Negligible obstructions and Turán exponents

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Abstract

We show that for every rational number $r \in (1,2)$ of the form $2 - a/b$, where $a, b \in \mathbb{N}^+$ satisfy $\lfloor a/b \rfloor^3 \le a \le b/(\lfloor b/a \rfloor + 1) + 1$, there exists a graph F_r such that the Turán number $\operatorname{ex}(n, F_r) = \Theta(n^r)$. Our result in particular generates infinitely many new Turán exponents. As a byproduct, we formulate a framework that is taking shape in recent work on the Bukh–Conlon conjecture.

1 Introduction

Given a family $\mathcal{F}$ of graphs, the Turán number $\operatorname{ex}(n, \mathcal{F})$ is defined to be the maximum number of edges in a graph on n vertices that contains no graph from the family $\mathcal{F}$ as a subgraph. The classical Erdős–Stone–Simonovits theorem shows that arguably the most interesting problems about Turán numbers, known as the degenerate extremal graph problems, are to determine the order of magnitude of $\operatorname{ex}(n, \mathcal{F})$ when $\mathcal{F}$ contains a bipartite graph. The following conjecture attributed to Erdős and Simonovits is central to Degenerate Extremal Graph Theory (see [15, Conjecture 1.6]).

Conjecture 1 (Rational Exponents Conjecture)**.** *For every finite family $\mathcal{F}$ of graphs, if $\mathcal{F}$ contains a bipartite graph, then there exists a rational $r \in [1,2)$ and a positive constant c such that* $\operatorname{ex}(n, \mathcal{F}) = cn^r + o(n^r)$.

Recently Bukh and Conlon made a breakthrough on the inverse problem [15, Conjecture 2.37].

Theorem 2 (Bukh and Conlon [3])**.** *For every rational number $r \in (1,2)$, there exists a finite family of graphs $\mathcal{F}_r$ such that* $\operatorname{ex}(n, \mathcal{F}_r) = \Theta(n^r)$.

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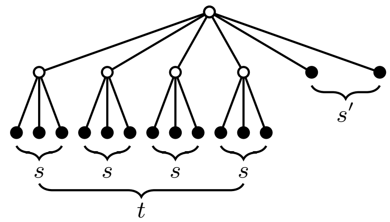

Figure 1: $T_{s,t,s'}$ with roots in black.

Motivated by another outstanding problem of Erdős and Simonovits (see [9, Section III] and [10, Problem 8]), subsequent work has been focused on the following conjecture, which aims to narrow the family $\mathcal{F}_r$ in Theorem 2 down to a single graph.

Conjecture 3 (Realizability of Rational Exponents)**.** *For every rational number* $r \in (1, 2)$*, there exists a bipartite graph* F_r *such that* $\mathrm{ex}(n, F_r) = \Theta(n^r)$*.*[1]

It is believed that the graph F_r in Conjecture 3 could be taken from a specific yet rich family of graphs, for which we give the following definitions.

Definition 4. A *rooted graph* is a graph F equipped with a subset $R(F)$ of vertices, which we refer to as *roots*. We define the pth *power* of F, denoted F^p, by taking the disjoint union of p copies of F, and then identifying each root in $R(F)$, reducing multiple edges (if any) between the roots.

Definition 5. Given a rooted graph F, we define the *density* ρ_F of F to be $\frac{e(F)}{v(F)-|R(F)|}$, where $v(F)$ and $e(F)$ denote the number of vertices and respectively edges of F. We say that a rooted graph F is *balanced* if $\rho_F > 1$, and for every subset S of $V(F) \setminus R(F)$, the number of edges in F with at least one endpoint in S is at least $\rho_F|S|$.

Indeed the following result on Turán numbers, which follows immediately from [3, Lemma 1.2], establishes the lower bound in Conjecture 3 for some power of a balanced rooted tree.

Lemma 6. *For every balanced rooted tree* F*, there exists* $p \in \mathbb{N}^+$ *such that* $\mathrm{ex}(n, F^p) = \Omega(n^{2-1/\rho_F})$*.*[2]

It is conjectured in [3] that the lower bound in Lemma 6 can be matched up to a constant factor.

Conjecture 7 (The Bukh–Conlon Conjecture)**.** *For every balanced rooted tree* F *and every* $p \in \mathbb{N}^+$*,* $\mathrm{ex}(n, F^p) = O(n^{2-1/\rho_F})$*.*

[1]Erdős and Simonovits asked a much stronger question: for every rational number $r \in (1, 2)$, find a bipartite graph F_r such that $\mathrm{ex}(n, F_r) = cn^r + o(n^r)$ for some positive constant c.

[2]A rooted tree is simply a rooted graph that is also a tree, not to be confused with a tree having a designated vertex.

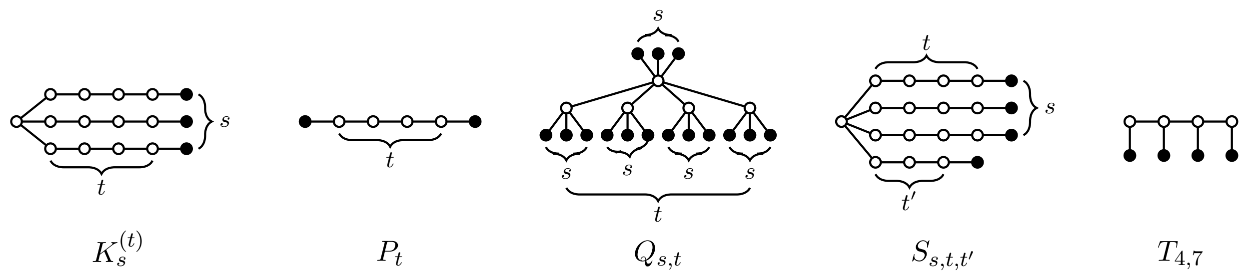

Figure 2: Balanced rooted trees, where s, t, t' refer to vertices, except t in $Q_{s,t}$.

Given the fact that every rational number bigger than 1 indeed appears as the density of some balanced rooted tree (see [3, Lemma 1.3]), Lemma 6 and Conjecture 7 would imply Conjecture 3.

Our main result establishes Conjecture 7 for certain balanced rooted trees $T_{s,t,s'}$ defined in Figure 1.

Theorem 8. *For every $s, t \in \mathbb{N}^+$ and $s' \in \mathbb{N}$, assume $t \geq s^3 - 1$ when $s - s' \geq 2$. If the rooted tree $F := T_{s,t,s'}$ is balanced, then for every $p \in \mathbb{N}^+$, $\mathrm{ex}(n, F^p) = O(n^{2-1/\rho_F})$, where $\rho_F = (st + t + s')/(t + 1)$.*

It is not hard to characterize the parameters s, t, s' for which $T_{s,t,s'}$ is balanced.

Proposition 9. *For every $s, t \in \mathbb{N}^+$ and $s' \in \mathbb{N}$, the rooted tree $F = T_{s,t,s'}$ is balanced if and only if $\rho_F \geq \max(s, s')$ and $\rho_F > 1$, or equivalently $s' - 1 \leq s \leq t + s'$ and $(t, s') \neq (1, 0)$.* □

Prior to our work, Conjecture 7 has been verified for the balanced rooted trees in Figure 2: the $K_s^{(0)}$ and P_t cases are classical results due to Kővári, Sós and Turán [22], and respectively Faudree and Simonovits [12]; $Q_{s,1}$ and $S_{2,1,0}$ are due to Jiang, Ma and Yepremyan [17]; $Q_{s,t}$ and $T_{4,7}$ are due to Kang, Kim and Liu [21]; $K_s^{(1)}$ and $S_{s,t,0}$ are due to Conlon, Janzer and Lee [5]; $K_s^{(2)}$ and $K_s^{(3)}$ are due to Jiang and Qiu [18]; $K_s^{(t)}$ is due to Janzer [16]; and $S_{s,t,t'}$ for all $t' \leq t$ is very recently settled by Jiang and Qiu [19].

These recent attacks on the Bukh–Conlon conjecture are full of interesting and promising techniques. In this paper, inspired by these previous attempts, we formulate an underlying framework that centers around a notion which we call negligible obstructions (Definitions 15 and 16). In this context, we develop a lemma (Lemma 17), which we call the negligibility lemma. To our best knowledge, ideas in our formulation of the framework can be traced back to the work of Conlon and Lee [6], and can be spotted throughout later work by various authors.

To establish an instance of the Bukh–Conlon conjecture, the negligibility lemma naturally leads to a two-step strategy: the identification of obstructions and the certification of their negligibility. By no means we claim that this strategy reduces the difficulty of Conjecture 7. Nevertheless we propose this strategy in hopes that it will bring us one step closer to pinning down a handful of essentially different techniques in this area, akin to the theory of flag algebras [23].

We illustrate the above two steps with the proof of Theorem 8. In contrast with all the previous work which has the inductive flavor of certifying negligibility of larger obstructions by that of the smaller, our implementation of the second step has a distinctive inductive pattern, which is elaborated at the end of Section 2. We point out that although Theorem 8 can be seen as an extension of [21, Section 3] which dealt with $Q_{s,t}$, our approach is quite different.

Turning to realizability of rational exponents, our main result Theorem 8 gives realizability of the following rational exponents.

Corollary 10. *For every rational number $r \in (1,2)$ of the form $2 - a/b$, where $a, b \in \mathbb{N}^+$, if*

$$\lfloor b/a \rfloor^3 \le a \le b/(\lfloor b/a \rfloor + 1) + 1, \tag{1}$$

then there exists a bipartite graph F_r such that $\mathrm{ex}(n, F_r) = \Theta(n^r)$.

Proof. In case $a = 1$, the assumption that $r > 1$ and (1) would contradict each other. Hereafter we assume that $a \ge 2$. Now take $s = \lfloor b/a \rfloor$, $t = a - 1$ and $s' = b - (a-1)(\lfloor b/a \rfloor + 1)$. Set $T = T_{s,t,s'}$. One can easily check that $s, t \in \mathbb{N}^+$, $\rho_T = (st + t + s')/(t+1) = b/a$ and so $\rho_T > 1, \rho_T \ge s$ and $s' \le b - (a-1)b/a = \rho_T$. Observe that (1) is equivalent to $t \ge s^3 - 1$ and $s' \ge 0$. In view of Proposition 9, T is balanced. The corollary follows from Lemma 6 and Theorem 8 immediately. □

As far as we know, all the rationals in $(1, 2)$ for which Conjecture 3 has been verified can be derived from Lemma 6 and the existing instances of Conjecture 7. For convenience, we say a fraction b/a is a *Bukh–Conlon density* if there exists a balanced rooted tree F such that $\rho_F = b/a$ and $\mathrm{ex}(n, F^p) = O(n^{2-1/\rho_F})$ for every $p \in \mathbb{N}^+$. Kang, Kim and Liu observed in [21, Lemma 4.3] that a graph densification operation due to Erdős and Simonovits [11] can be used to generate more Bukh–Conlon densities: whenever b/a is a Bukh–Conlon density, so is $m + b/a$ for every $m \in \mathbb{N}$.

It appears reasonable to restrict our attention to the fractions b/a of the form $m + s/a$ where $m \in \mathbb{N}^+$, for fixed $s, a \in \mathbb{N}$ with $s < a$. The results listed in Figure 2 yield Bukh–Conlon densities $m + s/a$ for every $m \in \mathbb{N}^+$ whenever $s\lceil (a-1)/(s+1) \rceil \le a - 1$.[3] For many choices of (s, a), for example $(4, 7)$, $(5, 8)$ or $(7, 10)$, it was not known whether $m + s/a$ is a Bukh–Conlon density for any $m \in \mathbb{N}^+$. For comparison, the family of fractions b/a given by (1) generates the Bukh–Conlon densities $m + s/a$ for all $m \ge a - s - 1$ whenever $a - 1 - \sqrt[3]{a} \le s \le a - 1$. In particular, our result gives new Bukh–Conlon densities of the form $m + 5/8$ and $m + 7/10$ as long as $m \ge 2$. Unfortunately our result does not give any Bukh–Conlon densities of the form $m + 4/7$. In view of the above discussion, we propose the following weaker conjecture about Bukh–Conlon densities.

Conjecture 11. *For every $s, a \in \mathbb{N}$ with $s < a$, there exists $m \in \mathbb{N}^+$ such that $m + s/a$ is a Bukh–Conlon density.*

[3] Combining [21, Lemma 4.3] with the results listed in Figure 2 (essentially with the one on $S_{s,t,t'}$), we know that $m + s/(st + t' + 1)$ is a Bukh–Conlon density for $m, s \in \mathbb{N}^+$ and $t, t' \in \mathbb{N}$ with $t' \le t$. For $m + s/a$ to be a fraction of such form, one needs $st + 1 \le a \le st + t + 1$ for some $t \in \mathbb{N}$, or equivalently $s\lceil (a-1)/(s+1) \rceil \le a - 1$.

We point out that one would settle the above conjecture if one could remove the technical condition $t \geq s^3 - 1$ when $s - s' \geq 2$ in Theorem 8.

The rest of the paper is organized as follows. In Section 2 we flesh out the aforementioned framework, and use it to prove Theorem 8. In Section 3 we prove the negligibility lemma that connects negligible obstructions with the Bukh–Conlon conjecture. In Sections 4 and 5 we certify the negligibility of two different obstructions needed for the proof of Theorem 8.

2 Negligible obstruction family

Throughout the rest of the paper, when we view a tree F as a rooted tree, by default the root set $R(F)$ of F consists exactly of the leaves of F. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G respectively.

To motivate the relevant concepts, it is instructive to think about finding a copy of F^p in an n-vertex d-regular graph G, where F is a tree and $d = \omega(n^{1-1/\rho_F})$. We mostly talk about embeddings rather than subgraphs.

Definition 12 (Embedding)**.** Given a tree F and a graph G, denote $\mathrm{Inj}(F, G)$ the set of *embeddings* from F to G, that is, the set of injections $\eta\colon V(F) \to V(G)$ such that $\eta(e) \in E(G)$ for every $e \in E(F)$. For a subset U of $R(F)$ and an injection $\sigma\colon U \to V(G)$, denote the set of embeddings from F to G relativized to σ by

$$\mathrm{Inj}(F, G; \sigma) = \{\eta \in \mathrm{Inj}(F, G)\colon \eta(u) = \sigma(u) \text{ for every } u \in U\}.$$

When we write these operators (and the ones coming later) in lowercase, we refer to their cardinalities, for example, $\mathrm{inj}(F, G) = |\mathrm{Inj}(F, G)|$ and $\mathrm{inj}(F, G; \sigma) = |\mathrm{Inj}(F, G; \sigma)|$.

Remark. We encourage the readers who are accustomed to counting subgraphs to think interchangeably the embedding counting $\mathrm{inj}(F, G)$ and the corresponding subgraph counting of F in G, as they only differ by a multiplicative factor depending only on F. We choose embeddings over subgraphs based on the pragmatic reason that it is more succinct to write in the language of embeddings when counting relativized to some injection σ.

Note that $\mathrm{inj}(F, G) \geq \Omega(nd^{e(F)})$ as one can embed F into G one vertex at a time. Because $nd^{e(F)} = \omega(n^{1+e(F)(1-1/\rho_F)}) = \omega(n^{1+e(F)-v(F)+|R(F)|}) = \omega(n^{|R(F)|})$, by the pigeonhole principle, there exists $\sigma\colon R(F) \to V(G)$ such that $\mathrm{inj}(F, G; \sigma) = \omega(1)$. Ideally the images of $V(F) \setminus R(F)$ under some p embeddings in $\mathrm{Inj}(F, G; \sigma)$ are pairwise (vertex) disjoint, and thus such p embeddings would give us a copy of F^p in G. To that end, we define the following notion.

Definition 13 (Ample embedding)**.** Given a tree F and a graph G, for $\eta \in \mathrm{Inj}(F, G)$, we say η is C-*ample* if there exist $\eta_1, \ldots, \eta_C \in \mathrm{Inj}(F, G)$ such that η_i and η are identical on $R(F)$, and the images of $V(F) \setminus R(F)$ under $\eta_1, \ldots, \eta_C$ are pairwise disjoint. Given $C \in \mathbb{N}$, denote $\mathrm{Amp}_C(F, G)$ the set

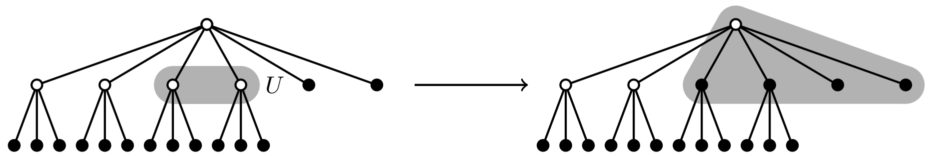

Figure 3: After adding U to the root set of $T_{3,4,2}$, the resulting rooted graph contains $K_{1,4}$ as a rooted subgraph.

of C-ample embeddings from F to G. For a subset U of $R(F)$ and an injection $\sigma\colon U \to V(G)$, the relativized version of $\mathrm{Amp}_C(F, G)$, denoted by $\mathrm{Amp}_C(F, G; \sigma)$, is just $\mathrm{Amp}_C(F, G) \cap \mathrm{Inj}(F, G; \sigma)$.

However it could happen that many embeddings in $\mathrm{Inj}(F, G; \sigma)$ map a nonempty subset of $V(F) \setminus R(F)$ in the same way, thus preventing us from finding a p-ample embedding in $\mathrm{Inj}(F, G; \sigma)$. These possible obstructions are encapsulated in the following definitions.

Definition 14 (Rooted subgraph)**.** Given two rooted graphs F_1 and F_2, we say that F_2 contains F_1 as a *rooted subgraph* if there exists an embedding η from F_1 to F_2 such that for every $v \in V(F_1)$, $\eta(v) \in R(F_2)$ if and only if $v \in R(F_1)$.

Definition 15 (Obstruction family)**.** Given a tree F, a family $\mathcal{F}_0$ of trees is an *obstruction family* for F if every member of $\mathcal{F}_0$ is isomorphic to a subtree of F that is not a single edge, and moreover for every nonempty proper subset U of $V(F)\setminus R(F)$, after adding U to the root set of F, the resulting rooted graph contains a member of $\mathcal{F}_0$ as a rooted subgraph. (See Figure 3 and Proposition 18 for a concrete example of an obstruction family.)

The following definition quantifies the conditions on the obstruction family for F that ensure the existence of a p-ample embedding of F in G.

Definition 16 (Negligible obstruction)**.** Given two trees F_0 and F, we say that F_0 is *negligible* for F if for every $p \in \mathbb{N}^+$ and $\varepsilon > 0$ there exist $c_0 > 0$ and $C_0 \in \mathbb{N}$ such that the following holds. For every $c > c_0$ and every n-vertex graph G with $n \geq n_0(c)$, if every vertex in G has degree between d and Kd, where $d = cn^{\alpha}$, $K = 5^{4/\alpha}$ and $\alpha = 1 - 1/\rho_F$, and moreover $\mathrm{amp}_p(F, G) = 0$, then $\mathrm{amp}_{C_0}(F_0, G) \leq \varepsilon n d^{e(F_0)}$. An obstruction family for F is negligible if every member of the family is negligible for F.

Remark. As we shall see later in Sections 4 and 5, when certifying the negligibility of an obstruction family, the concrete form of K is unimportant as long as it depends only on F. However, since we only need that specific K for Lemma 17 to work, we state it explicitly to avoid introducing an additional universal quantifier in Definition 16.

We wrap up the above discussion in the following lemma, and we postpone its proof to Section 3.

Lemma 17 (Negligibility lemma)**.** *Given a tree F, if there exists an negligible obstruction family $\mathcal{F}_0$ for F, then* $\mathrm{ex}(n, F^p) = O(n^{2-1/\rho_F})$ *for every* $p \in \mathbb{N}^+$.

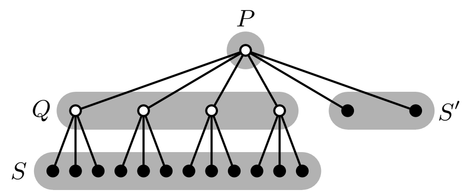

Figure 4: Vertex partition of $T_{s,t,s'}$.

The negligibility lemma provides us a two-step strategy to establish Conjecture 7 for a balanced rooted tree F: first identifying an obstruction family $\mathcal{F}_0$ for F, and second certifying the negligibility of $\mathcal{F}_0$. Although in the first step there might be multiple obstruction families for F, heuristically speaking it makes more sense to choose $\mathcal{F}_0$ that is minimal under inclusion, because certifying the negligibility of a member of $\mathcal{F}_0$ in the second step is where all the heavy lifting happens.

Coming back to the tree $T_{s,t,s'}$ defined in Figure 1, we choose the following obstruction family which is indeed minimal under inclusion.

Proposition 18. *The family* $\{K_{1,s+1}\} \cup \{T_{s,t-i,s'+i} \colon 1 \le i \le s-s'\}$ *is an obstruction family for* $T_{s,t,s'}$.

Proof. Let $F = T_{s,t,s'}$, and let U be a nonempty proper subset of $P \cup Q$, where $P = P(F)$ and $Q = Q(F)$ are vertex subsets of $V(F)$ defined as in Figure 4. Let F_+ be the rooted graph after adding U to the root set $R(F)$ of F. If U contains the vertex in P, then it is easy to see that, F_+ contains $K_{1,s+1}$ as a rooted subgraph. Otherwise $U \subseteq Q$. In this case, F_+ contains $T_{s,t-i,s'+i}$ as a rooted subgraph, where $i = |U|$. Finally notice that when $s'+i \ge s+1$, $T_{s,t-i,s'+i}$ contains $K_{1,s+1}$ as a rooted subgraph, and so does F_+ (see Figure 3 for an example). □

Theorem 8 follows immediately from the next theorem which certifies the negligibility of the obstruction family defined in Proposition 18 whenever $T_{s,t,s'}$ is balanced.

Theorem 19. *For $s, t \in \mathbb{N}^+$ and $s' \in \mathbb{N}$, suppose that $T := T_{s,t,s'}$ is balanced. When $s - s' \ge 2$, assume in addition that*

$$t \ge \left(1 - \tfrac{s'}{s+1}\right) k \left(k - \tfrac{1}{s}\right)(s+2-k) + \tfrac{1}{s}, \quad \text{for every } k \in \{2, \ldots, s-s'\}. \tag{2}$$

For every $p \in \mathbb{N}^+$ and $\varepsilon > 0$, there exists $c_0 > 0$ such that the following holds. For every $c > c_0$ and every n-vertex graph G with $n \ge n_0(c)$, if every vertex in G has degree between d and Kd, where $d = cn^\alpha$, $K = 5^{4/\alpha}$ and $\alpha = 1 - 1/\rho_T$, and moreover $\operatorname{amp}_p(T, G) = 0$, then

(a) $\operatorname{amp}_{C_*}(K_{1,s+1}, G) \le \varepsilon n d^{e(K_{1,s+1})}$, *where* $C_* = v(T^p_{s,t,s+1})$*; and*

(b) $\operatorname{amp}_{C_i}(F_i, G) \le \varepsilon n d^{e(F_i)}$, *where* $C_i = pv(T)^i$ *and* $F_i = T_{s,t-i,s'+i}$, *for* $1 \le i \le s - s'$.

Proof of Theorem 8. Suppose that $T := T_{s,t,s'}$ is balanced. When $s \le s'$, the obstruction family for T consists of a single $K_{1,s+1}$, which by Theorem 19(a) is negligible for T. When $s - s' = 1$, in view of Theorem 19, the obstruction family $\mathcal{F}_0$ defined in Proposition 18 is also negligible. When $s - s' \ge 2$, $\mathcal{F}_0$ is only negligible provided (2). One can check that $t \ge s^3 - 1$ ensures that (2) holds. Indeed, the right hand side of (2) is at most $k^2(s + 2 - k) + 1/s$, which, by the inequality of arithmetic and geometric means, is at most $(2(s + 2)/3)^3/2 + 1/s$, which is at most $s^3 - 1$ for $s \ge 3$. One can check directly in the $s = 2$ case that the right hand side of (2) is at most 7. In any case, it then follows from Lemma 17 that $\mathrm{ex}(n, T^p) = O(n^{2-1/\rho_F})$ for all p. ☐

Our proof of Theorem 19 is inductive in nature. In Section 4 we first establish the negligibility of $K_{1,s+1}$ in Theorem 19(a). In Section 5 we deduce the negligibility of F_i in Theorem 19(b) from that of $K_{1,s+1}$ and F_{i-1}. The inductive pattern here is counterintuitive in the sense that the negligibility of F_i, which is a subgraph of F_{i-1}, comes after that of F_{i-1}.

3 Proof of the negligibility lemma

In Section 2, we analyze the special case where the graph G is regular. In the context of degenerate extremal graph theory, it is indeed standard to assume that G is almost regular. This idea due to Erdős and Simonovits first appeared in [11]. We shall use the following variant (see also [20, Proposition 2.7] for a similar result).

Lemma 20 (Theorem 12 of Bukh and Jiang [4], only in arXiv version)**.** *For every $c > 0$ and $\alpha \in (0, 1]$, there exists $\tilde{n}_0 \in \mathbb{N}$ such that the following holds. Every $\tilde{n}$-vertex graph with $\tilde{n} \ge \tilde{n}_0$ and at least $(6c/\alpha)\tilde{n}^{1+\alpha}$ edges contains an n-vertex subgraph G with $n \ge (6c/\alpha)\tilde{n}^{\alpha/2}$ such that every vertex in G has degree between cn^α and Kcn^α, where $K = 5^{4/\alpha}$.*

We now formalize the discussion in Section 2 on finding a copy of F^p in G.

Definition 21 (Extension)**.** Given two trees F_1, F_2 and a graph G, for $\eta_1 \in \mathrm{Inj}(F_1, G)$ and $\eta_2 \in \mathrm{Inj}(F_2, G)$, we say η_2 *extends* η_1 if $\eta_1 = \eta_2 \circ \eta_{12}$ for some embedding $\eta_{12} \in \mathrm{Inj}(F_1, F_2)$. Given $C \in \mathbb{N}$, denote

$$\mathrm{Ext}_C(F_1, F_2, G) = \{\eta \in \mathrm{Inj}(F_2, G) \colon \eta \text{ extends } \eta_1 \text{ for some } \eta_1 \in \mathrm{Amp}_C(F_1, G)\}.$$

Proof of Lemma 17. Suppose that F is a tree, $p \in \mathbb{N}^+$ and $\mathcal{F}_0$ is a negligible obstruction family for F. Let $c > 0$ be a constant to be determined later. We would like prove that $\mathrm{ex}(\tilde{n}, F^p) < (6/\alpha)c\tilde{n}^{1+\alpha}$ for all $\tilde{n} \ge \tilde{n}_0(c)$, where $\alpha = 1 - 1/\rho_F$. By Lemma 20, it suffices to prove that every n-vertex graph G with $n \ge n_0(c)$, if every vertex in G has degree between cn^α and Kcn^α, where $K = 5^{4/\alpha}$, then G contains F^p as a subgraph.

Suppose that G is an n-vertex graph with $n \ge n_0(c)$ whose degrees are between d and Kd, where $d = cn^\alpha$. For the sake of contradiction, we assume that $\mathrm{amp}_p(F, G) = 0$. With hindsight, take

$$\varepsilon = \frac{K^{-e(F)}}{3\sum_{F_0\in\mathcal{F}_0} \operatorname{inj}(F_0, F)}.$$

Unwinding Definition 16, we obtain for every $F_0 \in \mathcal{F}_0$ two constants $c_{F_0} > 0$ and $C_{F_0} \in \mathbb{N}$. If we had chosen $c \geq \max\{c_{F_0} \colon F_0 \in \mathcal{F}_0\}$, then for every $F_0 \in \mathcal{F}_0$, $\operatorname{amp}_{C_{F_0}}(F_0, G) \leq \varepsilon n d^{e(F_0)}$, and in particular, $\operatorname{amp}_{C_0}(F_0, G) \leq \varepsilon n d^{e(F_0)}$, where $C_0 = \max(\{C_{F_0} \colon F_0 \in \mathcal{F}_0\} \cup \{p\})$.

Consider the embeddings in

$$I := \operatorname{Inj}(F, G) \setminus \bigcup_{F_0\in\mathcal{F}_0} \operatorname{Ext}_{C_0}(F_0, F, G). \tag{3}$$

Clearly $\operatorname{inj}(F, G) \geq (1 - o(1))nd^{e(F)}$, and moreover for every $F_0 \in \mathcal{F}_0$,

$$\operatorname{ext}_{C_0}(F_0, F, G) \leq \operatorname{inj}(F_0, F) \operatorname{amp}_{C_0}(F, G)(Kd)^{e(F)-e(F_0)} \leq \varepsilon \operatorname{inj}(F_0, F)K^{e(F)}nd^{e(F)}.$$

We can estimate the cardinality of I by

$$|I| \geq (1 - o(1))\, nd^{e(F)} - \varepsilon \sum_{F_0\in\mathcal{F}_0} \operatorname{inj}(F_0, F)K^{e(F)}nd^{e(F)} = (2/3 - o(1))nd^{e(F)},$$

and so $|I| \geq nd^{e(F)}/2 = c^{e(F)}n^{|R(F)|}/2$ if we had chosen $n_0(c)$ large enough.

By the pigeonhole principle, the cardinality of $I_\sigma := I \cap \operatorname{Inj}(F, G; \sigma)$ is at least $c^{e(F)}/2$ for some $\sigma \colon R(F) \to V(G)$. For every $U \subseteq V(F) \setminus R(F)$ and every injection $\tau \colon U \to V(G)$, set

$$I_\sigma(\tau) = \{\eta \in I_\sigma \colon \eta(u) = \tau(u) \text{ for every } u \in U\}.$$

Claim. For every $U \subseteq V(F) \setminus R(F)$ and $\tau \colon U \to V(G)$,

$$|I_\sigma(\tau)| \leq (C_0 v(F)^2)^{v(F)-|R(F)|-|U|}.$$

Proof of Claim. We prove by backward induction on $|U|$. Clearly $|I_\sigma(\tau)| \leq 1$ when the domain U of τ equals $V(F) \setminus R(F)$. Suppose U is a proper subset of $V(F) \setminus R(F)$. Recall from Definition 15 that after adding U to the root set of F, the resulting rooted graph contains a rooted subgraph F_0 that is isomorphic to a member of $\mathcal{F}_0$. Notice that $U_0 := V(F_0) \setminus R(F_0)$ is nonempty because F_0 is not a single edge.

Let $I'_\sigma(\tau)$ be a maximal subset of $I_\sigma(\tau)$ such that the images of U_0 under the embeddings in $I'_\sigma(\tau)$ are pairwise disjoint, and let V_0 be the union of these images. Since $I_\sigma(\tau) \subseteq I$ and I defined by (3) contains no extension of any C_0-ample embedding from F_0 to G, we bound $|I'_\sigma(\tau)| < C_0$, which implies that $|V_0| < C_0|U_0|$. For each $u \in U_0$ and $v \in V_0$, by the inductive hypothesis

$$|I_\sigma(\tau_{uv})| < (C_0 v(F)^2)^{v(F)-|R(F)|-|U|-1},$$

where $\tau_{uv}\colon U\cup\{u\}\to V(G)$ extends τ by mapping u to v additionally. The maximality of $I'_\sigma(\tau)$ means that for every $\eta\in I_\sigma(\tau)$ there is $u\in U_0$ such that $\eta(u)\in V_0$, and so $\eta\in I_\sigma(\tau_{uv})$ for some $v\in V_0$. Therefore

$$|I_\sigma(\tau)|\le\sum_{u\in U_0,v\in V_0}|I_\sigma(\tau_{uv})|<|U_0||V_0|(C_0v(F)^2)^{v(F)-|R(F)|-|U|-1},$$

which implies the inductive step as $|U_0|<v(F)$ and $|V_0|<C_0|U_0|$.

The same argument works for the last inductive step where $U=\varnothing$ because there is no p-ample embedding from F to G, and $C_0\ge p$. ⊡

In particular, $I_\sigma=I_\sigma(\tau)$ when the domain of τ is empty, and so $|I_\sigma|\le(C_0v(F)^2)^{v(F)-|R(F)|}$, which would yield a contradiction if we had chosen $c>(2(C_0v(F)^2)^{v(F)-|R(F)|})^{1/e(F)}$. □

4 Ample embeddings of stars

The negligibility of $K_{1,s+1}$ for $T_{s,t,s'}$ is established directly through the following technical lemma.

Lemma 22. *For $s,t\in\mathbb{N}^+$ and $s'\in\mathbb{N}$, set $s_0=\max(s',1)$, $F_0=K_{1,s_0}$, $F_1=K_{1,s+1}$ and $T=T_{s,t,s'}$. For every $p\in\mathbb{N}^+$ and $\varepsilon>0$, there exists $c_0>0$ such that for every n-vertex graph G, if $\mathrm{amp}_p(T,G)=0$ and $\mathrm{inj}(F_0,G)\ge c_0n^{s_0}$, then $\mathrm{amp}_{C_1}(F_1,G)\le\varepsilon\,\mathrm{inj}(F_1,G)$, where $C_1=v(T^p_{s,t,s_0})$.*

Our proof of Lemma 22 follows the outline of [5, Lemma 5.3]. Over there the conclusion, in our language, is that for every $\varepsilon>0$ there exists $C_1\in\mathbb{N}$ such that $\mathrm{amp}_{C_1}(F_1,G)\le\varepsilon\,\mathrm{inj}(F_1,G)$. One can work out the quantitative dependency $\varepsilon=\Theta(C_1^{1-s})$ from their argument. Although this dependency alone is enough for the negligibility of $K_{1,s+1}$, it becomes inadequate when we iteratively apply this bound later in Section 5. To decouple ε from C_1 in Lemma 22, we need the following classical result in degenerate extremal hypergraph theory.

Theorem 23 (Erdős [8])**.** *For every r-partite r-uniform hypergraph H there exists $\varepsilon>0$ so that* $\mathrm{ex}(n,H)=O(n^{r-\varepsilon})$.[4] □

Proof of Lemma 22. Let $C_1\in\mathbb{N}$ be at least $v(T^p_{s,t,s_0})$. Suppose that G is an n-vertex graph with $\mathrm{amp}_p(T,G)=0$ and $\mathrm{inj}(F_0,G)\ge c_0n^{s_0}$, where c_0 is to be chosen. As we only deal with embeddings to G in the following proof, we omit G in $\mathrm{Inj}(\cdot,G),\mathrm{Amp}_\cdot(\cdot,G)$ and $\mathrm{Amp}_\cdot(\cdot,G;\cdot)$.

Recall $s_0=\max(s',1)$. Clearly G contains no F^p as a subgraph, where $F=T_{s,t,s_0}$. Let U_0 denote an arbitrary vertex subset of size s_0 in G, and denote $N_G(U_0)$ the common neighborhood of U_0 in G. Let H be the $(s+1)$-uniform hypergraph on $V(G)$ given by

$$H=\{\eta(R(F_1))\colon\eta\in\mathrm{Amp}_{C_1}(F_1)\}.$$

[4]Given an r-uniform hypergraph H, the Turán number $\mathrm{ex}(n,H)$ is the maximum number of hyperedges in an r-uniform hypergraph on n vertices that contains no H as a subhypergraph.

The strategy is to use $\sum_{U_0} e(H[N_G(U_0)])$ and $\sum_{U_0} \binom{|N_G(U_0)|}{s+1}$ as intermediaries to connect $\mathrm{amp}_{C_1}(F_1)$ and $\mathrm{inj}(F_1)$, where $H[N_G(U_0)]$ is the subhypergraph of H induced on $N_G(U_0)$.

Claim 1. There exists $n_0 = n_0(s,t,p,C_1) \in \mathbb{N}$ such that for every U_0 with $|N_G(U_0)| \geq n_0$,

$$e(H[N_G(U_0)]) \leq \frac{\varepsilon}{4s_0^{s_0}} \binom{|N_G(U_0)|}{s+1}.$$

Proof of Claim 1. Recall the vertex partition $V(F) = P(F) \cup Q(F) \cup S(F) \cup S'(F)$ from Figure 4. This partition induces the vertex partition $V(F^p) = P(F^p) \cup Q(F^p) \cup S(F) \cup S'(F)$, where $P(F^p)$ denotes the union of the p disjoint copies of $P(F)$ in F^p, and $Q(F^p)$ is defined similarly.

Let H_0 be the $(s+1)$-uniform hypergraph on $P(F^p) \cup S(F)$ with each hyperedge given by the $s+1$ neighbors of a vertex of $Q(F^p)$ in F^p.

Observe that $H[N_G(U_0)]$ never contains H_0 as a subhypergraph. Suppose on the contrary that there exists an embedding η from H_0 to $H[N_G(U_0)]$,[5] then we can embed F^p in G by mapping $S'(F)$ to U_0, mapping $P(F^p) \cup S(F)$ according to η, and embedding the vertices in $Q(F^p)$ greedily. The last step of the embedding is possible because for every hyperedge $e \in H_0$, $\eta(e) = \eta'(R(F_1))$ for some $\eta' \in \mathrm{Amp}_{C_1}(F_1)$, and more importantly $C_1 \geq v(F^p)$.

Since H_0 is an $(s+1)$-partite hypergraph, the claim follows from Theorem 23 immediately. ⊡

We choose such $n_0 \in \mathbb{N}$ in Claim 1 and require in addition that $n_0 \geq s+1$. For convenience, set

$$\mathcal{U} = \{U_0 \subseteq V(G) \colon |U_0| = s_0, |N_G(U_0)| \geq n_0\}.$$

Claim 2. The number of C_1-ample embeddings from F_1 to G satisfies

$$\mathrm{amp}_{C_1}(F_1) \leq \frac{s_0^{s_0}(s+1)!}{C_1^{s_0-1}} \sum_{U_0} e(H[N_G(U_0)]).$$

Proof of Claim 2. Let σ denote an arbitrary injection from $R(F_1)$ to $V(G)$, and denote for short $a(\sigma) = \mathrm{amp}_{C_1}(F_1;\sigma)$. Note that $a(\sigma)$ has the dichotomy that either $a(\sigma) = 0$ or $a(\sigma) \geq C_1 \geq s_0$, which implies that $\binom{a(\sigma)}{s_0} \geq (a(\sigma)/s_0)^{s_0} \geq C_1^{s_0-1} a(\sigma)/s_0^{s_0}$ in either case. Through counting the disjoint union $\bigsqcup_{U_0} H[N_G(U_0)]$ in two ways, one can show that

$$(s+1)! \sum_{U_0} e(H[N_G(U_0)]) = \sum_{\sigma} \binom{a(\sigma)}{s_0} \geq \frac{C_1^{s_0-1}}{s_0^{s_0}} \sum_{\sigma} a(\sigma) = \frac{C_1^{s_0-1}}{s_0^{s_0}} \mathrm{amp}_{C_1}(F),$$

which implies the desired inequality in the claim. ⊡

Claim 3. The number of embeddings from F_1 to G satisfies

$$\mathrm{inj}(F_1) \geq \frac{(s+1)!}{2C_1^{s_0}} \sum_{U_0 \in \mathcal{U}} \binom{|N_G(U_0)|}{s+1}.$$

[5]Given two hypergraphs H_1 and H_2 of the same uniformity, an embedding from H_1 to H_2 is just an injection $\eta \colon V(H_1) \to V(H_2)$ such that $\eta(e) \in H_2$ for every $e \in H_1$.

Proof of Claim 3. We count in two ways the disjoint union $\bigsqcup_{U_0 \in \mathcal{U}} I(U_0)$, where

$$I(U_0) := \{\eta \in \mathrm{Inj}(F_1) \setminus \mathrm{Amp}_{C_1}(F_1) \colon \eta(R(F_1)) \subseteq N_G(U_0)\}.$$

On the one hand, for a fixed U_0 with $|N_G(U_0)| \geq n_0$, every subset of $N_G(U_0)$ of size $s+1$ that is not an hyperedge of $H[N_G(U_0)]$ gives rise to at least $s_0(s+1)!$ many $\eta \in I(U_0)$, and it follows form Claim 1 that $e(H[N_G(U_0)]) \leq \frac{1}{2}\binom{|N_G(U_0)|}{s+1}$. Thus we get

$$|I(U_0)| \geq \frac{s_0(s+1)!}{2}\binom{|N_G(U_0)|}{s+1}, \quad \text{for every } U_0 \in \mathcal{U}.$$

On the other hand, for every $\eta \in \mathrm{Inj}(F_1) \setminus \mathrm{Amp}_{C_1}(F_1)$, there are at most $\binom{C_1}{s_0}$ many U_0 such that $\eta(R(F_1)) \subseteq N_G(U_0)$, hence

$$\mathrm{inj}(F_1) \geq \mathrm{inj}(F_1) - \mathrm{amp}_{C_1}(F_1) \geq \frac{1}{\binom{C_1}{s_0}} \sum_{U_0} |I(U_0)| \geq \frac{s_0!}{C_1^{s_0}} \sum_{U_0} |I(U_0)|,$$

which implies the desired inequality in the claim. ⊡

A simple double counting argument shows that

$$\mathrm{inj}(F_0) = s_0! \sum_{U_0} |N_G(U_0)|.$$

Recall the assumption that $\mathrm{inj}(F_0) \geq c_0 n^{s_0}$. Thus the average $\bar{N}$ of $|N_G(U_0)|$ satisfies

$$\bar{N} = \frac{\mathrm{inj}(F_0)}{s_0!\binom{n}{s_0}} \geq c_0.$$

We can choose $c_0 > 0$ large enough so that $\binom{\bar{N}}{s+1} \geq (1 + 4s_0^{s_0} C_1/\varepsilon)\binom{n_0}{s+1}$. By Jensen's inequality, we have

$$\sum_{U_0} \binom{|N_G(U_0)|}{s+1} \geq \binom{n}{s_0}\binom{\bar{N}}{s+1} \geq (1 + 4s_0^{s_0} C_1/\varepsilon) \sum_{U_0 \notin \mathcal{U}} \binom{|N_G(U_0)|}{s+1},$$

which implies that

$$\sum_{U_0 \notin \mathcal{U}} \binom{|N_G(U_0)|}{s+1} \leq \frac{\varepsilon}{4s_0^{s_0} C_1} \sum_{U_0 \in \mathcal{U}} \binom{|N_G(U_0)|}{s+1}.$$

Applying Claim 2 and then Claim 1, we get

$$\begin{aligned}\frac{C_1^{s_0-1}}{s_0^{s_0}(s+1)!}\mathrm{amp}_{C_1}(F_1) &\leq \sum_{U_0} e(H[N_G(U_0)]) \\ &\leq \sum_{U_0 \notin \mathcal{U}} \binom{|N_G(U_0)|}{s+1} + \frac{\varepsilon}{4s_0^{s_0} C_1} \sum_{U_0 \in \mathcal{U}} \binom{|N_G(U_0)|}{s+1} \leq \frac{\varepsilon}{2s_0^{s_0} C_1} \sum_{U_0 \in \mathcal{U}} \binom{|N_G(U_0)|}{s+1},\end{aligned}$$

which implies

$$\mathrm{amp}_{C_1}(F_1) \leq \frac{(s+1)!\varepsilon}{2C_1^{s_0}} \sum_{U_0 \in \mathcal{U}} \binom{|N_G(U_0)|}{s+1}.$$

Comparing it with Claim 3, we get the desired inequality in Lemma 22. □

Proof of Theorem 19(a). For $s, t \in \mathbb{N}^+$ and $s' \in \mathbb{N}$, set $s_0 = \max(s', 1)$, and $T = T_{s,t,s'}$. Since T is balanced, by Proposition 9, $s_0 \leq s + 1$ and $\rho_T \geq s_0$, the latter of which implies that $1 + s_0\alpha \geq s_0$, where $\alpha = 1 - 1/\rho_T$.

Let $p \in \mathbb{N}^+, C_* = v(T^p_{s,t,s+1}) \geq v(T^p_{s,t,s_0})$ and $\varepsilon > 0$, and let $c_0 > 0$ be a constant to be determined later. Suppose that $c > c_0$ and G is an n-vertex graph with $n \geq n_0(c)$ whose degrees are between d and Kd, where $d = cn^\alpha$ and $K = 5^{4/\alpha}$, and moreover $\text{amp}_p(T, G) = 0$. Clearly, $\text{inj}(K_{1,s+1}, G) \leq n(Kd)^s$. We apply Lemma 22 and obtain $c_1 > 0$ so that if $\text{inj}(K_{1,s_0}, G) \geq c_1 n^{s_0}$ then

$$\text{amp}_{C_*}(K_{1,s+1}, G) \leq \varepsilon \, \text{inj}(K_{1,s+1}, G) \leq \varepsilon n(Kd)^{s+1} = \varepsilon K^{s+1} n d^{e(K_{1,s+1})}.$$

Since $1 + s_0\alpha \geq s_0$, we have

$$\text{inj}(K_{1,s_0}, G) \geq (1 - o(1)) n d^{s_0} = (1 - o(1)) c^{s_0} n^{1+s_0\alpha} \geq (1 - o(1)) c^{s_0} n^{s_0}.$$

Thus the condition $\text{inj}(K_{1,s_0}, G) \geq c_1 n^{s_0}$ can be met by choosing $c_0 = c_1^{1/s_0}$ and $n_0(c)$ sufficiently large. ☐

5 Ample embeddings of subtrees

5.1 Preliminary propositions

For the proof of Theorem 19(b), we need the classical sunflower lemma due to Erdős and Rado [7] and its immediate consequence for sequences (see [2] for the recent breakthrough on the sunflower conjecture and related background).

Definition 24 (Sequential sunflower)**.** Suppose that $W \subseteq V^k$ is a system of sequences. A subset S of W is a *sequential sunflower* with *kernel* $I \subsetneq [k]$ if for every pair of distinct sequences $(s_1, \ldots, s_k), (s'_1, \ldots, s'_k) \in S$, the subsequences $(s_i)_{i \in I}$ and $(s'_i)_{i \in I}$ are equal, but the sets $\{s_i \colon i \notin I\}$ and $\{s'_i \colon i \notin I\}$ are disjoint.

Proposition 25. *Fix $k, C \in \mathbb{N}^+$. Suppose that $W \subseteq V^k$ is a system of sequences such that each sequence in W consists of k distinct elements. If W contains no sequential sunflower of size C, then $|W| < (k!)^2(k!C - 1)^k$.*

Proof. Consider the system F of subsets of V defined by

$$F = \{\{s_1, \ldots, s_k\} \colon (s_1, \ldots, s_k) \in W\}.$$

Clearly $|W| \leq k!|F|$. We claim that F contains no sunflower of size $k!C$. Recall that a sunflower is a collection of sets whose pairwise intersection is constant. Assuming the claim, the classical sunflower lemma precisely states that $|F| < k!(k!C - 1)^k$, which implies the desired inequality. Suppose on the contrary that $E \subseteq F$ is a sunflower of size $k!C$ with kernel K. Consider the subsystem of sequences $W_0 = \{(s_1, \ldots, s_k) \in W : \{s_1, \ldots, s_k\} \in E\}$. Clearly $|W_0| \geq k!C$. By the pigeonhole principle, there exist a set $W_1 \subseteq W_0$ of size C and $I \subsetneq [k]$ such that for every $s \in W_1$, $\{s_i : i \in I\} = K$ and $(s_i)_{i \in I}$ is a constant subsequence. As E is a sunflower, one can check that W_1 is a sequential sunflower of size C, which is a contradiction. □

We also need the following classical theorem due to Kővári, Sós and Turán [22] on the Zarankiewicz problem.

Proposition 26. *Fix $s, t \in \mathbb{N}^+$. Suppose that H is a bipartite graph with two parts U and W such that every vertex in W has degree at least s. If H contains no complete bipartite subgraph with s vertices in U and t vertices in W, then $e(H) \leq K|U||W|^{1-1/s}$, where $K = s\sqrt[s]{(t-1)/s!}$.* □

The following result is a generalization of a result due to Füredi [14]. Our proof of the generalization follows the proof of Füredi's result by Alon, Krivelevich, and Sudakov [1] using dependent random choice (see [13] for a survey on dependent random choice). We denote $d_G(v)$ the degree of a vertex v in G.

Proposition 27. *Fix $k, r \in \mathbb{N}^+$ such that $k < r$. Suppose that F is a bipartite graph with parts U_0, W_0 such that every vertex in W_0 has degree at most r in F. For every bipartite graph G with parts U, W, if there is no embedding $\eta \in \mathrm{Inj}(F, G)$ such that $\eta(U_0) \subseteq U$ and $\eta(W_0) \subseteq W$, then*

$$\sum_{u \in U} d_G(u)^k \leq \left(K_1|U|^k + K_2|W|^k\right)|U|^{1-k/r},$$

where $K_1 = |W_0|^k/(r!)^{k/r}$ and $K_2 = (|U_0| - 1)^{k/r}$.

Proof. Assume for the sake of contradiction that

$$\sum_{u \in U} d(u)^k > (r!)^{-k/r}|W_0|^k|U|^{k+1-k/r} + (|U_0| - 1)^{k/r}|U|^{1-k/r}|W|^k.$$

Pick a subset $W_1 \subseteq W$ of size r uniformly at random with repetition. Set $U(W_1) \subseteq U$ to be the common neighborhood of W_1 in G, and let X denote the cardinality of $U(W_1)$. By linearity of expectation and Hölder's inequality,

$$\begin{aligned} \mathrm{E}[X] = \sum_{u \in U} \left(\frac{d(u)}{|W|}\right)^r &\geq \frac{\left(\sum_{u \in U} d(u)^k\right)^{r/k}}{|U|^{r/k-1}|W|^r} \\ &> \frac{(r!)^{-1}|W_0|^r|U|^{r+r/k-1} + (|U_0| - 1)|U|^{r/k-1}|W|^r}{|U|^{r/k-1}|W|^r} \geq \frac{|U|^r}{r!}\left(\frac{|W_0|}{|W|}\right)^r + |U_0| - 1. \end{aligned}$$

Let Y denote the random variable counting the number of subsets $S \subseteq U(W_1)$ of size r with fewer than $|W_0|$ common neighbors in G. For a given such S, the probability that it is a subset of $U(W_1)$ is less than $(|W_0|/|W|)^r$. Since there are at most $\binom{|U|}{r}$ subsets S of size r, it follows that

$$\mathrm{E}[Y] < \binom{|U|}{r}\left(\frac{|W_0|}{|W|}\right)^r \leq \frac{|U|^r}{r!}\left(\frac{|W_0|}{|W|}\right)^r.$$

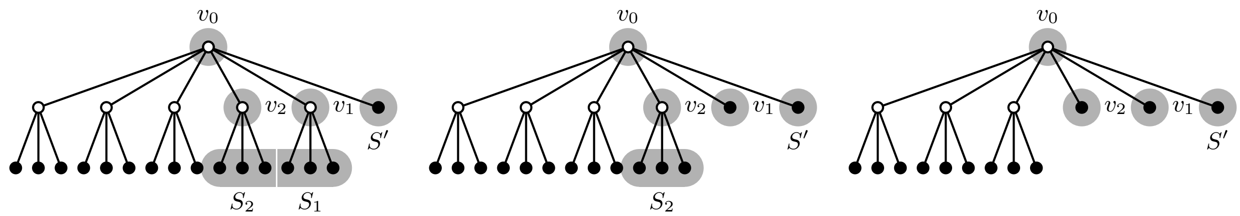

Figure 5: F_0, F_1 and F_2.

By linearity of expectation,

$$\mathrm{E}[X-Y] > \frac{|U|^r}{r!}\left(\frac{|W_0|}{|W|}\right)^r + |U_0| - 1 - \frac{|U|^r}{r!}\left(\frac{|W_0|}{|W|}\right)^r = |U_0| - 1.$$

Hence there exists a choice of W_1 for which $X - Y \geq |U_0|$. Delete one vertex from each subset S of $U(W_1)$ of size r with fewer than m common neighbors. We let U' be the remaining subset of $U(W_1)$. The set $U' \subseteq U$ has at least $|U_0|$ vertices, and every subset of U' of size r has at least $|W_0|$ common neighbors. One can then greedily find an embedding $\eta \in \mathrm{Inj}(F, G)$ such that $\eta(U_0) \subseteq U'$ and $\eta(W_0) \subseteq W$. □

5.2 Proof of Theorem 19(b)

We inductively deduce the negligibility of F_i by that of F_{i-1}, where $F_i = T_{s,t-i,s'+i}$. In each inductive step, we also need to set aside the embeddings from F_i to G that extend the ample embeddings from $K_{1,s+1}$ to G which were already dealt with in Lemma 22. Recall $\mathrm{Ext}_C(F_1, F_2, G)$ from Definition 21, and that $\mathrm{ext}_C(F_1, F_2, G)$ denotes its cardinality.

Lemma 28. *Fix $s, t, p, k \in \mathbb{N}^+$ and $s' \in \mathbb{N}$ such that $s' < s$, $k \leq s$ and $k < t$. Set $F_i = T_{s,t-i,s'+i}$ and $C_i = pv(F_0)^i$, for $0 \leq i \leq k$, and set $F_k^- = T_{s,t-k,s'}$ and $\alpha = 1 - 1/\rho_{F_0}$. When $k = 1$, assume that $\alpha \geq 1 - 1/s$; and when $k \geq 2$, assume that*

$$t \geq \left(1 - \tfrac{s'}{s+1}\right) k \left(k - \tfrac{1}{s}\right)(s + 2 - k) + \tfrac{1}{s} \tag{4}$$

For every $c > 1$, $C_* \in \mathbb{N}$ and n-vertex graph G, if every vertex in G has degree between d and Kd, where $d = cn^\alpha$ and $K = 5^{4/\alpha}$, and moreover $\mathrm{amp}_{C_0}(F_0, G) = 0$, then

$$\begin{aligned}\mathrm{amp}_{C_k}(F_k, G) &- \mathrm{ext}_{C_*}(K_{1,s+1}, F_k, G) \\ &\leq (K_* + o(1))\left(\tfrac{1}{c}\,\mathrm{inj}(F_k^-, G)d^k + \tfrac{1}{c} n d^{e(F_k)} + \sum_{i=1}^{k-1} d^{s(i-k)}\,\mathrm{amp}_{C_i}(F_i, G)\right),\end{aligned}$$

where $K_* = K_*(C_*, s, t, p, k, K)$ is a positive constant.

Proof. As we mostly deal with embeddings to G, we omit G in $\mathrm{Inj}(\cdot, G)$, $\mathrm{Amp}(\cdot, G)$ and $\mathrm{Ext}(\cdot,\cdot,G)$ and their relativized versions.

Let $v_0, v_1, \ldots, v_k$ be defined for $F_0, \ldots, F_k$ as in Figure 5, and let S_i be the set of roots which are adjacent to v_i for $i \in [k]$. We view F_i as a subtree of F_{i-1} induced on $V(F_{i-1}) \setminus S_i$. Let σ denote an arbitrary injection from $R(F_k) \setminus \{v_1, \ldots, v_k\}$ to $V(G)$, and set

$$\widetilde{A}_\sigma = \mathrm{Amp}_{C_k}(F_k; \sigma) \quad \text{and} \quad \widetilde{I}_\sigma^\times = \mathrm{Ext}_{C_*}(K_{1,s+1}, F_k) \cap \mathrm{Inj}(F_k; \sigma).$$

For short, denote $\vec{v} := (v_1, \ldots, v_k)$ and $\eta(\vec{v}) := (\eta(v_1), \ldots, \eta(v_k))$ for every $\eta \in \mathrm{Inj}(F_k)$. Let $\widetilde{H}_\sigma$ be the bipartite graph with two parts

$$\widetilde{U}_\sigma = \{\eta(v_0) \colon \eta \in \widetilde{A}_\sigma\} \quad \text{and} \quad \widetilde{W}_\sigma = \{\eta(\vec{v}) \colon \eta \in \widetilde{A}_\sigma\}$$

whose edge set is given by

$$\widetilde{H}_\sigma = \{(\eta(v_0), \eta(\vec{v})) \colon \eta \in \widetilde{A}_\sigma\}.$$

Claim 1. The size of $\widetilde{A}_\sigma$ is bounded by that of $\widetilde{H}_\sigma$ as follows:

$$|\widetilde{A}_\sigma| \le C_*^{t-k} |\widetilde{H}_\sigma| + |I_\sigma^\times|.$$

Proof of Claim 1. In view of the definition of $I_\sigma^\times$, $\widetilde{A}_\sigma \setminus I_\sigma^\times$ contains no extension of any C_*-ample embedding from $K_{1,s+1}$ to G. Therefore for every edge $(u, \vec{w})$ in $\widetilde{H}_\sigma$, there are at most C_*^{t-k} many $\eta \in \widetilde{A}_\sigma \setminus I_\sigma^\times$ with $(\eta(v_0), \eta(\vec{v})) = (u, \vec{w})$. ⊡

Sample a subset U_σ of $\widetilde{U}_\sigma$ of size m_0 chosen uniformly at random, where m_0 will be chosen later. We denote H_σ the bipartite subgraph H_σ of $\widetilde{H}_\sigma$ induced on $U_\sigma \cup \widetilde{W}_\sigma$, and we partition H_σ into H_σ^- and H_σ^+, where H_σ^- consists of edges $(u, \vec{w})$ in H_σ such that the degree of $\vec{w}$ is at most sk in H_σ, and H_σ^+ is the complement of H_σ^- in H_σ. We estimate the number of edges in H_σ^- and H_σ^+ in the following two claims respectively.

Claim 2. For every σ, the number of edges in H_σ^- satisfies

$$(1 - o(1)) d^{sk} |H_\sigma^-| \le K_0 n^{sk} + \sum_{i=1}^{k-1} K_i d^{si} |\{\eta \in \mathrm{Amp}_{C_i}(F_i; \sigma) \colon \eta(v_0) \in U_\sigma\}|,$$

where $K_0 = sk(k!)^2(k!C_{k-1} - 1)^k$, and $K_i = \binom{k}{i} K^{si}$ for $i \in [k-1]$.

Proof of Claim 2. For every edge $(u, \vec{w})$ in H_σ^-, we choose some $\eta \in \widetilde{A}_\sigma$ with $(u, \vec{w}) = (\eta(v_0), \eta(\vec{v}))$, and then this chosen η gives rise to $(1 - o(1)) d^{sk}$ many $\eta' \in \mathrm{Inj}(F_0; \sigma)$ such that $\eta' \supseteq \eta$ and $(u, \vec{w}) = (\eta'(v_0), \eta'(\vec{v}))$. We collect these η' in $B_\sigma \subseteq \mathrm{Inj}(F_0; \sigma)$ after going through all edges $(u, \vec{w})$ in H_σ^-. Note that

$$(1 - o(1)) d^{sk} |H_\sigma^-| \le |B_\sigma|, \tag{5}$$

and B_σ has the distinctness property in the sense that

$$\text{no two distinct embeddings in } B_\sigma \text{ are identical on } \{v_0, v_1, \dots, v_k\} \cup S_1 \cup \dots \cup S_k. \tag{6}$$

Let σ' denote an arbitrary injection from $R(F_0)$ to $V(G)$ such that $\sigma' \supseteq \sigma$, and define $B_{\sigma'} = B_\sigma \cap \mathrm{Inj}(F_0; \sigma')$. We claim that, for every $I \subsetneq [k]$, the cardinality of

$$B^I_{\sigma'} := \{\eta' \in B_{\sigma'} \colon \text{there exist } \eta' = \eta'_1, \eta'_2, \dots, \eta'_{C_i} \in B_{\sigma'} \text{ such that } \eta'_1(\vec{v}), \dots, \eta'_{C_i}(\vec{v}) \text{ form a sequential sunflower of size } C_i \text{ whose kernel is } I\}.$$

satisfies

$$\sum_{\sigma'} |B^I_{\sigma'}| \le (Kd)^{si} |\{\eta \in \mathrm{Amp}_{C_i}(F_i; \sigma) \colon \eta(v_0) \in U_\sigma\}|, \quad \text{where } i = |I|. \tag{7}$$

Indeed, without loss of generality, we may assume that $I = [k] \setminus [k-i]$ for some $i \in \{0, \dots, k-1\}$. By the definition of $B^I_{\sigma'}$, for every $\eta' \in B^I_{\sigma'}$, there exist $\eta' = \eta'_1, \eta'_2, \dots, \eta'_{C_i} \in B_{\sigma'}$ such that $\eta'_1(\vec{v}), \dots, \eta'_{C_i}(\vec{v})$ form a sequential sunflower of size C_i whose kernel $I \subsetneq [k]$ is of size i. For every $j \in [C_i]$, let η_j be the restriction of η'_j to $V(F_i)$. Unwinding the definition of a sequential sunflower, we know that $\eta_1, \dots, \eta_{C_i}$ are identical on $\{v_{k-i+1}, \dots, v_k\}$, and moreover the images of $\{v_1, \dots, v_{k-i}\}$ under $\eta_1, \dots, \eta_{C_i}$ are disjoint. Since $\eta'_j \in B_{\sigma'} \subseteq B_\sigma$, according to our choice of B_σ, we know that that $(\eta_j(v_0), \eta_j(\vec{v})) \in H^-_\sigma$, and so the restriction of η_j to $V(F_k)$ is a C_k-ample embedding from F_k to G. Using the assumption that $C_k = pv(F_0)^k \ge C_i v(F_i)$, one can show that $\eta_1, \dots, \eta_{C_i}$ are C_i-ample embeddings from F_i to G.

To sum up, for every $\eta' \in B^I_{\sigma'}$ we find $\eta \in \mathrm{Amp}_{C_i}(F_i; \sigma)$ such that $\eta' \supseteq \eta$, $(\eta(v_0), \eta(\vec{v})) \in H^-_\sigma$ and in particular $\eta(v_0) \in U_\sigma$. By the distinctness property (6) of B_σ, we know that a different $\eta' \in B^I_{\sigma'} \subseteq B_\sigma$ gives a different such η. However we might find the same η when σ' starts to vary. Because for every $\eta' \in \bigcup_{\sigma'} B^I_{\sigma'}$ and every $j \in I$ the restriction of η' to $\{v_j\} \cup S_j$ is an embedding from $K_{1,s}$ to G, the same η can be found for at most $(Kd)^{si}$ times, which implies (7).

Finally we estimate the cardinality of

$$I^\times_{\sigma'} := B_{\sigma'} \setminus \bigcup_{I \subsetneq [k]} B^I_{\sigma'}$$

by that of $W := \{\eta'(\vec{v}) \colon \eta' \in I^\times_{\sigma'}\}$. For every sequence $\vec{w} \in W$, as the degree of $\vec{w}$ is at most sk in H_σ, together with the distinctness property (6) of B_σ, we know that $|I^\times_\sigma| \le sk|W|$. By the definitions of $I^\times_{\sigma'}$ and $B^I_{\sigma'}$, one can check that W contains no sequential sunflower of size $\max(C_0, \dots, C_{k-1}) = C_{k-1}$. Thus Proposition 25 implies $|I^\times_{\sigma'}| \le sk|W| \le K_0$, and so

$$|B_{\sigma'}| \le K_0 + \sum_{I \subsetneq [k]} |B^I_{\sigma'}|.$$

Because the total number of $\sigma' \colon R(F_0) \to V(G)$, such that $\sigma' \supseteq \sigma$, is at most n^{sk}, summing the last inequality over all σ', together with (5), yields

$$(1 - o(1)) d^{sk} |H^-_\sigma| \le |B_\sigma| \le K_0 n^{sk} + \sum_{\sigma'} \sum_{I \subsetneq [k]} |B^I_{\sigma'}|,$$

which implies the desired inequality in view of (7) and the assumption that $\mathrm{amp}_{C_0}(F_0) = 0$. $\boxdot$

Claim 3. For every σ, the number of edges in H_σ^+ satisfies

$$|H_\sigma^+| \le \begin{cases} K_k m_0 n^{1-1/s} & \text{if } k = 1; \\ K_k \left(m_0^{(sk-1)(1-\frac{k-1}{s+1})} m_0 d^{k-1} + m_0^{sk+\frac{s(k-1)}{s+1}} \right) & \text{otherwise,} \end{cases}$$

where $K_k = K_k(s, t, p, k, K)$ is a positive constant.

Proof of Claim 3. Let U_0 denote an arbitrary vertex subset of U_σ of size sk in H_σ, and denote $N_\sigma(U_0) \subseteq \widetilde{W}_\sigma$ the common neighborhood of U_0 in H_σ^+. Let $W(U_0)$ be the k-uniform hypergraph defined by

$$W(U_0) = \{\{w_1, \ldots, w_k\} \colon (w_1, \ldots, w_k) \in N_\sigma(U_0)\}.$$

We observe that $W(U_0)$ contains no matching of size $C_0 = p$. Indeed, let $U_0 = U_{0,1} \cup \cdots \cup U_{0,k}$ be an arbitrary fixed partition of U_0 into k subsets each of size s. Notice that every hyperedge $\{w_1, \ldots, w_k\}$ in $W(U_0)$ gives rise to $\eta' \in \mathrm{Inj}(F_0; \sigma)$ such that $\eta'(v_i) = w_i$ for $i \in [k]$, $\eta'(S_i) = U_{0,i}$, and the restriction of η' to $V(F_k)$ is a C_k-ample embedding from F_k to G. If $W(U_0)$ contains a matching of size p, then one can find a p-ample embedding of F_0, which is a contradiction. Now we treat the $k = 1$ case and the $k \ge 2$ case separately.

Case 1: $k = 1$. In this case, $W(U_0)$ is a 1-uniform hypergraph, and it contains less than p vertices for every U_0. Therefore H_σ^+ contains no complete bipartite subgraph with s vertices in U_σ and p vertices in $\widetilde{W}_\sigma$. Proposition 26 shows that $|H_\sigma^+| \le K_k |U_\sigma| |\widetilde{W}_\sigma|^{1-1/s}$, where $K_k = s\sqrt[s]{(p-1)/s!}$, which implies the desired inequality in view of the fact that $|\widetilde{W}_\sigma| \le n$.

Case 2: $k \ge 2$. Using the assumption that $d_{H_\sigma^+}(\vec{w}) > sk$ for every $\vec{w} \in \widetilde{W}_\sigma$, a simple double counting argument shows that

$$\sum_{U_0} |N_\sigma(U_0)| = \sum_{\vec{w} \in \widetilde{W}_\sigma} \binom{d_{H_\sigma^+}(\vec{w})}{sk} \ge \sum_{\vec{w} \in \widetilde{W}_\sigma} d_{H_\sigma^+}(\vec{w}) = |H_\sigma^+|,$$

which, together with the fact that $|W(U_0)| \ge |N_\sigma(U_0)|/(k!)$, implies that

$$|H_\sigma^+| \le k! \sum_{U_0} |W(U_0)|.$$

For convenience, denote $N(U_0)$ the vertex set of the k-uniform hypergraph $W(U_0)$. As $W(U_0)$ contains no matching of size p, clearly we have

$$|W(U_0)| \le k(p-1)|N(U_0)|^{k-1}.$$

It suffices to estimate $\sum_{U_0}|N(U_0)|^{k-1}$. Clearly $|N(U_0)| \le Kd$, and so $\sum_{U_0}|N(U_0)|^{k-1} \le m_0^{sk}(Kd)^{k-1}$, which gives the following weaker bound on $|H_\sigma^+|$,

$$|H_\sigma^+| \le K_k m_0^{sk} d^{k-1}. \tag{8}$$

To to get a better estimate on $\sum_{U_0}|N(U_0)|^{k-1}$, we squeeze a bit more out of the assumption that G contains no F_0^p as a subgraph by iteratively applying Proposition 27.

Let $V(\widetilde{W}_\sigma) \subseteq V(G)$ be the set of vertices that ever appear in any sequence in $\widetilde{W}_\sigma$. For every subset of $U \subseteq U_\sigma$, we denote $N'_G(U)$ the set of vertices in $V(\widetilde{W}_\sigma)$ that are adjacent to every vertex in U in the graph G. Notice that $N(U_0) \subseteq N'_G(U_0)$ for every U_0. We prove inductively for every $i \in \mathbb{N}^+$ that

$$\sum_{U\subseteq U_\sigma:\,|U|=i} |N'_G(U)|^{k-1} \le \frac{K_{\max}^{i-1}}{i!}\left(m_0^{(i-1)\left(1-\frac{k-1}{s+1}\right)} m_0(Kd)^{k-1} + (i-1)m_0^{i+\frac{(k-1)s}{s+1}}\right), \tag{9}$$

where $K_{\max} = v(T_{s,t,0}^p)^{k-1}$, and in particular

$$\sum_{U_0}|N(U_0)|^{k-1} \le \frac{K_{\max}^{sk-1}}{(sk)!}\left(m_0^{(sk-1)\left(1-\frac{k-1}{s+1}\right)} m_0(Kd)^{k-1} + (sk-1)m_0^{sk+\frac{(k-1)s}{s+1}}\right),$$

which implies the desired inequality in Claim 3. The base case $i = 1$ is evident as the maximum degree of G is at most Kd. For the inductive step, consider an arbitrary $U \subseteq U_\sigma$ of size $i-1$ and denote u an arbitrary vertex in $U_\sigma \backslash U$. Clearly $|N'_G(U\cup\{u\})| = d_{G(U)}(u)$, where $G(U)$ is the bipartite subgraph of G induced on U_σ and $N'_G(U)$. Observe that there is no embedding $\eta \in \mathrm{Inj}(T_{s,t,0}, G(U))$ such that $\eta(R(T_{s,t,0}^p)) \subseteq U_\sigma$, because otherwise one can extend $\eta \in \mathrm{Inj}(T_{s,t,0}, G(U))$ to $\eta' \in \mathrm{Inj}(F_0^p)$ such that η' and σ are identical on $S'(F_0^p) = S'(F_0)$ (see Figure 4 for the definitions of $S'(F_0)$ and $Q(F_0)$). As every vertex in $Q(T_{s,t,0}^p)$ has degree $s+1$, and $|U_\sigma| = m_0$, Proposition 27 shows that

$$\sum_{u\in U_\sigma\backslash U} |N'_G(U\cup\{u\})|^{k-1} = \sum_{u\in U_\sigma\backslash U} d_{G(U)}(u)^{k-1} \le K_{\max}(m_0^{k-1} + |N'_G(U)|^{k-1})m_0^{1-\frac{k-1}{s+1}}.$$

Let U' denote an arbitrary subset of U_σ of size i. Summing the above inequality over all $U \subseteq U_\sigma$ of size $i-1$, we obtain from the inductive hypothesis that

$$\begin{aligned}\sum_{U'}|N'_G(U')|^{k-1} &\le i^{-1}\sum_U K_{\max}\left(m_0^{k-\frac{k-1}{s+1}} + m_0^{1-\frac{k-1}{s+1}}|N'_G(U)|^{k-1}\right)\\ &\le \frac{K_{\max}}{i!}m_0^{i+\frac{(k-1)s}{s+1}} + \frac{K_{\max}}{i}m_0^{1-\frac{k-1}{s+1}}\sum_U|N'_G(U)|^{k-1}\\ &\le \frac{K_{\max}m_0^{i+\frac{(k-1)s}{s+1}}}{i!} + \frac{K_{\max}^{i-1}}{i!}\left(m_0^{(i-1)\left(1-\frac{k-1}{s+1}\right)}m_0(Kd)^{k-1} + (i-2)m_0^{i+\frac{(k-1)(s-1)}{s+1}}\right),\end{aligned}$$

which implies (9) after checking that $i+\frac{(k-1)s}{s+1} \ge i+\frac{(k-1)(s-1)}{s+1}$. This finishes the proof of Claim 3. ⊡

Now we assemble Claims 1 to 3 together. We first treat the $k \geq 2$ case, and we take $m_0 = \lfloor n^{(s-(s+1)\alpha)k} \rfloor$. As $s' < s$, one can check that $\rho_{F_0} < s+1$, and so $\alpha < s/(s+1)$, which implies $s-(s+1)\alpha > 0$, hence $m_0 = \omega(1)$. We claim that the condition (4) on t implies that

$$m_0^{(sk-1)\left(1-\frac{k-1}{s+1}\right)} \leq n^\alpha \quad \text{and} \quad m_0^{sk+\frac{s(k-1)}{s+1}} \leq m_0 n^\alpha d^{k-1}. \tag{10}$$

Indeed, using $\alpha = 1 - 1/\rho_{F_0} = (st+s'-1)/(st+t+s')$, one can check that (4) is equivalent to

$$(s-(s+1)\alpha)k(sk-1)\left(1-\tfrac{k-1}{s+1}\right) \leq \alpha,$$

which implies the first inequality in (10). To check that the second inequality follows from the first inequality in (10), in view of the fact that $n^\alpha d^{k-1} \geq n^{k\alpha}$ as $d = cn^\alpha$ and $c > 1$, it suffices to check that $(sk-1)(1-(k-1)/(s+1)) \geq (sk + s(k-1)/(s+1) - 1)/k$, which is equivalent to $sk(k-1)(s-k+1) + (k-1)^2 \geq 0$, which clearly holds.

Using (10), we can simplify Claim 3 to $|H_\sigma^+| \leq 2K_k m_0 n^\alpha d^{k-1}$. Combining this with Claim 2, we obtain that for every σ that

$$\begin{aligned}(1-o(1))|H_\sigma| &\leq (1-o(1))|H_\sigma^-| + |H_\sigma^+| \\ &\leq K_0 n^{sk} d^{-sk} + \sum_{i=1}^{k-1} K_i d^{s(i-k)} |\{\eta \in \mathrm{Amp}_{C_i}(F_i;\sigma) \colon \eta(v_0) \in U_\sigma\}| + 2K_k m_0 n^\alpha d^{k-1}.\end{aligned}$$

Recall that U_σ is a subset of $\widetilde{U}_\sigma$ of size m_0 chosen uniformly at random, and H_σ is the bipartite subgraph of $\widetilde{H}_\sigma$ induced on $U_\sigma \cup \widetilde{W}_\sigma$. Observe that the expectation of $|H_\sigma|$ is $m_0|\widetilde{H}_\sigma|/|\widetilde{U}_\sigma|$, and the expectation of $|\{\eta \in \mathrm{Amp}_{C_i}(F_i;\sigma) \colon \eta(v_0) \in U_\sigma\}|$ is $m_0 \,\mathrm{amp}_{C_i}(F_i;\sigma)/|\widetilde{U}_\sigma|$. Thus taking the expectation of the above inequality, and then multiplying both sides by $|\widetilde{U}_\sigma|/m_0$, gives

$$(1-o(1))|\widetilde{H}_\sigma| \leq K_0 |\widetilde{U}_\sigma| m_0^{-1} n^{sk} d^{-sk} + \sum_{i=1}^{k-1} K_i d^{s(i-k)} \,\mathrm{amp}_{C_i}(F_i;\sigma) + 2K_k |\widetilde{U}_\sigma| n^\alpha d^{k-1}. \tag{11}$$

In case $|\widetilde{U}_\sigma| < m_0$, we simply bound

$$|\widetilde{H}_\sigma| \leq m_0 (Kd)^k \leq n^{(s-(s+1)\alpha)k}(Kd)^k. \tag{12}$$

Since every $u \in \widetilde{U}_\sigma$ corresponds to $\eta \in \mathrm{Inj}(F_k^-;\sigma)$ such that $\eta(v_0) = u$, where F_k^- is the subgraph of F_k induced on $V(F_k) \setminus \{v_1, \ldots, v_k\}$, clearly we have $|\widetilde{U}_\sigma| \leq \mathrm{inj}(F_k^-;\sigma)$. Using $d = cn^\alpha$, one can routinely check that $m_0^{-1} n^{sk} d^{-sk}$ and $n^\alpha d^{k-1}$ in (11) are at most $(1+o(1))d^k/c$, and that $n^{(s-(s+1)\alpha)k} d^k$ in (12) is at most $n^{1-(|R(F_k)-k|)} d^{e(F_k)}/c$. Therefore Claim 1 and (11) and (12) imply

$$|\widetilde{A}_\sigma| \leq (1+o(1))K_* \left(\tfrac{1}{c}\,\mathrm{inj}(F_k^-;\sigma) d^k + \tfrac{1}{c} n^{1-(|R(F_k)|-k)} d^{e(F_k)} + \sum_{i=1}^{k-1} d^{s(i-k)} \,\mathrm{amp}_{C_i}(F_i;\sigma) \right) + |I_\sigma^\times|,$$

where $K^* = K_*(C_*, s, t, p, k, K)$ is a positive constant. Summing the last inequality over all injections $\sigma\colon R(F_k) \setminus \{v_1, \dots, v_k\} \to V(G)$ yields that $\operatorname{amp}_{C_k}(F_k) - \operatorname{ext}_{C_*}(K_{1,s+1}, F_k)$ is at most

$$(1+o(1))K_* \left(\tfrac{1}{c} \operatorname{inj}(F_k^-) d^k + \tfrac{1}{c} n d^{e(F_k)} + \sum_{i=1}^{k-1} d^{s(i-k)} \operatorname{amp}_{C_i}(F_i) \right).$$

Finally for the $k = 1$ case, we simply take $m_0 = |\widetilde{U}_\sigma|$, in other words, $U_\sigma = \widetilde{U}_\sigma$. Notice that every vertex $\vec{w} \in \widetilde{W}_\sigma$ has degree at least C_k in $\widetilde{H}_\sigma$ because $\vec{w} = \eta(\vec{v})$ for some C_k-ample . Therefore $\widetilde{H}_\sigma = H_\sigma^+$. By Claims 1 and 3 and the assumption that $1 - 1/s \le \alpha$, we can similarly get that

$$\operatorname{amp}_{C_k}(F_k) - \operatorname{ext}_{C_*}(K_{1,s+1}, F_k) \le K_* \operatorname{inj}(F_1^-) n^{1-1/s} \le K_* \operatorname{inj}(F_1^-) d^k / c,$$

where $K_* = K_*(C_*, s, t, k, p)$ is a positive constant. This finishes the proof of Lemma 28. □

Remark. If we use in the proof of Lemma 28 the weaker bound (8) on $|H_\sigma^+|$ instead, we would need to impose a condition on t that is more restricted than (4).

Proof of Theorem 19(b). Assume that $s - s' \ge 1$. Denote $F_k = T_{s,t-i,s'+i}$ for $0 \le k \le s - s'$. In particular, $F_0 = T_{s,t,s'}$. Let $p \in \mathbb{N}^+$ and $C_* = v(T_{s,t,s+1}^p)$. Set $C_k = pv(F_0)^i$ for $k \le s - s'$. Let $c_0 \ge 1/\varepsilon$ be the constant to be chosen. Suppose that $c > c_0$ and G is an n-vertex graph with $n \ge n_0(c)$ such that every vertex in G has degree between d and Kd, where $d = cn^\alpha$ and $K = 5^{4/\alpha}$, and moreover $\operatorname{amp}_{C_0}(F_0, G) = 0$. We break into two cases.

Case 1: $k = 1$. Let c_0 be at least the constant already obtained from Theorem 19(a). By the choice of c_0, we know that $\operatorname{amp}_{C_*}(K_{1,s+1}, G) \le \varepsilon n d^{e(K_{1,s+1})}$. Since F_0 is balanced, by Proposition 9, $\rho_{F_0} \ge s$, which implies that $1 - 1/s \le \alpha$, where $\alpha = 1 - 1/\rho_{F_0}$. By Lemma 28, we obtain

$$\operatorname{amp}_{C_k}(F_k, G) \le (1+o(1))K_* \left(\tfrac{1}{c} \operatorname{inj}(F_k^-, G) d^k + \tfrac{1}{c} n d^{e(F_k)} \right) + \operatorname{ext}_{C_*}(K_{1,s+1}, F_k, G),$$

where $K_* = K_*(s, t, p)$ is a positive constant. As $1/c < \varepsilon$, $\operatorname{inj}(F_k^-, G) \le n(Kd)^{e(F_k^-)} \le n(Kd)^{e(F_k)-k}$, and $\operatorname{ext}_{C_*}(K_{1,s+1}, F_k, G) \le \operatorname{inj}(K_{1,s+1}, F_k) \operatorname{amp}_{C_*}(K_{1,s+1}, G)(Kd)^{e(F_i) - e(K_{1,s+1})}$, we estimate

$$\operatorname{amp}_{C_k}(F_k, G) \le \left((\varepsilon + o(1)) K_* (K^{e(F_k)} + 1) + \varepsilon \operatorname{inj}(K_{1,s+1}, F_k) K^{e(F_k)} \right) n d^{e(F_k)},$$

which implies the desired inequality in Theorem 19(b).

Case 2: $2 \le k \le s - s'$. By induction, let c_0 be such that $\operatorname{amp}_{C_i}(F_i, G) \le \varepsilon n d^{e(F_i)}$ for $i < k$. Note that the assumption (2) in Theorem 19 ensures the condition (4) in Lemma 28. By Lemma 28, we similarly obtain that

$$\operatorname{amp}_{C_k}(F_k, G) \le \left((\varepsilon + o(1)) K_* (K^{e(F_k)} + k) + \varepsilon \operatorname{inj}(K_{1,s+1}, F_k) \right) n d^{e(F_k)},$$

where $K_* = K_*(s, t, p)$ is a positive constant. □

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