classical Menger theorem, 2-connectivity is also equivalent to the property that every two distinct vertices are joined by two paths sharing no other vertices.

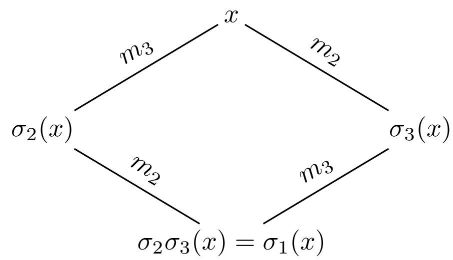

FIGURE 1. A 4-cycle in G_p formed by four vertices lying over a cusp $Kx \in \overline{G}_p$ with $x = (0, x_2, x_3)$.

Set

$$K \coloneqq \{1, \sigma_1, \sigma_2, \sigma_3\}.$$

It is easy to verify the relations $\sigma_i^2 = 1$ for $i \in \{1, 2, 3\}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i = \sigma_k$ for distinct $i, j, k \in \{1, 2, 3\}$. Thus we have a group isomorphism

$$K \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

Furthermore, the group K commutes with the three Vieta involutions m_1, m_2, m_3, and acts on X_p freely. Indeed, if a non-identity element $\sigma \in K$ fixes a point $x \in X_p$, then two coordinates of x vanish, and it follows from the Markoff equation that the third coordinate has to vanish.

Set $Y_p \coloneqq X_p/K$. We can define a quotient graph structure on Y_p as follows. The vertex set of the quotient graph $\overline{G}_p$ is the set Y_p of K-orbits, and two K-orbits $Kx, Ky \in Y_p$ are connected by an edge if $m_i(x) = \sigma(y)$ for some $i \in \{1, 2, 3\}$ and some $\sigma \in K$. Since the group K commutes with Vieta involutions, this is independent of the choice of representative. Indeed, if Kx is adjacent to Ky in $\overline{G}_p$, then between the set Kx and the set Ky, each of which has 4 vertices, there is an induced matching of size 4 in G_p.

Let $x = (x_1, x_2, x_3) \in G_p$. We call x a *cusp* if $x_i = 0$ for some $i \in \{1, 2, 3\}$, and a *regular vertex* otherwise. For a vertex $Kx \in \overline{G}_p$, we call Kx a *cusp* (resp. a *regular vertex*) if one, equivalently every, vertex in Kx is a cusp (resp. a regular vertex) in G_p. This definition is independent of the choice of representative because K acts by coordinate-wise sign-change and therefore preserves the vanishing or non-vanishing properties of each coordinate.

Lemma 2.1. Let Kx be a cusp in $\overline{G}_p$. Then the four vertices in Kx form a 4-cycle in G_p.

Proof. Without loss of generality, we assume that $x = (0, x_2, x_3)$ with $x_2, x_3 \neq 0$. It is easy to verify that

$$m_2(x) = (0, -x_2, x_3) = \sigma_3(x)$$

and

$$m_3(x) = (0, x_2, -x_3) = \sigma_2(x).$$

Thus, we have

$$\sigma_2\sigma_3(x) = \sigma_2 m_2(x) = m_2\sigma_2(x),$$

and

$$\sigma_2\sigma_3(x) = \sigma_3\sigma_2(x) = \sigma_3 m_3(x) = m_3\sigma_3(x).$$

Since $\sigma_1 = \sigma_2\sigma_3 = \sigma_3\sigma_2$, we obtain the 4-cycle in Figure 1 consisting of four vertices in Kx. □

Let π be the canonical projection $\pi\colon X_p \to Y_p$. We prove a lifting property that plays an important role in the proof of our main result.

Lemma 2.2. If a connected subgraph $H \subseteq \overline{G}_p$ contains a cusp Kc, then the subgraph of G_p induced by $\pi^{-1}(V(H))$, denoted by $G_p[\pi^{-1}(V(H))]$, is connected.

Proof. Take any vertex x in the induced subgraph $G_p[\pi^{-1}(V(H))]$. By definition, we have $Kx \in H$. Since H is connected, we choose a path $\overline{P}$ in H from Kx to the cusp Kc. We can lift this path to a path P in G_p from x to $\sigma(c)$ for some $\sigma \in K$. Thus, every vertex in the induced subgraph $G_p[\pi^{-1}(V(H))]$ is connected to one vertex of Kc in G_p. Now, by applying Lemma 2.1, the four vertices of Kc form a 4-cycle in G_p, in particular, a connected subgraph of G_p. It follows that any two vertices of $G_p[\pi^{-1}(V(H))]$ are connected to each other via the 4-cycle. Hence, we see that the induced subgraph $G_p[\pi^{-1}(V(H))]$ is connected. □

Next we prove a technical lemma, which asserts that cusps must exist on both sides of a bridge in $\overline{G}_p$.

Lemma 2.3. Let Q be a connected component of $\overline{G}_p$ containing a bridge $\overline{e}$. Suppose that

$$Q - \overline{e} = A \sqcup B.$$

Then each of A and B contains a cusp.

The proof of Lemma 2.3 is the most technical part of this paper. Our strategy is to construct a vector-valued function D on the oriented edges of the quotient graph $\overline{G}_p$ such that D is nonzero on every non-loop edge incident to a regular vertex. We then derive a contradiction by showing that D must vanish on a bridge. To construct D, we first introduce several functions on the regular vertices of G_p that provide the essential ingredients. For a regular vertex $x \in G_p$, we define

$$U_1(x) \coloneqq -\frac{2x_1 - x_2x_3}{x_2x_3^3}, \qquad U_2(x) \coloneqq \frac{(2x_2 - x_1x_3)(x_3^2 - x_1^2)}{x_1^3x_3^3}, \quad U_3(x) \coloneqq \frac{2x_3 - x_1x_2}{x_1^3x_2},$$
$$V_1(x) \coloneqq \frac{(2x_1 - x_2x_3)(x_3^2 - x_2^2)}{x_2^3x_3^3}, \quad V_2(x) \coloneqq -\frac{2x_2 - x_1x_3}{x_1x_3^3}, \qquad V_3(x) \coloneqq \frac{2x_3 - x_1x_2}{x_1x_2^3}.$$

Then for every $i \in \{1,2,3\}$, we set

$$F_i \coloneqq (U_i, V_i).$$

In the following lemma, we collect several key properties of the vector-valued functions F_i.

Lemma 2.4. Let x be a regular vertex in G_p. Then the following statements hold.

(1) For every $i \in \{1,2,3\}$, if $m_i(x)$ is also a regular vertex, then we have

$$F_i(m_i(x)) = -F_i(x).$$

(2) The function F_i is K-invariant in the sense that for any $\sigma \in K$ and any $i \in \{1,2,3\}$,

$$F_i(\sigma(x)) = F_i(x).$$

(3) We have

$$F_1(x) + F_2(x) + F_3(x) = (0,0).$$

(4) For every $i \in \{1,2,3\}$, if $m_i(x) \neq x$, then $F_i(x) \neq (0,0)$.

Proof. (1) By direct calculations, we have

$$U_1(m_1(x)) = -\frac{2(x_2x_3 - x_1) - x_2x_3}{x_2x_3^3} = \frac{2x_1 - x_2x_3}{x_2x_3^3} = -U_1(x)$$

and

$$U_2(m_2(x)) = \frac{(2(x_1x_3 - x_2) - x_1x_3)(x_3^2 - x_1^2)}{x_1^3x_3^3} = \frac{(x_1x_3 - 2x_2)(x_3^2 - x_1^2)}{x_1^3x_3^3} = -U_2(x).$$

By symmetry, we can deduce $U_3(m_3(x)) = -U_3(x)$ from $U_1(m_1(x)) = -U_1(x)$. Similarly, we can show that $V_i(m_i(x)) = -V_i(x)$ for every $i \in \{1,2,3\}$. Therefore, we see that

$$F_i(m_i(x)) = -F_i(x),$$

for every $i \in \{1,2,3\}$.

(2) Every non-identity automorphism $\sigma \in K$ changes the signs of two coordinates of x. It is straightforward to check that the sign-change of the denominator of U_i (resp. V_i) cancels with that of its numerator. Hence the function F_i is K-invariant for $i \in \{1,2,3\}$.

(3) A direct calculation gives

$$U_1(x) + U_2(x) + U_3(x) = \frac{2(x_3^2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3)}{x_1^3x_2x_3^3}$$

and

$$V_1(x) + V_2(x) + V_3(x) = \frac{2(x_3^2 - x_2^2)(x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3)}{x_1x_2^3x_3^3}.$$

We have $x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 0$ for $x \in X_p$. It follows that

$$F_1(x) + F_2(x) + F_3(x) = (U_1(x) + U_2(x) + U_3(x), V_1(x) + V_2(x) + V_3(x)) = (0,0).$$

This proves (3).

(4) Suppose that $i = 1$ (resp. $i = 3$). Then the assumption $m_i(x) \neq x$ implies that $2x_1 - x_2x_3 \neq 0$ (resp. $2x_3 - x_1x_2 \neq 0$). It follows that $U_i(x) \neq 0$. If $i = 2$, the assumption $m_2(x) \neq x$ implies that $2x_2 - x_1x_3 \neq 0$. So $V_2(x) \neq 0$. Thus, in all cases, we have shown that

$$F_i(x) \neq (0, 0),$$

as desired. ☐

Now we are ready to prove Lemma 2.3.

Proof of Lemma 2.3. Suppose to the contrary that A contains no cusps. In other words, every vertex of A is a regular vertex.

For each quotient edge

$$\overline{f} = \{Kx, Ky\} \in E(A) \cup \{\overline{e}\},$$

we assign the label $i \in \{1, 2, 3\}$ if $m_i(x) = \sigma(y)$ for some $\sigma \in K$. To see the labeling is well-defined, suppose to the contrary that $m_i(x) = \sigma(y)$ and $m_j(x) = \tau(y)$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$ and some $\sigma, \tau \in K$. Without loss of generality, we assume that $i = 1$ and $j = 2$. Using the fact that $m_1(x) \neq m_2(x)$, we can deduce that $\sigma \neq \tau$. Set $\kappa \coloneqq \sigma\tau \neq 1$. Since σ and τ are involutions, we see that

$$\sigma(m_1(x)) = y = \tau(m_2(x)),$$

and it follows that

$$\kappa(m_2(x)) = m_1(x).$$

Assume that $x = (x_1, x_2, x_3)$. Then

$$m_1(x) = (x_2x_3 - x_1, x_2, x_3)$$

and

$$m_2(x) = (x_1, x_1x_3 - x_2, x_3).$$

The automorphism $\kappa \neq 1$ flips the sign of at least one of the first two coordinates of $m_2(x)$. So, in both cases, one can deduce that x is a cusp. This is a contradiction. Thus, we have shown that the labeling is well-defined.

Choose an orientation for each quotient edge $\overline{f} \in E(A) \cup \{\overline{e}\}$ such that $\overline{e}$ is directed out of A. Then every oriented $\overline{f}$ has a regular initial endpoint, say $Kx \in A$, as A has no cusps. If the label of $\overline{f}$ is i, then we assign it a value

$$D(\overline{f}) \coloneqq F_i(x).$$

By Lemma 2.4(2), the assignment of the value $D(\overline{f})$ is independent of the choice of representative. The *divergence* at a quotient vertex $Kx \in A$ is defined as

$$\operatorname{div}(Kx) \coloneqq \sum_{\overline{f} \in \delta^+(Kx)} D(\overline{f}) - \sum_{\overline{f} \in \delta^-(Kx)} D(\overline{f}),$$

where $\delta^+(Kx)$ is the outgoing arc set of Kx and $\delta^-(Kx)$ is the incoming arc set of Kx. By Lemma 2.4(1), the divergence is independent of the choice of the orientation of internal edges in A.

If no loop is incident with a quotient vertex $Kx \in A$, then we can imagine that all edges incident to Kx are directed out of Kx. So we have

$$\operatorname{div}(Kx) = F_1(x) + F_2(x) + F_3(x).$$

Therefore, by Lemma 2.4(3), we see that

$$\operatorname{div}(Kx) = (0, 0).$$

Otherwise suppose that Kx has a loop, say an incident quotient edge of label i. Then $m_i(x) = \tau(x)$ for some $\tau \in K$, and hence

$$F_i(x) = F_i(\tau(x)) = F_i(m_i(x)) = -F_i(x).$$

Since p is odd, we have $F_i(x) = (0, 0)$. Note that a loop makes zero net contribution to the divergence, while every incident non-loop edge of label j contributes $F_j(x)$. Together with Lemma 2.4(3), this shows that

$$\operatorname{div}(Kx) = (0, 0).$$

Note that for any quotient vertex $Kx \in A$, we have proved that $\operatorname{div}(Kx) = (0, 0)$ always holds.

Noticing that every quotient edge in $E(A) \cup \{\overline{e}\}$ has both endpoints in A, except for $\overline{e}$, summing the divergence over all vertices in A and utilizing the cancellation of the contributions from two endpoints of the same edge, we can deduce that

$$(0,0) = \sum_{Kx \in A} \operatorname{div}(Kx) = D(\overline{e}).$$

Assume that $\overline{e}$ has initial endpoint $Kv \in A$ and label i. Then $\{v, m_i(v)\}$ is a non-loop in G_p for otherwise $\overline{e}$ would not be a bridge in $\overline{G}_p$. Thus, we have

$$(0,0) = D(\overline{e}) = F_i(v) \neq (0,0)$$

by Lemma 2.4(4). This is a contradiction. Hence A contains a cusp. The same argument shows that B also contains a cusp. This finishes the proof of Lemma 2.3. □

3. Proof of the main results

Now we are ready to prove our main results.

Proof of Theorem 1.1. Let $e = \{x, y\}$ be an arbitrary non-loop edge of G_p. It suffices to show that e is contained in a cycle of G_p. We denote by $\overline{e}$ the quotient edge connecting Kx and Ky in $\overline{G}_p$.

Case 1: $\overline{e}$ is a non-loop non-bridge in $\overline{G}_p$.

Since $\overline{e}$ is not a bridge, there exists a path $\overline{P}$ in $\overline{G}_p - \overline{e}$ from Kx to Ky. Lift $\overline{P}$ to a path P in G_p from x to $\sigma(y)$ for some $\sigma \in K$. Note that $e \notin P$ because $\overline{e} \notin \overline{P}$. If $\sigma = 1$, then e lies in a simple cycle $P \cup \{e\}$ in G_p. If $\sigma \neq 1$, then we have a closed walk

$$y \xrightarrow{\quad e \quad} x \xrightarrow{\quad P \quad} \sigma(y) \xrightarrow{\quad \sigma(e) \quad} \sigma(x) \xrightarrow{\quad \sigma(P) \quad} \sigma^2(y) = y.$$

Because $\sigma \neq 1$, we have $\sigma(e) \neq e$ (for otherwise σ would swap the endpoints of e; then $\overline{e}$ would be a loop in $\overline{G}_p$, which contradicts our assumption). The resulting closed walk traverses e exactly once. Hence it contains a simple cycle containing e.

Case 2: $\overline{e}$ is a bridge in $\overline{G}_p$.

Let Q be the connected component containing $\overline{e}$ in $\overline{G}_p$. Suppose that

$$Q - \overline{e} = A \sqcup B.$$

Without loss of generality, assume that $Kx \in A$ and $Ky \in B$. Applying Lemma 2.3, both A and B contain a cusp. Hence, by Lemma 2.2, the subgraphs of G_p induced by $\pi^{-1}(V(A))$ and $\pi^{-1}(V(B))$, denoted by $G_p[\pi^{-1}(V(A))]$ and $G_p[\pi^{-1}(V(B))]$, are connected. Because $\overline{e}$ is a bridge in $\overline{G}_p$, the four edges in $\{\sigma(e) : \sigma \in K\}$ are distinct, and all join $G_p[\pi^{-1}(V(A))]$ with $G_p[\pi^{-1}(V(B))]$. Choose a non-identity automorphism $\sigma \in K$. Then the edge $\sigma(e)$ has endpoints $\sigma(x) \in \pi^{-1}(V(A))$ and $\sigma(y) \in \pi^{-1}(V(B))$. Then there exists a path P_A (resp. P_B) connecting x with $\sigma(x)$ (resp. y with $\sigma(y)$) in the connected induced subgraph $G_p[\pi^{-1}(V(A))]$ (resp. $G_p[\pi^{-1}(V(B))]$). Hence e lies on the simple cycle formed by

$$x \xrightarrow{\quad e \quad} y \xrightarrow{\quad P_B \quad} \sigma(y) \xrightarrow{\quad \sigma(e) \quad} \sigma(x) \xrightarrow{\quad P_A \quad} x.$$

Thus we establish the statement in this case.

Case 3: $\overline{e}$ is a loop in $\overline{G}_p$.

Because $\overline{e} = \{Kx, Ky\}$ is a loop, we have $Kx = Ky$. By definition, there exists a Vieta involution m_i for some $i \in \{1, 2, 3\}$ and $\sigma \in K$ such that

$$m_i(x) = y = \sigma(x).$$

If $\sigma = 1$, then e is a loop in G_p, which contradicts our choice of $e = \{x, \sigma(x)\}$. Thus we have $\sigma \neq 1$.

Without loss of generality, we assume that $i = 1$. Clearly $\sigma \neq \sigma_1$ for otherwise, comparing coordinates gives $x_2 = x_3 = 0$, which forces $x = (0, 0, 0)$. This is impossible. If $\sigma = \sigma_2$, then $x_3 = 0$; if $\sigma = \sigma_3$, then $x_2 = 0$. So in both cases, Kx is a cusp in $\overline{G}_p$. Using Lemma 2.1, we see that e lies on the 4-cycle in G_p. □

We can easily deduce Corollary 1.2 from Theorem 1.1 together with the sub-cubic degree bound.

Proof of Corollary 1.2. By [5], we have

$$|X_p| = \begin{cases} p^2 + 3p & \text{if } p \equiv 1 \pmod 4, \\ p^2 - 3p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

It follows that $|X_p| \geq 3$ for $p \geq 5$. It remains to show that G_p has no cut vertices. Suppose to the contrary that G_p has a cut vertex x. There are at least two connected components in $G_p - x$, and each component has to be joined to x by at least two edges by Theorem 1.1. Therefore, we have $\deg(x) \geq 4$, which contradicts the sub-cubic degree bound

$$\max_{x \in G_p} \deg(x) = 3,$$

with a loop counted only once. This finishes the proof. □

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Declaration on the Use of AI

During the preparation of this work, the authors used AI systems to assist in generating candidate proof strategies, particularly in identifying the functions used in the proof of Lemma 2.3. All AI-generated suggestions were verified and refined by the authors, who take full responsibility for the correctness and originality of the paper.

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