

On a hypergraph Turán problem of Balogh–Bohman–Bollobás–Zhao

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ABSTRACT. Let S and T be disjoint sets with $|S| = i$ and $|T| = r - 1$ for $2 \leq i \leq r - 1$, and let $B_i^{(r)}$ be the r -graph on $S \cup T$ whose edges are the r -subsets containing S or T . We study the deficit $q_{r,i} := 1 - \pi(B_i^{(r)})$ in its Turán density. Balogh, Bohman, Bollobás, and Zhao previously obtained bounds for these deficits with logarithmic gaps near both ends of the sequence $B_i^{(r)}$, namely, when $i = O(1)$ or $i = r - O(1)$. We close these gaps by showing that, as $r \rightarrow \infty$, for every fixed integer $a \geq 1$, $q_{r,a+1} = \Theta_a(r^{-a})$, and for every fixed integer $b \geq 2$, $q_{r,r-b} = \Theta_b(r^{-b} \log r)$.

1. Introduction

For a set V and an integer $r \geq 1$, denote by $\binom{V}{r}$ the family of all r -element subsets of V . An r -uniform hypergraph (or r -graph) G consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq \binom{V(G)}{r}$. We write $|G| = |E(G)|$ for the number of edges. For an r -graph H , the *Turán number* $\text{ex}(n, H)$ is the maximum number of edges in an H -free r -graph on n vertices. The *Turán density* $\pi(H)$ is defined as

$$\pi(H) := \lim_{n \rightarrow \infty} \text{ex}(n, H) / \binom{n}{r},$$

and the existence of this limit is guaranteed by a result of Katona, Nemetz, and Simonovits [7].

In the setting of graphs (i.e., $r = 2$), the asymptotic behavior of Turán numbers is well understood due to the celebrated Erdős–Stone–Simonovits theorem, which determines $\pi(H)$ in terms of the chromatic number $\chi(H)$ of the graph H . To be precise, they proved that $\pi(H) = 1 - 1/(\chi(H) - 1)$ for any graph H with $\chi(H) \geq 3$. For hypergraphs of uniformity $r \geq 3$, however, the situation is much more complex. No simple parameter analogous to the chromatic number is known to determine $\pi(H)$. Indeed, even the Turán density of the complete 3-graph $K_4^{(3)}$ (the tetrahedron) is not known exactly. The best known bounds are $5/9 \leq \pi(K_4^{(3)}) \leq 0.5616$, where the lower bound

is due to Turán [15] (see also [4]), and the upper bound was established by Razborov [13] using his flag algebra method.

For general r , the problem of determining the Turán density of $K_{r+1}^{(r)}$, the complete r -graph on $r+1$ vertices, is even less understood. For $r = 4$, Markström [10] proved that $\pi(K_5^{(4)}) \leq 1753/2380 \approx 0.73655$, while a construction of Giraud [6] gave the lower bound $11/16$. Considerable attention has also been devoted to estimating $\pi(K_{r+1}^{(r)})$ as $r \rightarrow \infty$, a problem closely related to the study of the so-called *Turán systems*, studied by Frankl and Rödl [5], Sidorenko [14], and others. Combining de Caen's classical upper bound [3] with Pikhurko's recent construction [12] for Turán systems shows that, as $r \rightarrow \infty$, the density $\pi(K_{r+1}^{(r)})$ is $1 - \Theta(1/r)$; see also the extension by Liu and Pikhurko [9].

In this note, we study the Turán problem for a family of hypergraphs related to $K_{r+1}^{(r)}$ defined as follows. Given integers r, s, t with $2 \leq s \leq r$, $2 \leq t \leq r-1$ and $s+t \geq r$, let $F_{s,t}^r$ be the r -graph with vertex set $S \cup T$, where $S \cap T = \emptyset$, $|S| = s$ and $|T| = t$, and with edge set

$$\left\{ e \in \binom{S \cup T}{r} : S \subseteq e \text{ or } T \subseteq e \right\}.$$

Thus $F_{2,r-1}^r = K_{r+1}^r$. In this paper, we focus on the deficit

$$q_{r,i} := 1 - \pi(F_{i,r-1}^r) \quad \text{for} \quad 2 \leq i \leq r-1.$$

Throughout the paper, subscripts on asymptotic notation indicate the parameters on which the implicit constants may depend; asymptotic notation without subscripts has absolute implicit constants.

The study of these hypergraphs was motivated in part by a problem raised by Mubayi (see Problem 25 in the survey [11]), who asked to determine the growth rate of $1 - \pi(F_{i,r-1}^r)$ as $r \rightarrow \infty$ with $i = i(r)$, and in particular whether

$$q_{r,r-1} = 1 - \pi(F_{r-1,r-1}^r) = \Theta\left(\frac{\log r}{r}\right).$$

Writing $F_{i,r-1}^r$ as $B_i^{(r)}$, Balogh, Bohman, Bollobás, and Zhao [1] proved the following theorem.

THEOREM 1.1 ([1]). *Fix $a, b \geq 1$. As $r \rightarrow \infty$, we have*

$$\Theta_a(r^{-a}) \leq q_{r,a+1} \leq \Theta_a(r^{-a} \log r) \quad \text{and} \quad \Theta_b(r^{-b}) \leq q_{r,r-b} \leq \Theta_b(r^{-b} \log r).$$

Thus their theorem leaves logarithmic gaps near both ends of the sequence $B_2^{(r)}, \dots, B_{r-1}^{(r)}$. We refer readers to the survey of Keevash [8] for an explanation of why logarithmic deficits are natural near the right end of the sequence. To put our results in perspective, we briefly survey earlier results related to $q_{r,i}$. First, the case $i = 2$ is classical. Since $F_{2,r-1}^r = K_{r+1}^r$, the deficit $q_{r,2}$, as mentioned before, is known to be $\Theta(1/r)$. For general fixed i , Balogh, Bohman, Bollobás, and Zhao [1] also introduced certain modifications of $B_i^{(r)}$ and obtained matching logarithmic estimates for those modified

hypergraphs. It should be noted that the independent neighborhood problem studied by Bohman, Frieze, Mubayi, and Pikhurko [2], corresponding to $\pi(F_{r,r-1}^r)$, is distinct from the problem of determining $q_{r,i}$ considered here.

The purpose of this note is to close the logarithmic gaps of Theorem 1.1 for fixed $i = a + 1 \geq 2$ at the left end, and for fixed $b = r - i \geq 2$ at the right end (of the sequence $B_i^{(r)}$ for $2 \leq i \leq r - 1$). Our main results are as follows.

THEOREM 1.2. *For every fixed integer $a \geq 1$, as $r \rightarrow \infty$ we have*

$$q_{r,a+1} = O_a(r^{-a}).$$

THEOREM 1.3. *For every fixed integer $b \geq 2$, set $C = C_{r,b} := \binom{r-1}{b}$. As $r \rightarrow \infty$, we have*

$$q_{r,r-b} \geq (1 - o(1)) \frac{W(C/(r-b))}{C},$$

where W denotes the principal branch of the Lambert W -function, that is, the inverse of $f(x) = xe^x$ on $[0, \infty)$.

In particular, for every fixed $b \geq 2$, as $r \rightarrow \infty$,

$$q_{r,r-b} \geq (1 - o(1)) \frac{(b-1) \log r}{\binom{r-1}{b}} = \Omega_b(r^{-b} \log r).$$

Together, Theorem 1.2, Theorem 1.3, and Theorem 1.1 imply the following corollary.

COROLLARY 1.4. *For all fixed integers $a \geq 1$ and $b \geq 2$, as $r \rightarrow \infty$ we have*

$$q_{r,a+1} = \Theta_a(r^{-a}) \quad \text{and} \quad q_{r,r-b} = \Theta_b(r^{-b} \log r).$$

This solves two instances of Problem 25 in the survey [11] due to Mubayi. Note that Mubayi's problem of determining $q_{r,r-1}$ remains open up to a logarithmic factor.

The rest of this note is organized as follows. In Section 2, we introduce notation and the concept of shadow-complete systems. In Section 3, we prove Theorem 1.2 by relating $F_{s,t}^r$ -free r -graphs to shadow-complete systems. In Section 4, we prove Theorem 1.3 by applying a clique-counting method to obtain the required lower bound on $q_{r,r-b}$ for fixed $b \geq 2$.

2. Preliminaries

We use the following notation. For an integer $n \geq 1$, write $[n] = \{1, 2, \dots, n\}$. The *complement* $\overline{G}$ of an r -graph G has vertex set $V(G)$ and edge set $E(\overline{G}) = \binom{V(G)}{r} \setminus E(G)$. For an r -graph G and a set $U \subseteq V(G)$, the *induced subgraph* $G[U]$ has vertex set U and edges $\{e \in E(G) : e \subseteq U\}$. For $0 \leq s \leq r$, the *s -shadow* of G is

$$\partial_s(G) = \left\{ S \in \binom{V(G)}{s} : \text{there exists } e \in E(G) \text{ such that } S \subseteq e \right\}.$$

For an $(r-1)$ -set S in an r -graph G , its *neighborhood* is

$$N_G(S) = \{x \notin S : S \cup \{x\} \in G\},$$

and its *degree* is denoted by $d_G(S) = |N_G(S)|$. For an $(r-b)$ -set S in an r -graph G , the *b-uniform link* of S is

$$L_G(S) := \left\{ A \in \binom{V(G) \setminus S}{b} : S \cup A \in G \right\}.$$

We need the following Chernoff-type bound. For completeness, we present a self-contained proof here.

LEMMA 2.1. *Let $X_1, X_2, \dots, X_n$ be independent random variables with $0 \leq X_i \leq 1$. Define $S = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[S]$. Then for any $0 < \delta < 1$,*

$$\mathbb{P}[S \leq (1-\delta)\mu] \leq \exp(-\mu(1-\delta)\log(1-\delta) - \mu\delta).$$

PROOF. For any $t > 0$, Markov's inequality yields

$$\mathbb{P}[S \leq (1-\delta)\mu] = \mathbb{P}[e^{-tS} \geq e^{-t(1-\delta)\mu}] \leq e^{t(1-\delta)\mu} \mathbb{E}[e^{-tS}].$$

By independence and convexity of $x \mapsto e^{-tx}$ on $[0, 1]$,

$$\mathbb{E}[e^{-tX_i}] \leq 1 - \mu_i + \mu_i e^{-t} \leq \exp(\mu_i(e^{-t} - 1)),$$

where $\mu_i = \mathbb{E}[X_i]$. Hence $\mathbb{E}[e^{-tS}] \leq \exp(\mu(e^{-t} - 1))$. Thus

$$\mathbb{P}[S \leq (1-\delta)\mu] \leq \exp(\mu(t(1-\delta) + e^{-t} - 1)).$$

Choose $t = -\log(1-\delta) > 0$. Then $e^{-t} = 1-\delta$ and

$$\begin{aligned} t(1-\delta) + e^{-t} - 1 &= -(1-\delta)\log(1-\delta) + (1-\delta) - 1 \\ &= -(1-\delta)\log(1-\delta) - \delta. \end{aligned}$$

Substituting yields the claimed bound. $\square$

We next introduce a simple criterion for $F_{s,t}^r$ -free r -graphs.

LEMMA 2.2. *Let $2 \leq s, t \leq r-1$ and $s+t \geq r$. The r -graph G is $F_{s,t}^r$ -free if and only if for every pair of disjoint sets $S, T \subseteq V(G)$ with $|S| = s$ and $|T| = t$, there is an edge $e \in \overline{G}$ such that*

$$S \subseteq e \subseteq S \cup T \quad \text{or} \quad T \subseteq e \subseteq S \cup T.$$

PROOF. The candidate copy of $F_{s,t}^r$ on $S \cup T$ consists exactly of the r -sets contained in $S \cup T$ that contain S or contain T . Hence G contains this copy if and only if all these candidate edges lie in G . Equivalently, G avoids this copy if and only if at least one candidate edge lies in the complement $\overline{G}$. $\square$

Inspired by this lemma, we introduce the following “shadow-complete” concept.

DEFINITION 2.3. Let $n \geq r \geq s \geq 2$ and $R \geq 0$ be integers. An r -graph G is (s, R) -shadow-complete if for every $(r + R)$ -set $U \subseteq V(G)$ and every s -set $S \subseteq U$, there is an edge $e \in G$ such that $S \subseteq e \subseteq U$. Let

$$\theta(n, r, s, R) := \frac{1}{\binom{n}{r}} \min \left\{ |G| : G \subseteq \binom{[n]}{r} \text{ is } (s, R)\text{-shadow-complete} \right\},$$

and define

$$\theta(r, s, R) := \lim_{n \rightarrow \infty} \theta(n, r, s, R).$$

This limit exists by the usual averaging argument. Indeed, if $m < n$, then taking a uniformly random m -vertex induced subgraph of an optimal n -vertex (s, R) -shadow-complete r -graph shows that

$$\theta(m, r, s, R) \leq \theta(n, r, s, R).$$

Thus $\theta(n, r, s, R)$ is nondecreasing in n , and in particular $\theta(n, r, s, R) \leq \theta(r, s, R)$ for every $n \geq r$.

We note that the above limit estimation provides an important upper bound on $q_{r,i}$ for our approach (but not an exact characterization).

PROPOSITION 2.4. For $2 \leq i \leq r - 1$,

$$q_{r,i} \leq \theta(r, i, i - 1).$$

PROOF. Let G be $(i, i - 1)$ -shadow-complete. Given any pair of disjoint subsets $S, T \subseteq V(G)$ with $|S| = i$ and $|T| = r - 1$, set $U = S \cup T$. Then $|U| = r + i - 1$. By shadow-completeness, there exists $e \in G$ with $S \subseteq e \subseteq U$. By Lemma 2.2, $\overline{G}$ is $F_{i, r-1}^r$ -free. Taking asymptotic densities gives the claim. $\square$

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by providing the desired upper bound on $q_{r,a+1}$ via $(a + 1, a)$ -shadow-complete r -graphs. Our construction follows the recursive idea used by Pikhurko [12] in his construction of Turán systems. The goal in [12] is to construct an r -graph such that every $(r + a)$ -set U contains an r -edge. Very roughly speaking, it first uses a random hypergraph on the initial part of an ordered vertex set, and then treats the uncovered sets recursively on the remaining vertices.

We use a similar two-step scheme for the desired $(a + 1, a)$ -shadow-complete r -graphs. The key difference is that, for each $(r + a)$ -set U , we must cover every $(a + 1)$ -subset of U , rather than merely ensuring the presence of an edge in U . This conceptually will be carried out in the first step – Lemma 3.1 (using general parameters s, R instead of $a + 1, a$, respectively), where we use the probabilistic method to cover all $(a + 1)$ -subsets for most k -sets, where k is chosen to be an intermediate parameter satisfying $a + 1 \leq k < r - a$. These uncovered sets are then handled in the second step via a recursive shadow-complete construction on the remaining vertices.

LEMMA 3.1. *Let R, s, k be integers with $R \geq 1, k \geq s$, and let $h \geq s$. Let c satisfy $h \leq c \leq \binom{k}{R}$, and let Γ be the random R -graph on a fixed k -set in which each R -edge is selected independently with probability $p = c/\binom{k}{R}$. Then*

$$(1) \quad \mathbb{P}[\alpha(\Gamma) \geq k - s] \leq c^h h^{-h} e^{h-c} + \binom{k}{s} \left(\frac{sR}{k - R + 1} \right)^{h+1},$$

where $\alpha(\Gamma)$ denotes the independence number of Γ .

PROOF. For an s -set Y , let A_Y be the event that every selected edge of Γ meets Y . The event $\alpha(\Gamma) \geq k - s$ is the union of the events A_Y over all $Y \in \binom{[k]}{s}$. We proceed to estimate the probability of A_Y conditioned on the event $e(\Gamma) = j$. If A_Y occurs, then all j selected R -sets lie in the family of R -sets meeting Y . If $k - s < R$, then the desired bound (1) is trivial since $sR/(k - R + 1) \geq 1$ (if $R \geq 2$, this is evident; if $R = 1$, then $k = s$ and thus this is equal). Otherwise,

$$\frac{\binom{k-s}{R}}{\binom{k}{R}} = \prod_{\ell=0}^{R-1} \frac{k-s-\ell}{k-\ell} \geq \left(\frac{k-s-R+1}{k-R+1} \right)^R,$$

and we have

$$\begin{aligned} \mathbb{P}[A_Y \mid e(\Gamma) = j] &\leq \left(1 - \frac{\binom{k-s}{R}}{\binom{k}{R}} \right)^j \\ &\leq \left(1 - \left(\frac{k-s-R+1}{k-R+1} \right)^R \right)^j \leq \left(\frac{sR}{k-R+1} \right)^j, \end{aligned}$$

where the last inequality holds by Bernoulli inequality $x^R \geq R(x-1) + 1$ for $x \geq 0$ and $R \geq 1$. If $c = h$, then $c^h h^{-h} e^{h-c} = 1$, and the desired bound is trivial. Hence we may assume $c > h$. Since $e(\Gamma)$ has mean c and $h < c$, applying Lemma 2.1 with $\mu = c$ and $\delta = 1 - h/c \in (0, 1)$ yields

$$\mathbb{P}[e(\Gamma) \leq h] \leq \exp \{-c + h \log(ec/h)\} = c^h h^{-h} e^{h-c}.$$

On the event $j \geq h + 1$, take the union bound over Y . If $sR/(k - R + 1) \geq 1$, then the second term on the right-hand side of (1) is at least one, so the bound is trivial. Otherwise $sR/(k - R + 1) < 1$, and for $j \geq h + 1$ the preceding bound is at most its value at $j = h + 1$. This gives

$$\begin{aligned} \mathbb{P}[\alpha(\Gamma) \geq k - s] &= \mathbb{P}[\cup_Y A_Y] \\ &\leq \mathbb{P}[e(\Gamma) \leq h] \\ &\quad + \mathbb{P}[e(\Gamma) \geq h + 1] \cdot \mathbb{P}[\cup_Y A_Y \mid e(\Gamma) \geq h + 1] \\ &\leq c^h h^{-h} e^{h-c} + \sum_Y \mathbb{P}[A_Y \mid e(\Gamma) \geq h + 1] \\ &\leq c^h h^{-h} e^{h-c} + \binom{k}{s} \left(\frac{sR}{k - R + 1} \right)^{h+1}, \end{aligned}$$

which completes the proof. $\square$

We now proceed to prove a fixed-parameter upper bound for shadow-complete systems. Define

$$\Lambda_a(r) := \theta(r, a+1, a) \binom{r-1}{a}.$$

Our goal is to prove $\Lambda_a(r) = O_a(1)$, which would imply the upper bound for $q_{r,a+1}$ by Proposition 2.4. In the next lemma, we will first use Lemma 3.1 to obtain an intermediate bound on $\Lambda_a(r)$.

LEMMA 3.2. *Let $a \geq 1$, $r > a+1$, and $a+1 \leq k \leq r-a-1$. Let $D > 1$ and $m \in \mathbb{N}$ satisfy*

$$Dm(a+1) \leq \binom{k}{a}.$$

Set

$$p_0 = (De^{1-D})^{m(a+1)} + \binom{k}{a+1} \left(\frac{(a+1)a}{k-a+1} \right)^{m(a+1)+1}.$$

Then

$$(2) \quad \Lambda_a(r) \leq \binom{r-1}{a} \left(\frac{Dm(a+1)}{\binom{k}{a}} + p_0 \frac{\Lambda_a(r-k)}{\binom{r-k-1}{a}} \right).$$

PROOF. Let n be sufficiently large. We aim to construct an $(a+1, a)$ -shadow-complete r -graph on $[n]$ whose edge density satisfies the claimed bound. Label the vertices $1, 2, \dots, n$ in increasing order.

Let $\mathcal{S}$ be a random $(k-a)$ -graph on $[n]$ where each $(k-a)$ -set appears independently with probability $p = Dm(a+1)/\binom{k}{a}$. Define the first r -graph G_1 by including an r -set $\{x_1 < \dots < x_r\}$ if and only if its $k-a$ smallest vertices form an edge of $\mathcal{S}$. For a k -set $X = \{x_1 < \dots < x_k\}$, we say X is *bad* if there exists an $(a+1)$ -set $Z \subseteq X$ such that no member of $\mathcal{S}[X]$ contains Z . Equivalently, the a -graph $\Gamma_X := \{X \setminus Q : Q \in \mathcal{S}[X]\}$ on X has independence number at least $k-a-1$. By Lemma 3.1 (with $s = a+1$, $R = a$, $c = Dm(a+1)$, $h = m(a+1)$), the probability that a fixed k -set is bad is at most p_0 .

For each bad k -set $X = \{x_1 < \dots < x_k\}$ with $x_k \leq n-r+k$, choose an $(a+1, a)$ -shadow-complete $(r-k)$ -graph H_X on $\{x_k+1, \dots, n\}$ with

$$|H_X| = \theta(n-x_k, r-k, a+1, a) \binom{n-x_k}{r-k}.$$

Now let G_2 be the r -graph obtained by adding all r -sets $X \cup f$ with $f \in H_X$ for every bad k -set X with $x_k \leq n-r+k$. Finally, we set $G = G_1 \cup G_2$. We claim that G is the desired r -graph.

First, we verify that G is $(a+1, a)$ -shadow-complete. Take any $(r+a)$ -set $U = \{u_1 < \dots < u_{r+a}\}$ and any $(a+1)$ -set $Z \subseteq U$. Let $X = \{u_1, \dots, u_k\}$ be the k smallest vertices of U .

If X is not bad, then every $(a+1)$ -subset of X is contained in some edge of $\mathcal{S}[X]$. Choose an $(a+1)$ -set $Y \subseteq X$ that contains $Z \cap X$ (possible since $|X| = k \geq a+1$). Pick $Q \in \mathcal{S}[X]$ with $Y \subseteq Q$, and consider $e = Q \cup (U \setminus X)$.

Because $|Q| = k - a$ and $|U \setminus X| = r + a - k$, we have $|e| = r$. All elements of Q are $\leq x_k$ while those of $U \setminus X$ are $> x_k$, so the $k - a$ smallest vertices of e are exactly the elements of Q . Hence $e \in G_1[U]$, and $Z \subseteq Y \cup (U \setminus X) \subseteq e$.

If X is bad, then $|U \setminus X| = r + a - k \geq 2a + 1 \geq a + 1$ (because $k \leq r - a - 1$). Extend $Z \setminus X$ to an $(a + 1)$ -set $Y' \subseteq U \setminus X$, which is possible as $|U \setminus X| \geq a + 1$. Since $U \setminus X$ has size $r + a - k = (r - k) + a$, the recursive graph H_X (which is $(a + 1, a)$ -shadow-complete on its vertex set) provides an edge $f \in H_X$ with $Y' \subseteq f \subseteq U \setminus X$. Then $e = X \cup f$ belongs to $G_2[U]$ and contains Z by construction. Thus G is indeed $(a + 1, a)$ -shadow-complete.

It remains to bound the edge density. The expected density of G_1 is exactly $p = Dm(a + 1)/\binom{k}{a}$. For G_2 , condition on $\mathcal{S}$ and sum over the maximum element $y = x_k$ of a k -set X : there are $\binom{y-1}{k-1}$ choices of X with that maximum. Each such X , if bad, contributes a recursive graph on $n - y$ vertices. Hence

$$\mathbb{E}[|G_2|] \leq \sum_{y=k}^{n-r+k} \binom{y-1}{k-1} p_0 \cdot \theta(n-y, r-k, a+1, a) \binom{n-y}{r-k}.$$

The identity $\sum_{y=k}^{n-r+k} \binom{y-1}{k-1} \binom{n-y}{r-k} = \binom{n}{r}$ and $\theta(n-y, r-k, a+1, a) \leq \theta(r-k, a+1, a)$ give

$$\frac{\mathbb{E}[|G_2|]}{\binom{n}{r}} \leq p_0 \cdot \theta(r-k, a+1, a).$$

Therefore, there is an instance of the random choice of $\mathcal{S}$ satisfying

$$\frac{|G|}{\binom{n}{r}} \leq \frac{Dm(a+1)}{\binom{k}{a}} + p_0 \theta(r-k, a+1, a).$$

Letting $n \rightarrow \infty$ and using $\Lambda_a(r) = \theta(r, a+1, a) \binom{r-1}{a}$ yields the desired inequality. $\square$

To illustrate the construction, consider $a = 2$, $k = 5$, and $r = 8$. Let $U = \{u_1 < \dots < u_{10}\}$ and $X = \{u_1, \dots, u_5\}$. Suppose that $\mathcal{S}[X] = \binom{X}{3}$, so X is not bad. For instance, take $Z = \{u_2, u_7, u_9\}$ and $Q = \{u_1, u_2, u_3\}$. Then $Q \in \mathcal{S}[X]$ contains $Z \cap X$, and the 8-set $Q \cup (U \setminus X) = \{u_1, u_2, u_3, u_6, u_7, u_8, u_9, u_{10}\}$ belongs to $G_1[U]$ and contains Z . Similarly, every 3-subset of U is contained in an edge of $G_1[U]$, as required by $(3, 2)$ -shadow-completeness. For a bad set X , the construction instead uses the recursive system H_X to find a 3-edge $f \subseteq U \setminus X$ containing $Z \setminus X$, so that $X \cup f \in G_2[U]$ contains Z .

Using this lemma, we can promptly derive the following estimate on $\Lambda_a(r)$.

PROPOSITION 3.3. *For every fixed integer $a \geq 1$, there is a constant $F_a > 0$ such that $\Lambda_a(r) \leq F_a$ holds for all integers $r > a$. In particular, we have $\theta(r, a+1, a) \leq F_a \binom{r-1}{a}^{-1}$.*

PROOF. Fix $a \geq 1$. Choose $D > 1$ and m large enough so that

$$(2(a+1))^a (De^{1-D})^{m(a+1)} \leq \frac{1}{4}.$$

Increase m if necessary so that $m(a+1) > a+1$. For all sufficiently large r , take $k = r - \left\lfloor \frac{r}{a+1} \right\rfloor$. Then $r - k \geq a+1$ and the hypotheses of Lemma 3.2 hold. Moreover,

$$\frac{\binom{r-1}{a}}{\binom{k}{a}} = O_a(1), \quad \frac{\binom{r-1}{a}}{\binom{r-k-1}{a}} = O_a((a+1)^a),$$

and the second term in p_0 tends to 0 as $r \rightarrow \infty$, because a and m are fixed and $m(a+1) + 1 > a+1$. Therefore, after increasing the threshold for r , Lemma 3.2 implies

$$\Lambda_a(r) \leq A_a + \frac{1}{2} \Lambda_a \left(\left\lfloor \frac{r}{a+1} \right\rfloor \right)$$

for a constant A_a depending only on a . Choose r_0 large enough so that $\lfloor r/(a+1) \rfloor > a$ for all $r \geq r_0$ and the preceding inequality is valid whenever $r \geq r_0$. Since there are only finitely many $a < r < r_0$, put

$$B_a := \max_{a < r < r_0} \Lambda_a(r),$$

with the convention that $B_a = 0$ if the set is empty. By the preceding inequality, an easy inductive argument shows that

$$\Lambda_a(r) \leq 2A_a + B_a$$

for all $r \geq r_0$ (and also for all $r > a$), and hence the proposition follows. $\square$

Proof of Theorem 1.2. The result follows by combining Proposition 2.4 with Proposition 3.3. $\square$

4. Proof of Theorem 1.3

We now prove the lower bound for Theorem 1.3. We need the following lemma, which can be derived from the clique-counting theorem of de Caen in the form presented in [1].

LEMMA 4.1 ([3, 1]). *Fix integers $b \geq 2$ and $t \geq b$. Let K be a b -uniform hypergraph on N vertices with missing-edge density α , and set*

$$\hat{\alpha} = 1 - (1 - \alpha) \frac{N - b + 1}{N} = \alpha + (1 - \alpha) \frac{b - 1}{N}.$$

If $1 - \binom{j-1}{b-1} \hat{\alpha} \geq 0$ for every $b \leq j \leq t$, then the density of t -vertex cliques in K satisfies

$$\frac{m_t}{\binom{N}{t}} \geq \prod_{j=b}^t \left(1 - \binom{j-1}{b-1} \hat{\alpha} \right).$$

For fixed b , sufficiently large t , and $\alpha \binom{t-1}{b-1} = o_t(1)$, letting $N \rightarrow \infty$, then the right-hand side is asymptotically $\exp\left(-(1 + o_t(1)) \alpha \binom{t}{b}\right)$, uniformly over such α .

PROOF. Let m_j denote the number of j -vertex cliques in K . We have

$$|E(K)| = (1 - \alpha) \binom{N}{b} = (1 - \hat{\alpha}) \binom{N}{b} \frac{N}{N - b + 1}.$$

A direct application of Corollary 10 of Balogh, Bohman, Bollobás, and Zhao [1] with the parameter $\hat{\alpha}$ would give the desired inequality

$$\frac{m_t}{\binom{N}{t}} \geq \prod_{j=b}^t \left(1 - \binom{j-1}{b-1} \hat{\alpha}\right).$$

For the asymptotic statement, we first send $N \rightarrow \infty$ while keeping b and t fixed. Then $\hat{\alpha} \rightarrow \alpha$. Under the assumption $\alpha \binom{t-1}{b-1} = o_t(1)$, all factors in the product are positive for sufficiently large t . Moreover,

$$\sum_{j=b}^t \binom{j-1}{b-1} \alpha = \alpha \binom{t}{b},$$

and the sum of squares is

$$\sum_{j=b}^t \left(\binom{j-1}{b-1} \alpha \right)^2 = O \left(\alpha \binom{t-1}{b-1} \right) \alpha \binom{t}{b} = o_t \left(\alpha \binom{t}{b} \right).$$

Taking logarithms of the product and expanding $\log(1 - x) = -x + O(x^2)$ yields

$$\log \left(\prod_{j=b}^t \left(1 - \binom{j-1}{b-1} \alpha\right) \right) = -(1 + o_t(1)) \alpha \binom{t}{b},$$

which exponentiates to the claimed asymptotic bound. The uniformity over α follows because the error terms depend only on t and $\alpha \binom{t-1}{b-1} = o_t(1)$. $\square$

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let G be an n -vertex r -graph with edge density $q = q(G)$ such that $\overline{G}$ is $F_{r-b, r-1}^r$ -free. Throughout this proof, all $o(1)$ terms are taken as $n \rightarrow \infty$ first and then $r \rightarrow \infty$, with b fixed. Fix an $(r - b)$ -set $S \subseteq V(G)$ and consider its b -uniform link $L_G(S)$. Let α_S denote its edge density. Averaging over all S gives $\mathbb{E}_S[\alpha_S] = q$.

A set $T \in \binom{V(G) \setminus S}{r-1}$ is called an S -**clique** if $\binom{T}{b} \cap L_G(S) = \emptyset$; equivalently, none of the r -edges $S \cup A$ with $A \in \binom{T}{b}$ belongs to G (so they all lie in $\overline{G}$). If T is an S -clique, then to avoid a copy of $F_{r-b, r-1}^r$ in $\overline{G}$ on the vertex set $S \cup T$, at least one of the $r - b$ edges $T \cup \{s\}$ for $s \in S$ must be present in G . Let $\kappa(S)$ denote the proportion of $(r - 1)$ -sets $T \subseteq V(G) \setminus S$ that are S -cliques.

We first derive a lower bound on $\mathbb{E}[\kappa(S)]$. If qC is already larger than a sufficiently large constant multiple of $\log(C/(r-b))$, then the desired bound is immediate. Hence we may assume

$$(3) \quad qC = O\left(\log \frac{C}{r-b}\right) = O_b(\log r).$$

Set $D := \binom{r-2}{b-1}$. Then $qD = \frac{b}{r-1} qC = o(1)$. Choose a parameter $\eta = \eta(r)$ such that

$$\eta \rightarrow 0, \quad \frac{qD}{\eta} = o(1), \quad \text{and} \quad \frac{qD}{\eta} qC = o(1).$$

For instance, under (3) one may take $\eta = (\log^2 r/r)^{1/3}$. Call an $(r-b)$ -set S *good* if $\alpha_S D \leq \eta$. By Markov's inequality and $\mathbb{E}_S[\alpha_S] = q$, the proportion μ of bad S satisfies

$$\mu \leq \frac{qD}{\eta} = o(1), \quad \text{and consequently} \quad \mu qC = o(1).$$

For every good S , we apply Lemma 4.1 to the link $L_{\overline{G}}(S)$ (note that α_S is the edge density of $L_G(S)$, hence the missing-edge density of $L_{\overline{G}}(S)$ is also α_S). Since $\alpha_S D \leq \eta = o(1)$, the asymptotic part of Lemma 4.1 implies that $\kappa(S)$ is asymptotically at least

$$\exp(-(1+o(1))\alpha_S C),$$

uniformly over good S . Applying Jensen's inequality over the good sets S gives

$$\mathbb{E}[\kappa(S)] \geq (1-\mu) \exp(-(1+o(1))C \mathbb{E}[\alpha_S \mid S \text{ good}]).$$

Since $(1-\mu)\mathbb{E}[\alpha_S \mid S \text{ good}] \leq \mathbb{E}[\alpha_S] = q$, and $\mu qC = o(1)$, we obtain

$$(4) \quad \mathbb{E}[\kappa(S)] \geq \exp(-(1+o(1))qC).$$

We now obtain an upper bound on $\mathbb{E}[\kappa(S)]$ by counting “witnesses”. An edge $e \in G$ can witness a pair (S, T) only if $e = T \cup \{s\}$ for some $s \in S$. For a given e , there are at most r choices for $s \in e$; then $T = e \setminus \{s\}$ is determined, and the remaining $r-b-1$ vertices of S must lie outside e . Hence the total number of ordered pairs (S, T) witnessed by edges of G is at most $r|G| \cdot \binom{n-r}{r-b-1}$.

The total number of ordered pairs (S, T) with $|S| = r-b$, $|T| = r-1$, and $S \cap T = \emptyset$ is $\binom{n}{r-b} \binom{n-r+b}{r-1}$. Thus the witness density is at most

$$\frac{r|G| \binom{n-r}{r-b-1}}{\binom{n}{r-b} \binom{n-r+b}{r-1}} = (r-b)q + o(1).$$

Every S -clique must be witnessed (otherwise $\overline{G}$ would contain a copy of $F_{r-b, r-1}^r$), so inequality (4) implies

$$\exp(-(1+o(1))qC) \leq (r-b)q.$$

Set $y = qC$. Then the inequality becomes $ye^{(1+o(1))y} \geq C/(r-b)$. Solving for y in terms of the Lambert W -function gives

$$q(G) \cdot C = qC = y \geq (1 - o(1)) W\left(\frac{C}{r-b}\right).$$

Since G was arbitrary, taking the infimum over all such deletion graphs and then letting $n \rightarrow \infty$ gives

$$q_{r,r-b} \geq (1 - o(1)) \frac{W(C/(r-b))}{C}.$$

For fixed $b \geq 2$, the asymptotic expansion $W(C/(r-b)) = (b-1) \log r + O_b(\log \log r)$ yields the simplified form

$$q_{r,r-b} \geq (1 - o(1)) \frac{(b-1) \log r}{\binom{r-1}{b}}.$$

The proof is complete. □

5. Concluding remarks

In this note, we have sharpened the asymptotic estimates for the deficits $q_{r,i}$ of the hypergraphs $B_i^{(r)}$ near both ends of the considered sequence of hypergraphs. More precisely, for every fixed $a \geq 1$, we have proved that $q_{r,a+1} = \Theta_a(r^{-a})$, closing the logarithmic gap left by the work of Balogh et al. [1]. At the opposite end, for every fixed codimension $b \geq 2$, we established $q_{r,r-b} = \Theta_b(r^{-b} \log r)$, matching the upper bound of Balogh et al. [1] through a new logarithmic lower bound obtained by clique-counting.

The only remaining case where the logarithmic factor is not yet resolved is the endpoint $i = r-1$ (equivalently $b = 1$). Here $C/(r-b) = 1$, so the Lambert W -term is the constant $W(1)$ and the clique-counting argument yields no logarithmic improvement. It would be interesting to determine the precise asymptotic behavior of $q_{r,r-1}$.

Another direction for future work is to investigate the behavior of $q_{r,i}$ when i grows with r in other regimes, for instance when $i = \Theta(r)$. Our results indicate a sharp change in behavior: for fixed i there is no logarithmic factor, while for $i = r-b$, with fixed $b \geq 2$, a logarithmic factor appears. A natural problem is to determine the precise threshold where this transition occurs.

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Declaration on the use of AI

The authors used AI-assisted tools solely in the proof exploration and presentation of Theorem 1.3. All other mathematical ideas and proofs were developed independently by the authors without the use of AI-assisted tools. In particular, the proof of Theorem 1.2 was inspired by the work of [12], and the underlying approach dates back to November 2024. Each author has carefully checked all mathematical content of the paper and accepts full responsibility for its correctness.

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