arXiv:1904.08126v2 [math.CO] 18 Apr 2019

A unified proof of conjectures on cycle lengths in graphs

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April 22, 2019

Abstract

In this paper, we prove a tight minimum degree condition in general graphs for the existence of paths between two given endpoints, whose lengths form a long arithmetic progression with common difference one or two. This allows us to obtain a number of exact and optimal results on cycle lengths in graphs of given minimum degree, connectivity or chromatic number.

More precisely, we prove the following statements by a unified approach.

1. Every graph G with minimum degree at least $k+1$ contains cycles of all even lengths modulo k; in addition, if G is 2-connected and non-bipartite, then it contains cycles of all lengths modulo k.
2. For all $k \geq 3$, every k-connected graph contains a cycle of length zero modulo k.
3. Every 3-connected non-bipartite graph with minimum degree at least $k+1$ contains k cycles of consecutive lengths.
4. Every graph with chromatic number at least $k+2$ contains k cycles of consecutive lengths.

The first statement is a conjecture of Thomassen, the second is a conjecture of Dean, the third is a tight answer to a question of Bondy and Vince, and the fourth is a conjecture of Sudakov and Verstraëte. All of the above results are best possible.

1 Introduction

The distribution of cycle lengths has been extensively studied in the literature and remains one of the most active and fundamental research areas in graph theory. In this paper, along the line of the previous work [15] of two of the authors, we investigate various relations between cycle lengths and basic graph parameters such as minimum degree. The core of the results in [15] is an optimal bound on the longest sequence of consecutive even cycles in bipartite graphs of given minimum degree. In the current paper, we extend this result from bipartite graphs to general graphs and use it as a primary tool to derive a number of tight results on cycle lengths in related to minimum degree, connectivity and chromatic number. This resolves several conjectures and open problems on cycles of consecutive lengths, cycle lengths modulo a fixed integer and some other related topics. For a thoughtful introduction on the background, we direct interested readers to [15, 26].

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Throughout this section, let k be a fixed but arbitrary positive integer, unless otherwise specified. For a path or a cycle P, the *length* of P, denoted by $|P|$, is the number of edges in P.

1.1 Paths and cycles of consecutive lengths

The study of cycles of consecutive lengths can be dated back to a conjecture of Erdős (see [2]) stating that every graph with minimum degree at least three contains two cycles of lengths differing by one or two. This was solved by Bondy and Vince [2] in the following stronger form: if all but at most two vertices of a graph G have degree at least three, then G contains two cycles whose lengths differ by one or two. Since then, this result has inspired extensive research on its generalization to k cycles of consecutive (even or odd) lengths, including results of Häggkvist and Scott [12], Verstraëte [25], Fan [10], Sudakov and Verstraëte [19], Ma [17], and Liu and Ma [15].

We say that k paths or k cycles $P_1, P_2, \ldots, P_k$ are *admissible* if $|P_1| \geq 2$ and $|P_1|, |P_2|, \ldots, |P_k|$ form an arithmetic progression of length k with common difference one or two. The following generalization of Erdős' conjecture was posted in [15], which was in attempt to attack some related problems.

Conjecture 1.1 (Liu and Ma [15])**.** *Every graph with minimum degree at least $k+1$ contains k admissible cycles.*

By considering the complete graph K_{k+1} or the complete bipartite graph $K_{k,n}$ for any $n \geq k$, we see that the condition for the minimum degree in Conjecture 1.1 is best possible. Being an evidence, Conjecture 1.1 was proved for all bipartite graphs in [15].

One of our main results is the following theorem on admissible paths with two given endpoints, from which Conjecture 1.1 can be inferred as a corollary.

Theorem 1.2. *Let G be a 2-connected graph and x, y distinct vertices of G. If every vertex of G other than x and y has degree at least $k+1$, then there exist k admissible paths from x to y in G.*

The case $k = 1$ of Theorem 1.2 is trivial, and the case $k = 2$ follows from a result of Fan in [10]. We remark that Theorem 1.2 is the main force that will be applied to prove all other results in this paper.

We now show that Conjecture 1.1 is an easy corollary of Theorem 1.2. (For possibly ambiguous notations, we refer readers to Section 2.)

Theorem 1.3. *Every graph G with minimum degree at least $k+1$ contains k admissible cycles.*

Proof. If G is 2-connected, let $B = G$ and choose xy to be an arbitrary edge in G; otherwise, let B be an end-block of G with cut-vertex x and choose $y \in N_G(x) \cap V(B - x)$. Clearly, B is 2-connected and every vertex of B other than x has degree at least $k+1$. By Theorem 1.2, B contains k admissible paths $P_1, ..., P_k$ from x to y. Since each $|P_i| \geq 2$, $P_i \cup xy$ for all $i \in [k]$ form k admissible cycles in G. ∎

Theorem 1.3 improves many previous results such as the results in [10, 12, 15, 25]. As the writeup of this paper was close to complete, we noticed that very recently, Chiba and Yamashita [4] independently proved Theorem 1.3 under an extra condition that G is 2-connected, by using a different approach from this paper.

One can ask for another natural question: what are necessary or sufficient conditions for the existence of k cycles of consecutive lengths? It is clear that such conditions should include non-bipartiteness. This was addressed by Bondy and Vince in [2], where they proved that any non-bipartite 3-connected graph contains two cycles of consecutive lengths. On the other hand, Bondy and Vince showed that the 3-connectivity is necessary by constructing infinitely many non-bipartite 2-connected graphs with arbitrarily large minimum degree, yet not containing two cycles of consecutive lengths.

More generally, Bondy and Vince [2] asked if there exists a (least) function $f(k)$ such that every non-bipartite 3-connected graph with minimum degree at least $f(k)$ contains k cycles of consecutive lengths. The existence of $f(k)$ was confirmed by Fan [10], where he proved $f(k) \leq 3\lceil k/2 \rceil$. On the other hand, the complete graph K_{k+1} shows $f(k) \geq k+1$.

Our next result determines $f(k) = k+1$ and hence provides the optimal answer to the above question of Bondy and Vince.

Theorem 1.4. *Every non-bipartite 3-connected graph with minimum degree at least $k+1$ contains k cycles of consecutive lengths.*

1.2 Cycle lengths modulo a fixed integer

Burr and Erdős initiated the study of cycle lengths modulo an integer k; they conjecture (see [8]) that for odd k there exists a constant c_k such that every graph with average degree at least c_k contains cycles of all lengths modulo k. This was proved by Bollobás [1] and then the value c_k was improved to be $O(k^2)$ by Thomassen in [21,22]. Thomassen also proposed two conjectures in [21] as follows.

Conjecture 1.5 (Thomassen [21])**.** *Every graph with minimum degree at least $k+1$ contains cycles of all even lengths modulo k.*

Conjecture 1.6 (Thomassen [21])**.** *Every 2-connected non-bipartite graph with minimum degree at least $k+1$ contains cycles of all lengths modulo k.*

We remark that 2-connectivity and non-bipartiteness are necessary for even k in Conjecture 1.6; see [15] for explanations. The minimum degree condition in Conjectures 1.5 and 1.6 are tight, since K_{k+1} has no cycle of length 2 modulo k, and $K_{k,n}$ has no cycle of length 2 modulo k for $n \geq k$ and odd k.

Results of Verstraëte [25], Fan [10], Diwan [7] and Ma [17] indicate that the minimum degree at least $O(k)$ suffices for both conjectures. For fixed $m \geq 3$ and large k, Sudakov and Verstraëte [20] determined the optimal minimum degree condition for cycles of length m modulo k up to a constant factor.

In [15], Liu and Ma confirmed both Conjectures 1.5 and 1.6 for even k. They also proved that minimum degree $k+4$ suffices for odd k, and observed that an affirmative of Conjecture 1.1 would imply both Conjectures 1.5 and 1.6 for odd k. Therefore, as an immediate corollary of Theorem 1.3, we obtain the following.

Theorem 1.7. *Conjectures 1.5 and 1.6 hold for any positive integer k.*

We would like to address that very recently, Chiba and Yamashita [4] independently proved Conjecture 1.6. Also very recently, Lyngsie and Merker [16] proved that for odd k, every 3-connected cubic graph of large order contains cycles of all lengths modulo k.

The case of cycles of length zero modulo k has received considerable attention. Thomassen [22] gave a polynomial-time algorithm for finding a cycle of length zero modulo k in any graph or a certificate that no such cycle exists. In 1988, Dean [5] proposed the following conjecture.

Conjecture 1.8 (Dean [5])**.** *For any positive integer $k \geq 3$, every k-connected graph contains a cycle of length zero modulo k.*

We point out that Conjecture 1.8 is tight, as for all odd k and $n \geq k-1$, the complete bipartite graph $K_{k-1,n}$ is $(k-1)$-connected but has no cycles of length zero modulo k. The case $k=3$ in Conjecture 1.8 was proved by Chen and Saito [3], and the case $k=4$ was solved by Dean, Lesniak and Saito [6]. To our best knowledge, this conjecture remains open for any $k \geq 5$ prior to this paper.

Taking of advantage of Theorem 1.2, we are able to resolve Conjecture 1.8 completely.

Theorem 1.9. *Conjecture 1.8 holds for any positive integer $k \geq 3$.*

It turns out that the case $k = 5$ is the most difficult case for our approach. We would like to point out that for $k \geq 6$, in many cases in fact we are able to find k admissible cycles. In particular, our proofs also show that when $k \geq 6$, k-connectivity can force cycles of all even lengths modulo k, unless the residue class 2 modulo k (see Theorem 5.16 for the precise statement).[1]

1.3 Consecutive cycle lengths and chromatic number

There has been extensive research on the relation between the chromatic number and cycle lengths. For a graph G, let $L_e(G)$ and $L_o(G)$ be the set of even and odd cycle lengths in G, respectively. Bollobás and Erdős conjectured and Gyarfás [11] proved that $\chi(G) \leq 2|L_o(G)| + 2$ for any graph G. Mihok and Schiermeyer [18] proved an analog for even cycles that $\chi(G) \leq 2|L_e(G)| + 3$ for any graph G. A strengthening of the above result was obtained in [15], where the number of even cycles $|L_e(G)|$ was replaced by the longest sequence of consecutive even cycle lengths in G. Confirming a conjecture of Erdős [9], Kostochka, Sudakov and Verstraëte [13] proved that every triangle-free graph G with $\chi(G) = k$ contains at least $\Omega(k^2 \log k)$ cycles of consecutive lengths.

For $k \geq 2$, let χ_k be the largest chromatic number of a graph which does not contain k cycles of consecutive lengths. The complete graph K_{k+1} shows that $\chi_k \geq k+1$. In [20], Sudakov and Verstraëte conjectured that the chromatic number of a graph can be bounded by the longest sequence of consecutive cycle lengths from above.

Conjecture 1.10 (Sudakov and Verstraëte [20])**.** *For every integer $k \geq 2$, $\chi_k = k + 1$.*

Using Theorem 1.2, we are able to prove Conjecture 1.10.

Theorem 1.11. *Conjecture 1.10 holds for every integer $k \geq 2$.*

The rest of the paper is organized as follows. In Section 2, we define the notations and include some preliminaries. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorems 1.4 and 1.11 by a unified approach via Theorem 1.2. In Section 5, we prove Theorem 1.9 by extensively applying Theorem 1.2 as well.

2 Preliminaries

All graphs in this paper are finite, undirected, and simple. Let H be a subgraph of a graph G. We say that H and a vertex $v \in V(G) - V(H)$ are *adjacent* in G if v is adjacent in G to some vertex in $V(H)$. Let $N_G(H) := \bigcup_{v \in V(H)} N_G(v) - V(H)$ and $N_G[H] := N_G(H) \cup V(H)$. For a subset S of $V(G)$, $G[S]$ denotes the subgraph induced by S in G, and $G - S$ denotes the subgraph $G[V(G) - S]$. For two distinct vertices x, y of G, we define $G + xy$ to be the graph with $V(G + xy) = V(G)$ and $E(G + xy) = E(G) \cup \{xy\}$. A *clique* in G is a subset of $V(G)$ whose vertices are pairwise adjacent in G. A vertex is a *leaf* in G if it has degree one in G. We say that a path P is *internally disjoint* from H if no vertex of P other than its endpoints is in $V(H)$. For a positive integer k, we write $[k]$ for the set $\{1, 2, ..., k\}$.

For a graph G and a subset S of $V(G)$, we say that a graph G' *is obtained from G by contracting S into a vertex s*, if $V(G') = (V(G) - S) \cup \{s\}$ and $E(G') = E(G - S) \cup \{vs : v \in V(G) - S$ is adjacent to S in $G\}$.

[1] To see the tightness, note that both of K_{k+1} (for even and odd k) and $K_{k,n}$ (for odd k and $n \geq k$) are k-connected and contain cycles of all lengths $2t$ modulo k, except cycles of lengths in the residue class 2 modulo k.

A vertex v of a graph G is a *cut-vertex* of G if $G - v$ contains more components than G. A *block* B in G is a maximal connected subgraph of G such that there exists no cut-vertex of B. So a block is an isolated vertex, an edge or a 2-connected graph. An *end-block* in G is a block in G containing at most one cut-vertex of G. If D is an end-block of G and a vertex x is the only cut-vertex of G with $x \in V(D)$, then we say that D is an *end-block with cut-vertex* x. Let $\mathcal{B}(G)$ be the set of blocks in G and $\mathcal{C}(G)$ be the set of cut-vertices of G. The *block structure* of G is the bipartite graph with bipartition $(\mathcal{B}(G), \mathcal{C}(G))$, where $x \in \mathcal{C}(G)$ is adjacent to $B \in \mathcal{B}(G)$ if and only if $x \in V(B)$. Note that the block structure of any graph G is a forest, and it is connected if and only if G is connected. For notations not defined here, we refer readers to [15].

The next result can be derived from a special case of [10, Theorem 2.5].

Theorem 2.1. *Let G be a 2-connected graph and x, y be distinct vertices of G. If every vertex in G other than x and y has degree at least 3, then there are two admissible paths from x to y in G.*

3 Admissible paths

In this section, we prove Theorem 1.2. We say that (G, x, y) is a *rooted graph* if G is a graph and x, y are two distinct vertices of G. The *minimum degree* of a rooted graph (G, x, y) is $\min\{d_G(v) : v \in V(G) - \{x, y\}\}$. We also say that a rooted graph (G, x, y) is 2*-connected* if $G + xy$ is 2-connected. Theorem 1.2 is an immediate corollary of the following theorem.

Theorem 3.1. *Let k be a positive integer. If (G, x, y) is a 2-connected rooted graph with minimum degree at least $k + 1$, then there exist k admissible paths from x to y in G.*

The rest of this section is devoted to a proof of Theorem 3.1. We need the following lemma.

Lemma 3.2. *Let (H, u, v) be a rooted graph and W be a subset of $V(H)$. Let s be a positive integer. Assume that there exist s admissible paths $P_1, ..., P_s$, where P_i is from u to some $w_i \in W$ for each $i \in [s]$. Assume that for each $i \in [s]$, $H - V(P_i - w_i)$ contains t paths $R^i_1, ..., R^i_t$ from w_i to v such that their lengths form an arithmetic progression with common difference one or two[2]. If $|R^1_j| = \cdots = |R^s_j|$ for every $j \in [t]$, then there exist $s + t - 1$ admissible paths in H from u to v.*

Proof. If each of A and B is an arithmetic progression with common difference one or two, then $A + B = \{a + b : a \in A, b \in B\}$ also forms an arithmetic progression with common difference one or two of size at least $|A| + |B| - 1$. So the set $\{P_i \cup R^i_j : i \in [s], j \in [t]\}$ contains $s + t - 1$ admissible paths between u and v in H. ∎

Throughout the rest of this section, let (G, x, y) be a counterexample of Theorem 3.1 with minimum $|V(G)| + |E(G)|$. That is, for any 2-connected rooted graph (H, u, v) with $|V(H)| + |E(H)| < |V(G)| + |E(G)|$, if the minimum degree of (H, u, v) is at least $\ell + 1$, then there exist ℓ admissible paths from u to v in H.

We now prove a sequence of lemmas and then, according to the order of some specified component (this will be clear after Lemma 3.7), the remaining proof will be divided into two subsections which we handle separately.

Lemma 3.3. *G is 2-connected, x and y are not adjacent in G, and $k \geq 3$.*

[2]Here, we allow that some path R^i_j has length one.

Proof. Theorem 3.1 is obvious when $k = 1$, and it follows from Theorem 2.1 when $k = 2$. So $k \geq 3$. Note that $|V(G)| \geq 4$, for otherwise, $|V(G)| = 3$ and (G, x, y) has minimum degree two and thus $k = 1$, a contradiction.

Since $G + xy$ is 2-connected, G is connected. Suppose that G is not 2-connected. Then there exist a cut-vertex b and two connected subgraphs G_1, G_2 of G on at least two vertices such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{b\}$. We may assume that $x \in V(G_1) - b$, $y \in V(G_2) - b$ and by symmetry, $|V(G_1)| \geq 3$. Then it is straightforward to see that (G_1, x, b) is 2-connected and has minimum degree at least $k+1$. By the minimality of G, there exist k admissible paths in G_1 from x to b. By concatenating each of these paths with a fixed path in G_2 from b to y, we obtain k admissible paths in G from x to y, a contradiction. Therefore G is 2-connected.

Suppose that x is adjacent to y in G. Let $G' = G - xy$. Since G is 2-connected, clearly (G', x, y) is 2-connected and has minimum degree at least $k+1$. By the minimality of G, G' (and thus G) contains k admissible paths from x to y. ∎

Lemma 3.4. *There is no clique in $G - y$ of size at least three containing x, and there is no clique in $G - x$ of size at least three containing y.*

Proof. Suppose to the contrary that there is a clique K in $G - y$ of size at least three containing x. We choose K such that $t = |K|$ is maximum. So $t \geq 3$. Since x and y are non-adjacent by Lemma 3.3, $y \notin K$. So there exists a component C of $G - K$ containing y.

Suppose $V(C) = \{y\}$. Then $N_G(y) \subseteq K - x$. Let $Y = N_G(y) \cap K$, and let $m = |Y|$. Since G is 2-connected, we have $m \geq 2$. For each vertex $v \in K$, let $\mathscr{D}_v$ denote the family of components $D \neq C$ of $G - K$ such that $v \in N_G(D)$. Let $\mathscr{D} = \bigcup_{v \in K} \mathscr{D}_v$, $\mathscr{D}' = \bigcup_{v \in Y} \mathscr{D}_v$ and $\mathscr{D}'' = \mathscr{D} - \mathscr{D}'$. If there is a vertex $v \in K - x$ such that $\mathscr{D}_v = \emptyset$ or there exists some $D \in \mathscr{D}_v$ with $|V(D)| = 1$, then $t \geq k + 1$, from which one can easily find k paths of lengths $2, 3, \ldots, k+1$ from x to y in $G[K \cup \{y\}]$, a contradiction. So for every $v \in K - x$, $\mathscr{D}_v \neq \emptyset$ and $|V(D)| \geq 2$ for every $D \in \mathscr{D}_v$.

Suppose that there exists some $D \in \mathscr{D}' \backslash \mathscr{D}_x$. Let v be a vertex in $N_G(D) \cap Y$ such that $D \in \mathscr{D}_v$. Since G is 2-connected, $N_G(D) - \{v\} \neq \emptyset$. Let G_1 be the graph obtained from $G[N_G[D]]$ by contracting $N_G(D) - \{v\}$ into a new vertex u_1. Since $D \notin \mathscr{D}_x$, we see $|N_G(D) - \{v\}| \leq t - 2$. So (G_1, u_1, v) is 2-connected and has minimum degree at least $k - t + 4$. By the minimality of G, G_1 contains $k - t + 3$ admissible paths from u_1 to v. Hence, $G[V(D) \cup K]$ contains $k - t + 3$ admissible paths P_i from a vertex $p_i \in N_G(D) - v$ to v internally disjoint from K for $i \in [k - t + 3]$. Since K is a clique, $K - v$ contains $t - 2$ paths from x to p_i with lengths $1, 2, \ldots, t-2$, respectively. By Lemma 3.2, by concatenating each of these paths with $P_i \cup \{vy\}$, we obtain k admissible paths from x to y in G, a contradiction.

Hence $\mathscr{D}' \subseteq \mathscr{D}_x$. Let G_2 be the graph obtained from $G - y$ by contracting Y into a new vertex u_2. Let $K' = G_2[(K - Y) \cup \{u_2\}]$. Then K' is a complete graph of order $t - m + 1 \geq 2$ in G_2. Any component $D \neq C$ of $G - K$ in G is also a component of $G_2 - V(K')$ in G_2. If $D \in \mathscr{D}'$, then D is adjacent in G_2 to both x and u_2; otherwise $D \in \mathscr{D}''$ and D is adjacent to at least two vertices of $K' - u_2$ in G_2 since G is 2-connected. Note that there exists a 2-connected end-block G'_2 containing x and u_2. So (G'_2, x, u_2) is 2-connected and has minimum degree at least $k - m + 2$. By the minimality of G, G_2 contains $k - m + 1$ admissible paths from x to u_2. Hence, $G - y$ contains $k - m + 1$ admissible paths P_i from x to a vertex $p_i \in Y$ for $i \in [k - m + 1]$ internally disjoint from Y. Since $G[Y \cup \{y\}]$ is complete, $G[Y \cup \{y\}]$ contains m paths from p_i to y with lengths $1, 2, \ldots, m$, respectively. By Lemma 3.2, we obtain k admissible paths from x to y in G, a contradiction.

Hence $|V(C)| \geq 2$. If C is 2-connected, then let $B = C$ and $b = y$; otherwise let B be an end-block of C with cut-vertex b such that $y \notin V(B) - \{b\}$. Suppose B is an edge vb. Then v has at least k neighbours in K. Since K is a clique, we can find k consecutive paths from x to v in $G[K \cup \{v\}]$.

Concatenating each of these paths with a fixed path in C from v to y, we find k admissible paths from x to y, a contradiction.

Hence B is 2-connected. Let P be a path in $C - V(B - b)$ from b to y. Since G is 2-connected, we have $N_G(B - b) \cap K \neq \emptyset$.

Suppose that $N_G(B - b) \cap (K - \{x\}) \neq \emptyset$. Let G_3 be the graph obtained from $G[V(B) \cup (N_G(B - b) \cap (K - \{x\}))]$ by contracting $N_G(B - b) \cap (K - \{x\})$ into a vertex u_3. By the maximality of K, every vertex in $V(B - b)$ is adjacent to at most $t - 1$ vertices in K. Then (G_3, u_3, b) is 2-connected and has minimum degree at least $k - t + 3$. By the minimality of G, G_3 contains $k - t + 2$ admissible paths from u_3 to b. Hence, $G[V(B) \cup (N_G(B - b) \cap (K - \{x\}))]$ contains $k - t + 2$ admissible paths P_i from some vertex $p_i \in N_G(B - b) \cap (K - \{x\})$ to b internally disjoint from K for $i \in [k - t + 2]$. Note that for each i, $G[K]$ contains $t - 1$ paths from x to p_i with lengths $1, 2, \ldots, t - 1$, respectively. By Lemma 3.2, by concatenating each of these paths with $P_i \cup P$, we obtain k admissible paths from x to y in G, a contradiction.

Therefore $N_G(B - b) \cap K = \{x\}$. Then the rooted graph $(G[V(B) \cup \{x\}], x, b)$ is 2-connected and has minimum degree at least $k + 1$. By the minimality of G, $G[V(B) \cup \{x\}]$ contains k admissible paths from x to b. By concatenating each of these paths with P, we obtain k admissible paths from x to y, a contradiction.

This proves that there is no clique in $G - y$ of size at least three containing x. Similarly, there is no clique in $G - x$ of size at least three containing y, completing the proof of Lemma 3.4. ∎

In the rest of this section, by symmetry between x and y, we may assume that $d_G(x) \leq d_G(y)$.

Lemma 3.5. *$G - y$ has a cycle of length four containing x.*

Proof. Suppose that x is not contained in any cycle of length four in $G - y$. Then

$$|N_G(v) \cap N_G(x)| \leq 1 \text{ for every } v \in V(G) - \{x, y\}. \tag{1}$$

Let G_1 be the graph obtained from G by contracting $N_G[x]$ into a new vertex x_1. By (1), G_1 is connected and the minimum degree of (G_1, x_1, y) is at least $k + 1$. If G_1 is not 2-connected, then x_1 is the unique cut-vertex of G_1 and we let B be the end-block of G_1 containing x_1 and y; otherwise G_1 is 2-connected and let $B = G_1$.

Suppose that B is not an edge. Then (B, x_1, y) is 2-connected and has minimum degree at least $k + 1$. By the minimality of G, B contains k admissible paths from x_1 to y. Then $G - x$ contains k admissible paths P_i from a vertex $p_i \in N_G(x)$ to y for all $i \in [k]$. By concatenating each of these paths with xp_i, we obtain k admissible paths from x to y in G, a contradiction.

Therefore B is an edge. Since $d_G(x) \leq d_G(y)$, we conclude that $N_G(x) = N_G(y)$. By Lemma 3.4 and $k \geq 3$, we see $V(G) \neq N_G[x] \cup \{y\}$. So there exists a component D of $G - N_G(x)$ not containing x and y. Since G is 2-connected, we have $|N_G(D)| \geq 2$. Fix a vertex u in $N_G(D)$. Let G_2 be the graph obtained from $G[N_G[D]]$ by contracting $N_G(D) - \{u\}$ into a new vertex v. Then by (1), (G_2, u, v) is 2-connected and has minimum degree at least $k + 1$. By the minimality of G, G_2 contains k admissible paths from u to v. So $G - \{x, y\}$ contains k admissible paths P_i from u to some vertex $p_i \in N_G(x) - \{u\}$ for $i \in [k]$. By concatenating each of these paths with xu and $p_i y$, we obtain k admissible paths from x to y in G, a contradiction. ∎

Lemma 3.6. *Let $C = xx_1ax_2x$ be a cycle of length four in $G - y$. Then every vertex in $V(G) - (V(C) \cup \{y\})$ is not adjacent in G to all of x_1, x_2, a.*

Proof. Suppose to the contrary that there exists a vertex $v \in V(G) - (V(C) \cup \{y\})$ adjacent in G to all of x_1, x_2, a. Let K be a maximal clique in $G - \{x, y, x_1, x_2\}$ such that $a \in K$ and every vertex in K is adjacent to both of x_1 and x_2. So $t = |K| \geq 2$. We have the following two facts:

(a) for any $u \in K$, $G[V(C) \cup K]$ contains $t+1$ admissible paths from x to u of lengths $2, 3, ..., t+2$, respectively;

(b) for any $i \in [2]$, $G[V(C) \cup K]$ contains t admissible paths from x to x_i of lengths $3, 4, ..., t+2$, respectively.

Let F be the component of $G - (V(C) \cup K)$ containing y.

Suppose $V(F) = \{y\}$. Then $N_G(y) \subseteq V(K) \cup \{x_1, x_2\}$. Since G is 2-connected, we have $|N_G(y)| \geq 2$. If $N_G(y) \neq \{x_1, x_2\}$, then there exists a triangle containing y in $G - x$, contradicting Lemma 3.4. Therefore $N_G(y) = \{x_1, x_2\}$. Since $d_G(x) \leq d_G(y)$, $N_G(x) = N_G(y) = \{x_1, x_2\}$. Let $G' = G - \{x, y\}$. It is clear that (G', x_1, x_2) is 2-connected and has minimum degree at least $k+1$. By the minimality of G, G' contains k admissible paths from x_1 to x_2. By concatenating each of these paths with xx_1 and x_2y, G contains k admissible paths from x to y, a contradiction.

So $|V(F)| \geq 2$. If F is 2-connected, let $B = F$ and $b = y$; otherwise let B be an end-block of F with cut-vertex b such that $y \notin V(B) - b$.

Suppose that B is an edge vb. If v is adjacent to x, then by Lemma 3.4, $N_G(v) \cap \{x_1, x_2\} = \emptyset$ and thus $t \geq |N_G(v) \cap K| \geq k - 1$. If v is not adjacent to x, then by the maximality of K, it holds that $t + 1 \geq |N_G(v) \cap (K \cup \{x_1, x_2\})| \geq k \geq 3$. So in both cases, we have $t \geq k - 1$ and there exists some $u \in N_G(v) \cap K$. By (a), there exist k admissible paths from x to y in G, a contradiction.

Therefore B is 2-connected. Let P be a path in $F - V(B - b)$ from b to y.

Suppose that $N_G(B - b) \cap K \neq \emptyset$. Let G_1 be the graph obtained from $G[V(B) \cup (N_G(B - b) \cap K)]$ by contracting $N_G(B - b) \cap K$ into a new vertex u_1. Let us consider the degree of any $v \in V(B - b)$ in G_1. If v is adjacent to both x_1, x_2, then by Lemma 3.4 and the maximality of K, v is not adjacent to x and is adjacent to at most $t - 1$ vertices in K, implying that $d_{G_1}(v) \geq k + 1 - t$; if v is adjacent to exactly one of x_1, x_2, then v is not adjacent to x and thus $d_{G_1}(v) \geq k + 1 - t$; if v is adjacent to none of x_1, x_2, then v may be adjacent to x and all vertices in K, which also shows that $d_{G_1}(v) \geq k + 1 - t$. So (G_1, u_1, b) is 2-connected and has minimum degree at least $k - t + 1$. By the minimality of G, G_1 contains $k - t$ admissible paths from u_1 to b. Hence, G contains $k - t$ admissible paths P_i from a vertex $p_i \in N_G(B - b) \cap K$ to b for $i \in [k - t]$ internally disjoint from $V(C) \cup K$. By (a), $G[V(C) \cup K]$ contains $t + 1$ paths from x to p_i with lengths $2, 3, \ldots, t + 2$, respectively. By Lemma 3.2, concatenating each of these path with $P_i \cup P$ leads to k admissible paths from x to y, a contradiction.

Therefore, $N_G(B - b) \subseteq \{x, x_1, x_2, b\}$. Since G is 2-connected, $N_G(B - b) \cap \{x, x_1, x_2\} \neq \emptyset$. If $x_1 \in N_G(B - b)$, then $(G[B \cup \{x_1\}], x_1, b)$ is 2-connected and has minimum degree at least k by Lemma 3.4. By the minimality of G, $G[B \cup \{x_1\}]$ contains $k - 1$ admissible paths from x_1 to b. By (b), there are t admissible paths from x to x_1 in $G[C \cup K]$. By concatenating each of the above paths with P, we obtain $k - 1 + t - 1 \geq k$ admissible paths from x to y in G, a contradiction. This shows that $x_1, x_2 \notin N_G(B - b)$. So $N_G(B - b) = \{x, b\}$. Then $(G[B \cup \{x\}], x, b)$ is a 2-connected rooted graph with minimum degree at least $k + 1$, from which one can obtain k admissible paths from x to y by the minimality of G. This completes the proof of Lemma 3.6. ∎

Lemma 3.7. *There exist a positive integer s and an induced complete bipartite subgraph Q with bipartition (Q_1, Q_2) in G satisfying that*

1. $x \in Q_2, y \notin V(Q), |Q_1| \geq |Q_2| = s + 1 \geq 2$, *and*

2. *for every $v \in V(G) - (V(Q) \cup \{y\})$,*

   (a) $|N_G(v) \cap Q| \leq s + 1$, $|N_G(v) \cap Q_1| \leq s + 1$, $|N_G(v) \cap Q_2| \leq s$, and

   (b) if v is adjacent to both of Q_1 and Q_2, then $|N_G(v) \cap Q_1| = 1$.

Proof. By Lemma 3.5 there exists a 4-cycle in $G-y$ containing x. Thus there exists a complete bipartite subgraph Q of $G-y$ with bipartition (Q_1, Q_2) such that $x \in Q_2, y \notin V(Q)$ and $|Q_1| \geq |Q_2| \geq 2$. We choose Q so that $|Q_2|$ is maximum and subject to this, $|Q_1|$ is maximum. Let s be a positive integer such that $|Q_2| = s+1$.

We claim that such Q and s satisfy the conclusion of this lemma. Statement 2(b) holds by Lemmas 3.4 and 3.6. By the choice of Q, for every $v \in V(G) - (V(Q) \cup \{y\})$, $|N_G(v) \cap Q_1| \leq s+1$ and $|N_G(v) \cap Q_2| \leq s$. This together with Statement 2(b), we know Statement 2(a) holds. By Lemmas 3.4 and 3.6, Q is an induced subgraph in G. The proof of Lemma 3.7 is completed. ∎

Throughout the remaining of the section, Q and s denote the induced complete bipartite subgraph and the positive integer s promised by Lemma 3.7, and let C be the component of $G - V(Q)$ containing y.

There are two possibilities for the size of C: $|V(C)| = 1$ or $|V(C)| \geq 2$. We now split the rest of the proof into two subsections based on these two cases. We shall derive a contradiction in each subsection and hence show that G is a not a counterexample to complete the proof of Theorem 3.1.

3.1 $|V(C)| = 1$

In this case we have $V(C) = \{y\}$. By Lemma 3.3, $xy \notin E(G)$. So by Lemma 3.4, y is adjacent to exactly one of Q_1 and Q_2. Since $d_G(y) \geq d_G(x)$, we derive that $N_G(x) = N_G(y) = Q_1$ and so $G[V(Q) \cup \{y\}]$ is complete bipartite. If $s \geq k-1$, then $G[V(Q) \cup \{y\}]$ contains k admissible paths from x to y of lengths $2, 4, \ldots, 2k$, respectively, a contradiction.

So $s \leq k-2$. This shows that $V(G) \neq V(Q) \cup \{y\}$, for otherwise every vertex in Q_1 has degree $s+2 \leq k$ in G. Hence there exists a component in $G - (V(Q) \cup \{y\})$.

Let D be an arbitrary component of $G - (V(Q) \cup \{y\})$. If there exists a vertex v of D of degree at most one in D, then by Lemma 3.7, $s+1 \geq |N_G(v) \cap V(Q)| \geq k$, a contradicting that $s \leq k-2$. So $|V(D)| \geq 2$ and every end-block of D is 2-connected. In addition, $N_G(x) = Q_1$, so $x \notin N_G(D)$.

We claim that $N_G(D) \cap Q_1 \neq \emptyset$. Suppose to the contrary that $N_G(D) \cap Q_1 = \emptyset$. Since G is 2-connected and $x \notin N_G(D)$, we have $|N_G(D) \cap (Q_2 - \{x\})| \geq 2$. Let u_1 be a vertex in $N_G(D) \cap (Q_2 - \{x\})$. Let G_1 be the graph obtained from $G[N_G[D]]$ by contracting $N_G(D) \cap (Q_2 - \{x, u_1\})$ into a new vertex v_1. Therefore (G_1, u_1, v_1) is 2-connected and has minimum degree at least $k-s+3$. By the minimality of G, G_1 contains $k-s+2$ admissible paths from u_1 to v_1. Hence, $G - \{x, y\}$ contains $k-s+2$ admissible paths P_i from u_1 to some vertex $p_i \in Q_2 - \{x, u_1\}$ internally disjoint from $V(Q)$ for $i \in [k-s+2]$. Let w be a vertex in Q_1. Since Q is complete bipartite, $Q - \{u_1, w\}$ contains $s-1$ paths from x to p_i of lengths $2, 4, \ldots, 2s-2$. By Lemma 3.2, concatenating each of these paths with P_i and u_1wy leads to k admissible paths from x to y, a contradiction.

We claim that $N_G(D) \cap (Q_2 - \{x\}) \neq \emptyset$. Suppose to the contrary that $N_G(D) \cap (Q_2 - \{x\}) = \emptyset$. Since G is 2-connected and $x \notin N_G(D)$, $|N_G(D) \cap Q_1| \geq 2$. Let u_2 be a vertex in $N_G(D) \cap Q_1$. Let G_2 be the graph obtained from $G[N_G[D]]$ by contracting $N_G(D) \cap (Q_1 - \{u_2\})$ into a new vertex v_2. If $|Q_1| \geq s+2$, then let $\epsilon = 0$; if $|Q_1| = s+1$, then let $\epsilon = 1$. So (G_2, u_2, v_2) is 2-connected and has minimum degree at least $k-s+1+\epsilon$. By the minimality of G, G_2 contains $k-s+\epsilon$ admissible paths from u_2 to v_2. Hence, $G - \{x, y\}$ contains $k-s+\epsilon$ admissible paths P_i from u_2 to some vertex $p_i \in Q_1 - u_2$ internally disjoint from $V(Q)$ for all $i \in [k-s+\epsilon]$. Since Q is complete bipartite, $Q - u_2$ contains $s+1-\epsilon$ paths from x to p_i of lengths $1, 3, \ldots, 2s+1$. By Lemma 3.2, concatenating each of these paths with P_i and u_2y leads to k admissible paths from x to y, a contradiction. This proves the claim.

Now we claim that there is a matching of size two in G between $V(D)$ and Q_1. Suppose that there is no matching of size two in G between $V(D)$ and Q_1. Then either $|N_G(D) \cap Q_1| = 1$ or $|N_G(Q_1) \cap V(D)| =$

1. In the former case, let u_3 and w_3 be the unique vertex in $N_G(D) \cap Q_1$; in the latter case, let u_3 be the unique vertex in $N_G(Q_1) \cap V(D)$ and let w_3 be a vertex in Q_1 adjacent in G to u_3. Recall that $N_G(D) \cap (Q_2 - \{x\}) \neq \emptyset$. Let G_3 be the graph obtained from $G[D \cup \{u_3\} \cup (N_G(D) \cap (Q_2 - \{x\}))]$ by contracting $N_G(D) \cap (Q_2 - \{x\})$ into a new vertex v_3. Then (G_3, u_3, v_3) is 2-connected and has minimum degree at least $k-s+2$. By the minimality of G, G_3 contains $k-s+1$ admissible paths from u_3 to v_3. Hence, $G-y$ contains $k-s+1$ admissible paths P_i from u_3 to some vertex $p_i \in Q_2 - \{x\}$ internally disjoint from $V(Q)$ for $i \in [k-s+1]$. Since Q is complete bipartite, $Q - w_3$ contains s paths from x to p_i of lengths $2, 4, \dots, 2s$. By Lemma 3.2, concatenating each of these paths with P_i and u_3w_3y, we obtain k admissible paths from x to y in G. This contradiction completes the proof of the claim.

Suppose that D is not 2-connected and there exists an end-block B of D with cut-vertex b such that $N_G(B-b) \cap V(Q) \subseteq Q_2 - \{x\}$. Recall that every end-block of D is 2-connected. So B is 2-connected. Let G_4 be the graph obtained from $G[V(B) \cup (N_G(B-b) \cap (Q_2 - \{x\}))]$ by contracting $N_G(B-b) \cap (Q_2 - \{x\})$ into a new vertex v_4. Then (G_4, b, v_4) is 2-connected and has minimum degree at least $k-s+2$. By the minimality of G, G_4 contains $k-s+1$ admissible paths from b to v_4. Hence, G contains $k-s+1$ admissible paths P_i from b to some vertex $p_i \in Q_2 - \{x\}$ internally disjoint from $V(Q)$ for $i \in [k-s+1]$. Since $N_G(D) \cap Q_1 \neq \emptyset$, there exists a path R in $G[(D - V(B-b)) \cup Q_1]$ from b to some vertex $a \in Q_1$ internally disjoint from $V(B) \cup V(Q)$. Since Q is complete bipartite, $Q - a$ contains s paths from x to p_i with fixed lengths $2, 4, \dots, 2s$. By Lemma 3.2, concatenating each of these paths with $P_i \cup R \cup ay$ leads to k admissible paths from x to y in G, a contradiction.

Therefore, either D is 2-connected, or every end-block B of D with cut-vertex b satisfies that $N_G(B-b) \cap Q_1 \neq \emptyset$.

We claim $|Q_1| = s+1$. Suppose to the contrary that $|Q_1| \geq s+2$. Recall that there exists a matching M of size two in G between $V(D)$ and Q_1. So there exists a vertex $u_5 \in N_G(D) \cap Q_1$ incident with an edge in M such that $N_G(D) \cap (Q_1 - \{u_5\}) \neq \emptyset$. Let G_5 be the graph obtained from $G[V(D) \cup (N_G(D) \cap Q_1)]$ by contracting $N_G(D) \cap (Q_1 - \{u_5\})$ into a new vertex v_5. Since M is a matching of size two in G between $V(D)$ and Q_1, if D is 2-connected, then (G_5, u_5, v_5) is 2-connected; if D is not 2-connected, then every end-block of D has a non-cut vertex adjacent in G_5 to one of u_5, v_5, so (G_5, u_5, v_5) is 2-connected. Moreover, by Lemma 3.7, G_5 has minimum degree at least $k-s+1$. By the minimality of G, G_5 contains $k-s$ admissible paths from u_5 to v_5. Hence, G contains $k-s$ admissible paths P_i from u_5 to $p_i \in V(Q_1 - u_5)$ internally disjoint from $V(Q)$ for $i \in [k-s]$. Since $|Q_1| \geq s+2$, $Q - u_5$ contains $s+1$ paths from x to p_i of lengths $1, 3, \dots, 2s+1$. By Lemma 3.2, concatenating each of these paths with $P_i \cup u_5y$, we obtain k admissible paths from x to y in G, a contradiction. This proves that $|Q_1| = s+1$.

Suppose that $s = 1$. Denote Q_1 by $\{u, v\}$. As $N_G(x) = N_G(y) = Q_1$, it is clear that $(G - \{x, y\}, u, v)$ is 2-connected and has minimum degree at least $k+1$. By the minimality of G, there are k admissible paths from u to v in $G - \{x, y\}$, which can be easily extended to k admissible paths from x to y in G, a contradiction.

Therefore we have $s \geq 2$. Let w be a vertex in $Q_2 - x$. Since $s \leq k-2$, w is adjacent in G to least two vertices in $V(G) - (V(Q) \cup \{y\})$. So there exists a non-empty set $\mathscr{D}$ of all components in $G - (V(Q) \cup \{y\})$ adjacent to w. Since every member of $\mathscr{D}$ is a component of $G - (V(Q) \cup \{y\})$, for every $D' \in \mathscr{D}'$, either D' is 2-connected or every end-block of D' has a non-cut-vertex adjacent to Q_1.

Let $H = \bigcup_{D' \in \mathscr{D}} V(D')$. Since every member D' of $\mathscr{D}$ is a component of $G - (V(Q) \cup \{y\})$, there exists a matching $M_{D'}$ of size two in G between $V(D')$ and Q_1, so we have $|N_G(H) \cap Q_1| \geq 2$. Let u_6 be a vertex in $N_G(H) \cap Q_1$ incident with an edge in M_{D_0} for some $D_0 \in \mathscr{D}$. Let G_6 be the graph obtained from $G[N_G[H]]$ by deleting $Q_2 - \{x, w\}$ and contracting $Q_1 - u_6$ into a new vertex v_6.

We claim that (G_6, u_6, v_6) is 2-connected. Let $G' = G_6 + u_6v_6$. We shall prove that G' is 2-

connected. It suffices to show that for every $D' \in \mathscr{D}$, $G'[V(D') \cup \{u_6, v_6, w\}]$ is 2-connected. Suppose to the contrary that there exists $D' \in \mathscr{D}$ such that $G'[V(D') \cup \{u_6, v_6, w\}]$ is not 2-connected. Note that $G'[\{u_6, v_6, w\}]$ is isomorphic to K_3 and every end-block of D' is adjacent to $\{u_6, v_6\}$. So G' is connected, and there exists a cut-vertex of c of $G'[V(D') \cup \{u_6, v_6, w\}]$ such that either $c \in \{u_6, v_6, w\}$ or c is a cut-vertex of D'. Since $V(D')$ is adjacent to w and $\{u_6, v_6\}$, if $c \in \{u_6, v_6, w\}$, then the component of $G'[V(D') \cup \{u_6, v_6, w\}] - c$ containing $V(D')$ also contains $\{u_6, v_6, w\}$, so this component contains equals $G'[V(D') \cup \{u_6, v_6, w\}] - c$, a contradiction. So c is a cut-vertex of D'. But every component of $D' - c$ contains a non-cut-vertex of D' in an end-block of D', so it is adjacent to $\{u_6, v_6\}$, and hence there exists a component of $G'[V(D') \cup \{u_6, v_6, w\}] - c$ contains every component of $D' - c$ and $\{u_6, v_6, w\}$. This shows that $G'[V(D') \cup \{u_6, v_6, w\}] - c$ is connected, a contradiction. So (G_6, u_6, v_6) is 2-connected.

Now we show the minimum degree of (G_6, u_6, v_6) is at least $k - s + 2$. Let $v \in V(G_6) - \{u_6, v_6, w\}$. Then either $N_G(v) \cap Q \subseteq Q_1$, $N_G(v) \cap Q \subseteq Q_2 - x$ or v is adjacent to both of Q_1 and $Q_2 - x$. By Lemma 3.7, in either case we can derive that $d_{G_6}(v) \geq k - s + 2$. In addition, since $|Q_1| = s + 1$, $d_{G_6}(w) \geq k - s + 2$. Hence, indeed the minimum degree of (G_6, u_6, v_6) is at least $k - s + 2$.

By the minimality of G, G_6 contains $k - s + 1$ admissible paths from u_6 to v_6. Hence, $G[N_G[H]]$ contains $k - s + 1$ admissible paths P_i from u_6 to some vertex $p_i \in Q_1 - u_6$ internally disjoint from $V(Q) - w$ for $i \in [k-s+1]$. Note that P_i possibly contains w. Since Q is complete bipartite, $Q - \{u_6, w\}$ contains s paths from x to p_i of lengths $1, 3, \dots, 2s - 1$. By Lemma 3.2, by concatenating each of these paths with $P_i \cup u_6 y$, we obtain k admissible paths from x to y in G, a contradiction. This finishes the proof of Subsection 3.1.

3.2 $|V(C)| \geq 2$

We first show that no vertex in $C - y$ has degree one in C. Suppose to the contrary that there exists $v \in V(C - y)$ with degree one in C. By Lemma 3.7, $s + 1 \geq |N_G(v) \cap V(Q)| \geq k$. If $N_G(v) \cap Q_1 \neq \emptyset$, then there are k paths from x to v in $G[Q \cup \{v\}]$ of lengths $2, 4, \dots, 2k$. If $N_G(v) \cap Q_1 = \emptyset$, then $N_G(v) \cap V(Q) \subseteq Q_2$, so $s \geq |N_G(v) \cap V(Q)| \geq k$ by Lemma 3.7, and hence there are k paths from x to v in $G[Q \cup \{v\}]$ of lengths $3, 5, \dots, 2k + 1$. In both cases, by concatenating each of these path with a path from v to y in C, we obtain k admissible paths from x to y in G, a contradiction. So no vertex in $C - y$ has degree one in C. In particular, every end-block of C is 2-connected, except possibly an end-block consisting of y and its unique neighbor in C.

We say a block of C is a *feasible* block if it is an end-block of C such that y is not a non-cut-vertex of this block. Note that feasible blocks exist, since either C has no cut-vertex, or C contains at least two end-blocks.

Let B be an arbitrary feasible block. If C is 2-connected, then $b = y$; otherwise let b be the cut-vertex of C contained in B.

We claim that $N_G(B - b) \subseteq Q_2 \cup \{b\}$. Suppose to the contrary that $N_G(B - b) \cap Q_1 \neq \emptyset$. Let G_1 be the graph obtained from $G[V(B) \cup (N_G(B - b) \cap Q_1)]$ by contracting $N_G(B - b) \cap Q_1$ into a new vertex x_1. So (G_1, x_1, b) is 2-connected and has minimum degree at least $k - s + 1$ by Lemma 3.7. By the minimality of G, G_1 has $k - s$ admissible paths from x_1 to b. Therefore there are $k - s$ admissible paths P_i from some vertex $p_i \in N_G(B - b) \cap Q_1$ to b internally disjoint from $V(Q)$ for $i \in [k - s]$. Also Q contains $s + 1$ paths from x to p_i of fixed lengths $1, 3, \dots, 2s + 1$. By Lemma 3.2, by concatenating each of these paths with P_i and a fixed path in $C - V(B - b)$ from b to y, we obtain k admissible paths from x to y in G, a contradiction. This proves $N_G(B - b) \subseteq Q_2 \cup \{b\}$.

Next we prove that $s = 1$. Suppose to the contrary that $s \geq 2$. Let R be a path in $C - V(B - b)$ from b to y. If $N_G(B - b) \cap Q_2 = \{x\}$, then $(N_G[B], x, b)$ is 2-connected and has minimum degree at least $k + 1$, so by the minimality of G, $G[V(B) \cup \{x\}]$ contains k paths from x to b, and hence concatenating each of them with R leads to k admissible paths from x to y in G, a contradiction. Therefore, $N_G(B-b) \cap Q_2 \neq \{x\}$. Let G_2 be the graph obtained from $G[V(B) \cup (N_G(B-b) \cap (Q_2 - \{x\}))]$ by contracting $N_G(B-b) \cap (Q_2 - \{x\})$ into a new vertex x_2. So (G_2, x_2, b) is 2-connected and has minimum degree at least $k-s+2$. By the minimality of G, G_2 has $k-s+1$ paths from x_2 to b. So there are $k-s+1$ paths P_i from some vertex $p_i \in N_G(B-b) \cap (Q_2 - \{x\})$ to b internally disjoint from $V(Q)$ for $i \in [k-s+1]$. Also Q contains s paths from x to p_i of lengths $2, 4, \ldots, 2s$. By Lemma 3.2, concatenating each of these paths with P_i and R, we obtain k admissible paths from x to y, a contradiction.

Hence $s = 1$. We denote Q_2 by $\{x, a\}$.

We claim that $N_G(B-b) \cap V(Q) = \{x, a\}$. Suppose to the contrary that $N_G(B-b) \cap V(Q) \neq \{x, a\}$. Recall that $N_G(B-b) \subseteq Q_2 \cup \{b\}$. So either $N_G(B-b) \cap V(Q) = \{a\}$ or $\{x\}$. In the former case, $(G[V(B) \cup \{a\}], a, b)$ is 2-connected and has minimum degree at least $k+1$. In the latter case, $(G[V(B) \cup \{x\}], x, b)$ is 2-connected and has minimum degree at least $k+1$. By the minimality of G, $G[V(B) \cup (N_G(B-b) \cap V(Q_2))]$ contains k paths from the unique vertex in $N_G(B-b) \cap V(Q_2)$ to b internally disjoint from $V(Q)$. These paths, together with a path in Q from x to $N_G(B-b) \cap V(Q_2)$ and a fixed path in $C - V(B-b)$ from b to y, lead to k admissible paths from x to y in G, a contradiction. This proves $N_G(B-b) \cap V(Q) = \{x, a\}$.

Case 1. $N_G(C-y) \cap Q_1 = \emptyset$.

Since $N_G(B-b) \cap V(Q) = Q_2 = \{a, x\}$, we have that $(G[V(B) \cup \{a\}], a, b)$ is 2-connected and has minimum degree at least k. By the minimality of G, $G[V(B) \cup \{a\}]$ contains $k-1$ admissible paths $P_1, \ldots, P_{k-1}$ from a to b. Let Y be a path from b to y in $C - V(B-b)$.

For any $v \in Q_1$, if $N_G(v) \subseteq Q_2 \cup \{y\}$, then the degree of v in G is three, so $k \leq 2$, contradicting Lemma 3.3. Therefore, there exists a component D of $G - V(Q \cup C)$ adjacent to v. Since $N_G(C-y) \cap Q_1 = \emptyset$, $N_G(Q_1) \cap V(C) \subseteq \{y\}$. So $(G - V(C), x, a)$ is 2-connected and has minimum degree at least k. By the minimality of G, there are $k-1$ admissible paths $R_1, ..., R_{k-1}$ from x to a in $G - V(C)$. Then by Lemma 3.2, $R_i \cup P_j \cup Y$ for all $i, j \in [k-1]$ give at least $2k-3 \geq k$ admissible paths from x to y, a contradiction. This completes the proof of Case 1.

Case 2. $N_G(C-y) \cap Q_1 \neq \emptyset$.

If C is 2-connected, then $C = B$ and $y = b$, contradicting $N_G(B-b) \cap V(Q) = \{x, a\}$. So C is not 2-connected. Let $B_1, B_2, \ldots, B_t$ be all end-blocks of C with cut-vertices $b_1, b_2, \ldots, b_t$, respectively. Note that $t \geq 2$.

Suppose that $y \notin \bigcup_{i=1}^{t}(V(B_i) - \{b_i\})$. So for every $i \in [t]$, B_i is a feasible block, and hence $N_G(B_i - b_i) \cap V(Q) = \{x, a\}$ which is disjoint from Q_1. Since $N_G(C-y) \cap Q_1 \neq \emptyset$, there is a vertex w in $V(C) - (\bigcup_{i=1}^{t}(V(B_i) - \{b_i\}) \cup \{y\})$ such that $N_G(w) \cap Q_1 \neq \emptyset$. Let c be a vertex in $N_G(w) \cap Q_1$. Using the block structure of C, there exist two end-blocks B_m, B_n for $1 \leq m < n \leq t$, such that there are two disjoint paths L_1, L_2 from b_m to w and from b_n to y internally disjoint from $V(B_n) \cup V(B_m)$, respectively. Since B_m and B_n are feasible, $N_G(B_m - b_m) \cap V(Q) = \{x, a\} = N_G(B_n - b_n) \cap V(Q)$. So both of $(G[V(B_m) \cup \{x\}], x, b_m)$ and $(G[V(B_n) \cup \{a\}], a, b_n)$ are 2-connected and have minimum degree at least k. By the minimality of G, there are $k-1$ admissible paths $P_1, \ldots, P_{k-1}$ from x to b_m in $G[V(B_m) \cup \{x\}]$; and there are $k-1$ admissible paths $R_1, \ldots, R_{k-1}$ from a to b_n in $G[V(B_n) \cup \{a\}]$. By Lemma 3.3, $k \geq 3$. So the set $\{P_i \cup L_1 \cup wca \cup R_j \cup L_2 : i, j \in [k-1]\}$ contains at least $2k-3 \geq k$ admissible paths from x to y in G, a contradiction.

So there exists an end-block, say B_t, of C such that $y \in V(B_t) - \{b_t\}$. We say that a block H of C other than B_1 is a *hub* if H is 2-connected and contains at most two cut-vertices of C, and every path in C from B_1 to B_t contains all cut-vertices of C contained in $V(H)$.

Suppose there exists a hub B^* of C. So there exists a cut-vertex x^* of C contained in B^* such that every path in C from b_1 to $V(B^*)$ contains x^*. If $B^* = B_t$, then let $y^* = y$; otherwise, let y^* be the cut-vertex of C contained in B^* such that every path in C from b_t to $V(B^*)$ contains y^*. Let Z_0 be a path in $C - (V(B_1 - b_1) \cup V(B^* - x^*))$ from b_1 to x^*, and let Z_1 be a path in C from y^* to y. Since $(G[B_1 \cup \{x\}], x, b_1)$ is 2-connected with minimum degree at least k, by the minimality of G, $G[B_1 \cup \{x\}]$ contains $k-1$ admissible paths $P_1, \ldots, P_{k-1}$ from x to b_1. If every vertex in $V(B^*) - \{x^*, y^*\}$ has at most one neighbor in Q, then (B^*, x^*, y^*) is 2-connected with minimum degree at least k. By the minimality of G, B^* contains $k-1$ admissible paths $R_1, \ldots, R_{k-1}$ from x^* to y^*. By Lemmas 3.2 and 3.3, the set $\{P_i \cup Z_0 \cup R_j \cup Z_1 : i, j \in [k-1]\}$ contains least $2k-3 \geq k$ admissible paths from x to y in G, a contradiction. Therefore some vertex $w \in V(B^*) - \{x^*, y^*\}$ satisfies $|N_G(w) \cap V(Q)| \geq 2$. Since $s = 1$, we have $|N_G(w) \cap V(Q)| = 2$ by Lemma 3.7. Let u, v be the vertices in $N_G(w) \cap V(Q)$. By Lemma 3.7, either $\{u, v\} \subseteq Q_1$, or by symmetry say $u \in Q_1$ and $v \in Q_2$. In the former case, there are two admissible paths $L_1 = xua$ and $L_2 = xuwva$ from x to a; in the latter case, since there is no triangle containing x in $G - y$ by Lemma 3.4, we must have $v = a$, which also gives two admissible paths $L_1 = xua$ and $L_2 = xuwa$ from x to a. Since $(G[B_1 \cup \{a\}], a, b_1)$ is 2-connected with minimum degree at least k, by the minimality of G, there exist $k-1$ admissible paths $N_1, \ldots, N_{k-1}$ from a to b_1 in $G[B_1 \cup \{a\}]$. Since B^* is 2-connected, there exists a path L' from x^* to y^* in $B - w$. By Lemma 3.2, the set $\{L_i \cup N_j \cup Z_0 \cup L' \cup Z_1 : i \in [2], j \in [k-1]\}$ contains k admissible paths from x to y in G, a contradiction.

So there exists no hub. In particular, B_t is not 2-connected, for otherwise B_t is a hub. Therefore $B_t = yb_t$ is an edge. So $B_1, ..., B_{t-1}$ are the all feasible blocks in C. Recall that $N_G(B_i - b_i) \cap V(Q) = \{a, x\}$ for all $i \in [t-1]$, which implies $d_G(x) \geq |Q_1| + t - 1$. Since there is no triangle containing y in $G - x$ by Lemma 3.4, we have $d_G(y) \leq |Q_1| + 1$. Hence $|Q_1| + t - 1 \leq d_G(x) \leq d_G(y) \leq |Q_1| + 1$. That is, $t \leq 2$. As $t \geq 2$, this forces $t = 2$, $d_G(x) = d_G(y) = |Q_1| + 1$. In other words, there is exactly one end-block B_1 of C other than $B_2 = yb_2$, $N_G(y) = Q_1 \cup \{b_2\}$ and $N_G(x) \subseteq Q_1 \cup V(B_1 - b_1)$. Note that the block structure of C is a path. Since there exists no hub, every block of C other than B_1 is an edge. If $V(C) = V(B_1 \cup B_2)$, then since $N_G(C - y) \cap Q_1 \neq \emptyset$ and $N_G(B_1 - b_1) \cap Q_1 = \emptyset$, b_2 must have a neighbor in Q_1. If $V(C) \neq V(B_1 \cup B_2)$, then $|N_G(b_2) \cap V(C)| = 2$, and since $d_G(b_2) \geq k + 1 \geq 4$, we have $|N_G(b_2) \cap V(Q)| \geq 2$. Recall that $N_G(x) \subseteq Q_1 \cup V(B_1 - b_1)$, so $xb_2 \notin E(G)$. So in either case, b_2 must have a neighbor w^* in Q_1. But $G[\{y, b_2, w^*\}]$ is a triangle, contradicting Lemma 3.4.

This completes the proof of Theorem 3.1 (and of Theorem 1.2). ∎

4 Consecutive cycles

In this section, we prove Theorems 1.4 and 1.11. This will be achieved in a unified approach, namely, by finding optimal number of cycles of consecutive lengths in 2-connected non-bipartite graphs (see Theorem 4.4).

We begin by introducing a concept on cycles, which is crucial in our approach. We say that a cycle C in a connected graph G is *non-separating* if $G - V(C)$ is connected. The study of non-separating cycles appears in the work of Tutte [24] and is furthered explored by Thomassen and Toft [23]. The proof of the following lemma can be found in [2] (though it was not formally stated).

Lemma 4.1 (Bondy and Vince [2])**.** *Every non-bipartite 3-connected graph contains a non-separating induced odd cycle.*

We also need the following lemma on non-separating odd cycles from [15], which is a slight modification of a result of Fan [10].

Lemma 4.2 (Liu and Ma [15]). *Let G be a graph with minimum degree at least four. If G contains a non-separating induced odd cycle, then G contains a non-separating induced odd cycle C, denoted by $v_0v_1...v_{2s}v_0$, such that either*

1. *C is a triangle, or*
2. *for every non-cut-vertex v of $G-V(C)$, $|N_G(v)\cap V(C)|\le 2$, and the equality holds if and only if $N_G(v)\cap V(C)=\{v_i,v_{i+2}\}$ for some i, where the indices are taken under the additive group $\mathbb{Z}_{2s+1}$.*

The next lemma can be viewed as a corollary of Theorem 3.1, which will be used for finding paths in a 2-connected graph with three special vertices.

Lemma 4.3. *Let $k\ge 2$ be a positive integer. Let G be a 2-connected graph and x,y,z be three distinct vertices in G. If every vertex of G other than z has degree at least $k+1$, then G contains $k-1$ admissible paths from x to y.*

Proof. Since every two vertices are contained in a cycle in a 2-connected graph, there is nothing to prove when $k=2$. So we may assume that $k\ge 3$. Note that $G-z$ is connected and has minimum degree at least k. If $G-z$ is 2-connected, then this lemma follows from Theorem 3.1. Hence we may assume that $G-z$ is not 2-connected.

Let B be an end-block of $G-z$ with cut-vertex b. Since every vertex in $V(B-b)$ has degree at least $k\ge 3$ in G, we see that B is 2-connected. Suppose that $|V(B-b)\cap\{x,y\}|=1$. Without loss of generality, we may assume that $x\in V(B-b)$. By Theorem 3.1, B has $k-1$ admissible paths from x to b. Concatenating each of these paths with a path in $(G-z)-V(B-b)$ from b to y gives $k-1$ admissible paths in G from x to y. Therefore, there exists an end-block B' with cut-vertex b' of $G-z$ such that $V(B'-b')\cap\{x,y\}=\emptyset$. It follows that $N_G(B'-b')=\{b',z\}$. Since G is 2-connected, G has two disjoint paths P_1,P_2 internally disjoint from $V(B')$ from x to b' and from y to z, respectively. Let u be a vertex in $B'-b'$ adjacent to z in G. By Theorem 3.1, B' has $k-1$ admissible paths $R_1,R_2,...,R_{k-1}$ from b' to u. Then the set $\{P_1\cup R_i\cup uz\cup P_2 : i\in[k-1]\}$ contains $k-1$ admissible paths in G from x to y. This completes the proof. ∎

We are ready to prove the main result of this section.

Theorem 4.4. *Let k be a positive integer and G be a 2-connected graph containing a non-separating induced odd cycle. If the minimum degree of G is at least $k+1$, then G contains k cycles of consecutive lengths.*

Proof. The theorem is obvious when $k=1$. For the case $k=2$, let C_0 be an induced non-separating odd cycle in G and $x,y\in V(C_0)$ such that x,y divide C_0 into two subpaths say P_1,P_2 of lengths differing by one. Since G has minimum degree at least three, each of x,y has at least one neighbor in $G-V(C_0)$ and thus there exists a path L from x to y in $G[(V(G)-V(C_0))\cup\{x,y\}]$. Then $L\cup P_1$ and $L\cup P_2$ are two cycles of consecutive lengths in G.

So we may assume that $k\ge 3$. By Lemma 4.2, there exists a non-separating induced odd cycle $C=v_0v_1...v_{2s}v_0$ in G satisfying the conclusions of Lemma 4.2. In particular, the minimum degree of $G-V(C)$ is at least $k-1$. Throughout the rest of the proof of this theorem, the subscripts will be taken under the additive group $\mathbb{Z}_{2s+1}$.

Suppose that C is a triangle $v_0v_1v_2v_0$. Consider the graph G' obtained from G by contracting v_1 and v_2 into a vertex u. Then G' is 2-connected with minimum degree at least k. By Theorem 3.1, there are $k-1$ admissible paths in G' from u to v_0. Note that each of these paths has length at least two, so it does not contain the edge uv_0, and each of those paths corresponds to a path in $G-V(C)$ from

v_0 to some $v_i \in \{v_1, v_2\}$. Concatenating with v_0v_i and $v_0v_{3-i}v_i$, these paths lead to cycles of at least k consecutive lengths in G.

Therefore we may assume that C is not a triangle and hence $s \geq 2$. For any two vertices v_i, v_j in C, denote $C'_{i,j}$ and $C''_{i,j}$ to be the shorter and longer paths in C from v_i to v_j, respectively.

Suppose that $G - V(C)$ is 2-connected. We first assume that for every $v \in V(G-C)$, $|N_G(v) \cap V(C)| \leq 1$. Then the minimum degree of $G-V(C) \geq k$. Since G has minimum degree at least $k+1 \geq 4$, there exist distinct vertices $x, y \in V(G-C)$ such that $xv_0, yv_s \in E(G)$. By Theorem 3.1, $G - V(C)$ contains $k-1$ admissible paths $P_1, ..., P_{k-1}$ from x to y. Note that $C'_{0,s}$ and $C''_{0,s}$ are two paths from v_0 to v_s of lengths s and $s+1$, respectively. Concatenating each of $C'_{0,s}$ and $C''_{0,s}$ with $v_0x \cup P_i \cup yv_s$ for all $i \in [k-1]$ leads to k cycles of consecutive lengths in G. Hence we may assume that there exists some $u \in V(G-C)$ adjacent in G to two vertices of C. By Lemma 4.2, without loss of generality, let $N_G(u) \cap V(C) = \{v_1, v_{2s}\}$ and let $w \in V(G-C) - \{u\}$ such that $wv_s \in E(G)$. Since $G - V(C)$ has minimum degree at least $k-1$, by Theorem 3.1, $G - V(C)$ contains $k-2$ admissible paths $R_1, ..., R_{k-2}$ from u to w. Observe that $uv_1 \cup C'_{1,s}, uv_{2s} \cup C'_{s,2s}, uv_{2s} \cup C''_{s,2s}$ and $uv_1 \cup C''_{1,s}$ are four paths from u to v_s of lengths $s, s+1, s+2$ and $s+3$, respectively. By concatenating each of these paths with $v_sw \cup R_i$ for $i \in [k-2]$, we obtain cycles of $k+1$ consecutive lengths in G.

Therefore $G - V(C)$ is not 2-connected. Let B be an end-block of $G - V(C)$ with cut-vertex b. Since every vertex in $B - b$ has degree at least $k-1 \geq 2$ in B, B is 2-connected.

Suppose that $|N_G(v) \cap V(C)| \leq 1$ for every vertex $v \in V(B-b)$. Then every vertex in B other than b has degree at least k in B. We first assume that there exist $x \in V(B-b)$ and $y \in V(G-C) - V(B-b)$ such that $v_jx, v_{j+s}y \in E(G)$ for some j, then by Theorem 3.1, B contains $k-1$ admissible paths $P_1, ..., P_{k-1}$ from x to b. Let P be a path in $G - (V(C) \cup V(B-b))$ from b to y. Note that $C'_{j,j+s}, C''_{j,j+s}$ are two paths of lengths $s, s+1$, respectively. Then, by concatenating each of these paths with P_i and P, we find k cycles in G with consecutive lengths. Hence, we may assume that for every integer j with $0 \leq j \leq 2s$, if v_j is adjacent to $V(B-b)$, then $N_G(v_{j+s}) \cap V(G-C) \subseteq V(B-b)$. Since G is 2-connected, there is some vertex v_{i^*} of C adjacent in G to $V(B-b)$. Since $k+1 \geq 4$, every vertex in $V(C)$ is adjacent in G to some vertex in $V(G-C)$. So $\emptyset \neq N_G(v_{i^*+s}) - V(C) \subseteq V(B-b)$. Hence we can inductively show that $N_G(v_{i^*+rs}) - V(C) \subseteq V(B-b)$ for every positive integer r. Since s is a generator of $\mathbb{Z}_{2s+1}$, $N_G(C) \subseteq V(B-b)$. This implies that b is a cut-vertex of G, contradicting the 2-connectivity of G.

Therefore there exists a vertex $x \in V(B-b)$ with at least two neighbors in $V(C)$. By Lemma 4.2, without loss of generality, we may assume that $N_G(x) \cap V(C) = \{v_1, v_{2s}\}$. Assume there exists some $y \in V(G-C) - V(B-b)$ such that $v_sy \in E(G)$. Since every vertex in $B - b$ has degree at least $k-1$ in B, by Theorem 3.1, B contains $k-2$ admissible paths $Q_1, ..., Q_{k-2}$ from x to b. Let Q be a fixed path in $G - (V(C) \cup V(B-b))$ from b to y. Note that $xv_1 \cup C'_{1,s}, xv_{2s} \cup C'_{s,2s}, xv_{2s} \cup C''_{s,2s}$ and $xv_1 \cup C''_{1,s}$ are four paths from x to v_s of lengths $s, s+1, s+2$ and $s+3$, respectively. By concatenating each of these paths with $Q_i \cup Q \cup yv_s$, we find $k+1$ cycles of consecutive lengths in G. Hence we have $N_G(v_s) \cap V(G-C) \subseteq V(B-b)$. Since $|N_G(v_s) \cap V(G-C)| \geq k-1 \geq 2$, there exists $z \in N_G(v_s) \cap V(B) - \{x, b\}$. If $k \leq 4$, then using the above four paths from x to v_s, together with v_sz and a path in B from z to x, we obtain cycles of four consecutive lengths in G. So we may assume $k \geq 5$. Note that every vertex of B other than b has degree at least $k-1 \geq 4$ in B. By Lemma 4.3, B has $k-3$ admissible paths $R_1, ..., R_{k-3}$ from x to z. Again, concatenating each of these paths with zv_s and the four paths from x to v_s, one can find cycles of k consecutive lengths in G. This completes the proof of Theorem 4.4. ∎

Using Theorem 4.4, we can derive Theorems 1.4 and 1.11 easily.

Theorem 1.4. *Every non-bipartite* 3*-connected graph with minimum degree at least* $k+1$ *contains* k

cycles of consecutive lengths.

Proof. This theorem immediately follows from Lemma 4.1 and Theorem 4.4. ∎

We say that a graph G is *k-critical*, if it has chromatic number k but every proper subgraph of G has chromatic number less than k.

We now prove Theorem 1.11, which we restate as the following.

Theorem 4.5. *For every positive integer k, every graph with chromatic number at least $k+2$ contains k cycles of consecutive lengths.*

Proof. Let G be any graph with chromatic number at least $k+2$. We may assume that $k \geq 2$, for otherwise the theorem is obvious. Then there exists a $(k+2)$-critical subgraph G' of G. It is easy to see that G' is 2-connected and has minimum degree at least $k+1$. It is known that for any integer $t \geq 4$, every t-critical graph contains a non-separating induced odd cycle (the case $t = 4$ was explicitly stated and proved by Krusenstjerna-Hafstrøm and Toft [14, Theorem 4], but their proof works for every $t \geq 4$ as well). Therefore G' contains a non-separating induced odd cycle. By Theorem 4.4, we see that G' (and thus G) contains k cycles of consecutive lengths. ∎

5 Dean's conjecture

In this section we prove Conjecture 1.8, which will be divided into several lemmas. For a brief overview of the coming proof, we would suggest readers to have a sketch on the proof of Theorem 5.15, which is a restatement of Theorem 1.9.

Define K_4^- to be the graph obtained from K_4 by deleting one edge. A *chord* of a cycle C in a graph G is an edge $e \in E(G) - E(C)$ such that the both ends of e belong to $V(C)$. For a positive integer $t \geq 4$, we define K_t' to be the graph obtained from K_t by deleting v_1v_4 and v_iv_j for every $i \in \{1, 2\}$ and $j \in \{5, 6, ..., t\}$, where $V(K_t) = \{v_k : 1 \leq k \leq t\}$.

Lemma 5.1. *Let d and t be integers with $d + 1 \geq t \geq 5$. Let G be a graph containing a K_4^- subgraph but not containing a K_t' subgraph. If G is $(t-1)$-connected, then G contains d cycles of consecutive lengths.*

Proof. Let $\{x, y, a, b\}$ be a set of four vertices of G inducing a K_4^- subgraph, where x is of degree two in this K_4^- subgraph and y is a neighbor of x. So there exists a clique in $G - \{x, y\}$ containing a, b. Let K be a maximal clique in $G - \{x, y\}$ containing a, b. Note that $|K| \leq t-3$, for otherwise $G[\{x, y\} \cup K]$ contains a K_t' subgraph. Hence $G - K$ is 2-connected since G is $(t-1)$-connected.

By the maximality of K, every vertex in $G - (K \cup \{x, y\})$ is adjacent in G to at most $|K| - 1$ vertices in K. So $((G - K) - xy, x, y)$ is a 2-connected rooted graph of minimum degree at least $d - |K| + 1$. By Theorem 3.1, there exist $d - |K|$ admissible x-y paths $P_1, P_2, ..., P_{d-|K|}$ in $(G - K) - xy$. Note that there exist $|K| + 1$ x-y paths $Q_1, Q_2, ..., Q_{|K|+1}$ in $G[K \cup \{x, y\}]$ with consecutive lengths. For every integers i, j with $1 \leq i \leq d - |K|$ and $1 \leq j \leq |K| + 1$, let $C_{i,j}$ be the cycle obtained by concatenating P_i and Q_j. Let $\mathcal{C} = \{C_{i,j} : 1 \leq i \leq d - |K|, 1 \leq j \leq |K| + 1\}$. If $P_1, P_2, ..., P_{d-|K|}$ have consecutive lengths, then $\mathcal{C}$ contains d cycles of consecutive lengths. If the lengths of $P_1, P_2, ..., P_{d-|K|}$ form an arithmatic progression of length two, then $\mathcal{C}$ contains $2d - |K| - 2 \geq d$ cycles of consecutive lengths. ∎

Lemma 5.2. *Let $d \geq 3$ be an integer. Let G be a 3-connected graph of minimum degree at least d. If G contains a K_3 subgraph but does not contain a K_4^- subgraph, then G contains d cycles of consecutive lengths.*

Proof. Let $\{a, b, c\}$ be a set of three vertices of G that induces a K_3 subgraph. Let G' be the graph obtained from G by contracting the edge bc into a new vertex a^* and deleting resulting loops and parallel edges. Since G is 3-connected, G' is 2-connected, so $(G' - aa^*, a, a^*)$ is a 2-connected rooted graph. Since G does not contain a K_4^- subgraph, $(G' - aa^*, a, a^*)$ has minimum degree at least d. By Theorem 3.1, there exist $d-1$ admissible a-a^* paths $P_1, P_2, ..., P_{d-1}$ in $G' - aa^*$. So there exist paths $P_1', P_2', ..., P_{d-1}'$ in G such that their lengths form an arithmetic progression of common difference one or two, and for every i with $1 \leq i \leq d-1$, P_i' is either an a-b path disjoint from c or an a-c path disjoint from b. For every integer i with $1 \leq i \leq d-1$, let $C_{i,1} = P_i' + ab$, $C_{i,2} = P_i' + ac$, $C_{i,3} = P_i' \cup abc$, and let $C_{i,4} = P_i' \cup acb$. Then the set $\{C_{i,j} : 1 \leq i \leq d-1, 1 \leq j \leq 4\}$ contains d cycles of consecutive lengths. ∎

Lemma 5.3. *Let ℓ be a positive integer. Let A be a subset of integers such that ℓ elements of A form an arithmetic progression of common difference r, where $r \in \{1, 2\}$.*

1. *If $r = 1$ and $\ell \geq 3$, then for every integer x, the set $\{a + x, a + x + 3 : a \in A\}$ contains $\ell + 3$ elements that form an arithmetic progression of common difference one.*
2. *If $r = 2$ and $\ell \geq 2$, then for every integer x, the set $\{a + x, a + x + 3 : a \in A\}$ contains $2\ell - 2$ elements that form an arithmetic progression of common difference one.*

Proof. Let $a_1, a_2, ..., a_\ell$ be ℓ elements of A forming an arithmetic progression of common difference r, where $r \in \{1, 2\}$. We may assume that for every i with $1 \leq i \leq \ell$, $a_i = a_1 + (i-1)r$. Let x be any integer, and let $S = \{a_i + x, a_i + x + 3 : 1 \leq i \leq \ell\}$.

If $r = 1$ and $\ell \geq 3$, then $S = \{i : a_1 + x \leq i \leq a_1 + \ell - 1 + x + 3\}$. If $r = 2$ and $\ell \geq 2$, then S contains $\{i : a_1 + 2 + x \leq i \leq a_1 + 2(\ell - 1) + x + 1\}$. This proves the lemma. ∎

Lemma 5.4. *Let d be an integer with $d \geq 6$. If G is a 3-connected non-bipartite graph with minimum degree at least d that does not contain a K_3 subgraph, then either*

1. *G contains d cycles of consecutive lengths, or*
2. *$d \in \{6, 7\}$ and there exists an induced cycle C in G, denoted by $v_0v_1v_2...v_{2s}v_0$, of length at least five such that*
   (a) *$G - V(C)$ is connected but not 2-connected,*
   (b) *every end-block of $G - V(C)$ is 2-connected, and*
   (c) *for every non-cut vertex v of $G - V(C)$, $|N_G(v) \cap V(C)| \leq 2$, and if $|N_G(v) \cap V(C)| = 2$, then $N_G(v) \cap V(C) = \{v_i, v_{i+2}\}$ for some $i \in \mathbb{Z}_{2s+1}$ and the indices are computed in $\mathbb{Z}_{2s+1}$.*

Proof. We may assume that G does not contain d cycles of consecutive lengths, for otherwise we are done.

By Lemmas 4.1 and 4.2, since G does not contain a K_3 subgraph, there exists an induced odd cycle C of length at least five, denoted by $v_0v_1...v_{2s}v_0$, such that for every non-cut vertex v of $G - V(C)$, $|N_G(v) \cap V(C)| \leq 2$, and if $|N_G(v) \cap V(C)| = 2$, then $N_G(v) \cap V(C) = \{v_i, v_{i+2}\}$ for some $i \in \mathbb{Z}_{2s+1}$ and the indices are computed in $\mathbb{Z}_{2s+1}$.

In particular, no vertex of $G - V(C)$ is of degree at most one in $G - V(C)$, since $d \geq 4$. So every end-block of $G - V(C)$ is 2-connected. Note that for every end-block B of $G - V(C)$, there exists at most one cut-vertex of $G - V(C)$ contained in $V(B)$, and if such vertex exists, we denote it by b_B.

So to prove this lemma, it suffices to prove that $d \in \{6, 7\}$ and $G - V(C)$ is not 2-connected.

We first suppose to the contrary that $G - V(C)$ is 2-connected.

Suppose that there exists a vertex $x \in V(G) - V(C)$ adjacent in G to at least two vertices in $V(C)$. By symmetry, we may assume that x is adjacent to v_0 and v_2. Since G is 3-connected and C is an induced cycle in G, v_{s+1} is adjacent in G to a vertex y in $V(G) - V(C)$. Since $|V(C)| \geq 5$, $v_{s+1} \notin \{v_0, v_2\}$, so $x \neq y$. Since $G - V(C)$ is 2-connected, every vertex in $G - V(C)$ has degree at least $d-2$, so by Theorem 3.1, there exist $d-3$ admissible paths $P_1, P_2, ..., P_{d-3}$ in $G - V(C)$ from x to y. Let Q_1, Q_2 be the subpaths of C with ends v_0 and v_{s+1} of length $s+1$ and s, respectively. Let Q_3, Q_4 be the subpaths of C with ends v_2 and v_{s+1} of lengths $s-1$ and $s+2$, respectively. Let $\mathcal{C} = \{(P_i \cup Q_j) + xv_0 + yv_{s+1}, (P_i \cup Q_k) + xv_2 + yv_{s+1} : 1 \leq i \leq d-3, 1 \leq j \leq 2, 3 \leq k \leq 4\}$. Then $\mathcal{C}$ contains $(d-3)+4-1 = d$ cycles of consecutive lengths, a contradiction.

So every vertex $x \in V(G) - V(C)$ is adjacent in G to at most one vertex in $V(C)$. Since G is connected, there exists a vertex x' in $V(G) - V(C)$ adjacent in G to a vertex in $V(C)$. By symmetry, we may assume that x' is adjacent to v_0. Since G is 3-connected and C is an induced cycle in G, v_{s-1} is adjacent in G to a vertex y' in $V(G)-V(C)$. Since $|V(C)| \geq 5$, $v_0 \neq v_{s-1}$, so $x \neq y$. Since every vertex in $G-V(C)$ has degree at least $d-1$, by Theorem 3.1, there exist $d-2$ admissible paths $P'_1, P'_2, ..., P'_{d-2}$ in $G - V(C)$ from x' to y'. Let Q'_1, Q'_2 be the subpaths of C with ends v_0 and v_{s-1} of length $s-1$ and $s+2$, respectively. Let $\mathcal{C}' = \{(P'_i \cup Q'_j) + xv_0 + yv_{s-1} : 1 \leq i \leq d-2, 1 \leq j \leq 2\}$. Since $d-2 \geq 3$, $\mathcal{C}'$ contains $\min\{d-2+3, 2(d-2)-2\} \geq d$ cycles of consecutive lengths by Lemma 5.3, a contradiction.

Hence $G - V(C)$ is not 2-connected. It suffices to prove $d \in \{6,7\}$. Suppose to the contrary that $d \geq 8$.

Suppose that there exist an end-block B of $G - V(C)$ and a vertex $x \in V(B) - \{b_B\}$ adjacent in G to v_{i_x} in $V(C)$ for some $0 \leq i_x \leq 2s$, such that v_{i_x+s-1} is adjacent in G to a vertex $y \in V(G) - (V(C) \cup (V(B) - \{b_B\}))$, where the indices are computed in $\mathbb{Z}_{2s+1}$. Since (B, x, b_B) is a 2-connected rooted graph of minimum degree at least $d-2$, Theorem 3.1 implies that there exist $d-3$ admissible paths in B from x to b_B, and hence by concatenating each of them with a fixed path in $G-(V(C) \cup (V(B)-\{b_B\}))$ from b_B to y, there exist $d-3$ admissible paths $P''_1, P''_2, ..., P''_{d-3}$ in $G-V(C)$ from x to y. Let Q''_1, Q''_2 be the subpaths of C with ends v_{i_x} and v_{i_x+s-1} of length $s-1$ and $s+2$, respectively. Let $\mathcal{C}'' = \{(P''_i \cup Q''_j) + xv_{i_x} + yv_{i_x+s-1} : 1 \leq i \leq d-3, 1 \leq j \leq 2\}$. Since $d-3 \geq 3$, by Lemma 5.3, $\mathcal{C}''$ contains $\min\{(d-3)+3, 2(d-3)-2\} = \min\{d, 2d-8\}$ cycles of consecutive lengths. Since G does not contain d cycles of consecutive lengths, $2d-8 < d$. Hence $d \in \{6,7\}$, a contradiction.

Hence for every end-block B of $G - V(C)$ and every vertex $x \in V(B) - \{b_B\}$ adjacent in G to v_{i_x} for some $0 \leq i_x \leq 2s$, $N_G(v_{i_x+s-1}) \subseteq V(B) - \{b\}$, where the indices are computed in $\mathbb{Z}_{2s+1}$. Similarly, for every end-block B of $G - V(C)$ and every vertex $x \in V(B) - \{b_B\}$ adjacent in G to v_{i_x} for some $0 \leq i_x \leq 2s$, $N_G(v_{i_x-(s-1)}) \subseteq V(B) - \{b\}$, where the indices are computed in $\mathbb{Z}_{2s+1}$.

For every end-block B of $G-V(C)$, let $S_B = \{i : 0 \leq i \leq 2s, v_i \in N_G(B - b_B)\}$. Note that for every end-block B and $i \in \mathbb{Z}_{2s+1}$, if $i \in S_B$, then $\{i+(s-1), i-(s-1)\} \subseteq S_B - S_{B'}$ for every end-block B' of $G-V(C)$ other than B. So if $s-1$ is relatively prime to $2s+1$, then $S_B = \{i : 0 \leq i \leq 2s\}$ for every end-block B, but there are at least two end-blocks of $G - V(C)$, a contradiction.

Hence $s-1$ is not relatively prime to $2s+1$. So there exists a prime p that divides $s-1$ and $2s+1$. Hence p divides $(2s+1)-2(s-1) = 3$. That is, $p = 3$, and 3 is the greatest common divisor of $2s+1$ and $s-1$. So for every end-block B and $i \in \mathbb{Z}$, if $i \in S_B$, then since S_B contains $i + t(s-1)$ for every integer t, where the computation is in $\mathbb{Z}_{2s+1}$, S_B contains $i + 3t'$ for every integer t'. Hence for every $i \in \{0,1,2\}$, either $S_B \supseteq \{i+3t : t \in \mathbb{Z}\}$ or $S_B \cap \{i+3t : t \in \mathbb{Z}\} = \emptyset$, where the computation is in $\mathbb{Z}_{2s+1}$. Since there are at least two end-blocks in $G - V(C)$, there exists an end-block B^* such that there uniquely exists $i^* \in \{0,1,2\}$ such that $S_{B^*} \cap \{i+3t : t \in \mathbb{Z}\} \neq \emptyset$. This implies that every vertex in $B^* - b_{B^*}$ is adjacent in G to at most one vertex in $V(C)$.

By symmetry, we may assume that $i^* = 0$, and there exist $x^*, y^* \in V(B^*) - \{b_{B^*}\}$ such that $x^*v_0 \in E(G)$ and $y^*v_{s-1} \in E(G)$. Since G is 3-connected, x^* and y^* can be chosen to be distinct vertices. Hence x^*, y^*, b_{B^*} are distinct vertices. Since B is 2-connected and every vertex in B^* other than b_{B^*} is of degree at least $d-1$ in B^*, by Lemma 4.3 there exist $d-3$ admissible paths $P_1^*, P_2^*, ..., P_{d-3}^*$ in B^* from x^* to y^*. Let Q_1^*, Q_2^* be the subpaths of C with ends v_0 and v_{s-1} of length $s-1$ and $s+2$, respectively. Let $\mathcal{C}^* = \{(P_i^* \cup Q_j^*) + v_0x^* + v_{s-1}y^* : 1 \leq i \leq d-3, 1 \leq j \leq 2\}$. Since $d-3 \geq 3$, by Lemma 5.3, $\mathcal{C}^*$ contains $\min\{d-3+3, 2(d-3)-2\}$ cycles of consecutive lengths. Since G does not contain d cycles of consecutive lengths, $2d-8 < d$. Hence $d \in \{6,7\}$, a contradiction. This proves the lemma. ∎

Lemma 5.5. *Let d be an integer with $d \geq 6$. If G is a 3-connected non-bipartite graph with minimum degree at least d that does not contain a K_3 subgraph, then G contains d admissible cycles. Furthermore, if $d \geq 8$, then G contains d cycles of consecutive lengths; and if $d \in \{6,7\}$, then G contains cycles of all lengths modulo d.*

Proof. We may assume that G does not contain d cycles of consecutive lengths, for otherwise we are done. By Lemma 5.4, $d \in \{6,7\}$ and there exists an induced cycle C in G, denoted by $v_0v_1v_2...v_{2s}v_0$, of length at least five such that

- $G - V(C)$ is connected but not 2-connected,
- every end-block of $G - V(C)$ is 2-connected, and
- for every non-cut vertex v of $G - V(C)$, $|N_G(v) \cap V(C)| \leq 2$, and if $|N_G(v) \cap V(C)| = 2$, then $N_G(v) \cap V(C) = \{v_i, v_{i+2}\}$ for some $i \in \mathbb{Z}_{2s+1}$ and the indices are computed in $\mathbb{Z}_{2s+1}$.

Since $d \in \{6,7\}$, to prove this lemma, it suffices to prove that G contains d admissible cycles, and prove that G contains cycles of all lengths modulo d. Note that if $d = 7$ and G contains d admissible cycles, then G contains cycles of all lengths modulo d.

Suppose to the contrary that either G does not contain d admissible cycles, or $d = 6$ and G does not contain cycles of all lengths modulo d.

Since $G - V(C)$ is not 2-connected and every end-block of $G - V(C)$ is 2-connected, for every end-block B of $G - V(C)$, there exists exactly one vertex b_B in B such that b_B is a cut-vertex of $G - V(C)$. For every end-block B of $G - V(C)$, let u_B be a vertex in $B - \{b_B\}$ such that $|N_G(u_B) \cap V(C)|$ is as large as possible. Note that for every end-block B of $G - V(C)$, (B, u_B, b_B) is a 2-connected rooted graph of minimum degree at least $d - |N_G(u_B) \cap V(C)|$, so there exist $d - |N_G(u_B) \cap V(C)| - 1$ admissible paths $P_{B,1}, P_{B,2}, ..., P_{B,d-|N_G(u_B) \cap V(C)|-1}$ in B from u_B to b_B. In addition, for every end-block B of $G - V(C)$, $1 \leq |N_G(u_B) \cap V(C)| \leq 2$ since $u_B \in V(B) - \{b_B\}$.

Suppose that there exists an end-block B_1 of $G - V(C)$ such that $|N_G(u_{B_1}) \cap V(C)| = 1$. Let B_2 be an end-block of $G - V(C)$ other than B_1. Let $x \in N_G(u_{B_1}) \cap V(C)$. Since G is 3-connected, u_{B_2} can be chosen such that $N_G(u_{B_2}) \cap V(C) - \{x\} \neq \emptyset$. Let $y \in N_G(u_{B_2}) \cap V(C) - \{x\}$. Let Q be a path in C from x to y. Let R be a path in $G - V(C)$ from b_{B_1} to b_{B_2}. Let $\mathcal{C} = \{(P_{B_1,i} \cup R \cup P_{B_2,j} \cup Q) + xu_{B_1} + yu_{B_2} : 1 \leq i \leq d - |N_G(u_{B_1}) \cap V(C)| - 1, 1 \leq j \leq d - |N_G(u_{B_2}) \cap V(C)| - 1\}$. So $\mathcal{C}$ contains $(d - |N_G(u_{B_1}) \cap V(C)| - 1) + (d - |N_G(u_{B_2}) \cap V(C)| - 1) - 1 = 2d - 4 - |N_G(u_{B_2}) \cap V(C)| \geq 2d - 6 \geq d$ admissible cycles. Hence $d = 6$ and G does not contain d cycles of consecutive lengths. So the lengths of the cycles in $\mathcal{C}$ form an arithmetic progression of common difference two. It follows that the set $\{P_{B_1,i} \cup R \cup P_{B_2,j} : 1 \leq i \leq d - |N_G(u_{B_1}) \cap V(C)| - 1, 1 \leq j \leq d - |N_G(u_{B_2}) \cap V(C)| - 1\}$ contains $2d-6$ paths whose of lengths form an arithmetic progression of common difference two from u_{B_1} to u_{B_2} in $G - V(C)$. Let Q_o be the odd path from x to y in C and Q_e be the even path from x to y in C. By concatenating each of Q_o and Q_e with $P_{B_1,i} \cup R \cup P_{B_2,j}$, we could obtain $2d-6$ cycles of consecutive odd lengths and $2d-6$ cycles of consecutive even lengths. Since $d = 6$ is even, G contains cycles of all lengths modulo d.

Hence for every end-block B of $G - V(C)$, $|N_G(u_B) \cap V(C)| = 2$. Let B_3, B_4 be two distinct end-blocks of $G - V(C)$. By symmetry, we may assume that $N_G(u_{B_3}) \cap V(C) = \{v_0, v_2\}$. Since $|N_G(u_{B_4}) \cap V(C) \cap \{v_0, v_1\}| \leq 1$, there exists $z \in N_G(u_{B_4}) \cap V(C) - \{v_0, v_1\}$. Let Q_1 be the path in C from v_0 to z containing $v_0v_1v_2$, and let Q_2 be the subpath of Q_1 from v_2 to z. Note that $|E(Q_1)| = |E(Q_2)|+2$. Let R' be a path in $G - V(C)$ from b_{B_3} to b_{B_4}. Let $\mathcal{C}' = \{(P_{B_3,i} \cup R' \cup P_{B_4,j} \cup Q_1) + u_{B_3}v_0 + u_{B_4}z, (P_{B_3,i} \cup R' \cup P_{B_4,j} \cup Q_2) + u_{B_3}v_2 + u_{B_4}z : 1 \leq i \leq d - |N_G(u_{B_3}) \cap V(C)| - 1, 1 \leq j \leq d - |N_G(u_{B_4}) \cap V(C)| - 1\}$. Since $|E(Q_1)| = |E(Q_2)|+2$, $\mathcal{C}'$ contains $(d - |N_G(u_{B_3}) \cap V(C)| - 1) + (d - |N_G(u_{B_4}) \cap V(C)| - 1) - 1 + 2 - 1 = 2d - 6 \geq d$ admissible cycles. So $d = 6$ and G does not contain d cycles of consecutive lengths. Hence the lengths of these cycles form an arithmetic progression of common difference two. It follows that $P_{B_3,i} \cup R' \cup P_{B_4,j}$ for all $1 \leq i \leq d - |N_G(u_{B_3}) \cap V(C)| - 1$ and $1 \leq j \leq d - |N_G(u_{B_4}) \cap V(C)| - 1$ contains $2d - 7$ paths whose of lengths form an arithmetic progression of common difference two between u_{B_3} to u_{B_4} in $G - V(C)$. Let Q'_o and Q'_e be the odd path and even path in C from z to v_0, respectively. By concatenating each of Q'_o and Q'_e with $P_{B_3,i} \cup R' \cup P_{B_4,j}$, we obtain $2d - 7$ cycles of consecutive odd lengths and $2d - 7$ cycles of consecutive even lengths. Since $d = 6$ is even, G contains cycles of all lengths modulo d, a contradiction. This proves the lemma. ∎

For every positive integer d, let $K^-_{d,d}$ be the graph obtained from $K_{d,d}$ by deleting an edge.

Lemma 5.6. *Let $d \geq 3$ be an integer. Let G be a $\max\{d,5\}$-connected graph of girth exactly four and of minimum degree at least d that does not contain a cycle of length five. If G does not contain a $K^-_{d,d}$ subgraph, then G contains d admissible cycles.*

Proof. Suppose to the contrary that G does not contain d admissible cycles.

Since the girth of G equals four, G contains a $K_{2,2}$ subgraph. So there exists a complete bipartite subgraph Q of G with bipartition (Q_1, Q_2) such that

(i) $2 \leq |Q_1| \leq |Q_2|$,

(ii) subject to (i), $|Q_1|$ is maximum, and

(iii) subject to (i) and (ii), $|Q_2|$ is maximum.

Let $s = |Q_1|$. Note that $2 \leq s \leq d-1$ since G does not contain a $K_{d,d}$ subgraph. Since G does not contain a K_3 subgraph, Q is an induced subgraph of G, and for every vertex v of $G - V(Q)$, either $N_G(v) \cap Q_1 = \emptyset$ or $N_G(v) \cap Q_2 = \emptyset$.

For any $v \in V(G) - V(Q)$, $|N_G(v) \cap Q_1| \leq s-1$ by (iii), and $|N_G(v) \cap Q_2| \leq s$ by (ii). If there exists a vertex $z \in V(G) - V(Q)$ such that z is adjacent in G to at least s vertices in $V(Q)$, then let $Z = \{z\}$; otherwise, let Z be the empty set. Note that if $Z \neq \emptyset$, then $N_G(z) \cap V(Q) \subseteq Q_2$ and $|Q_2| \geq s+1$, since G is of girth four and by (i)-(iii).

Suppose there exists $i \in \{1,2\}$ such that there exists a component M of $G - (Q_i \cup Z)$ disjoint from Q_{3-i}. Since G is d-connected, $|Q_i| + |Z| \geq d$. If $|Q_i| = s$, then since $d \geq s+1$, $Z \neq \emptyset$ and $|Q_i| = s = d-1$, so $i = 1$ and $|Q_{3-i}| \geq s+1 = d$, and hence $G[V(Q) \cup Z]$ contains a $K^-_{d,d}$ subgraph, a contradiction. Hence $|Q_i| \geq s+1$. In particular, $i = 2$. Note that when $s = 2$, $|Q_i| \geq s+2$, for otherwise $|Q_i| + |Z| \leq (s+1)+1 = 4$, contradicting that G is 5-connected. Since $G - Z$ is 2-connected, there exist distinct vertices q_i, q'_i in $Q_i \cap N_G(M)$. If $|Q_i| = 2$, let $A_i = \{q_i\}$; if $|Q_i| \geq 3$, let $A_i = \{q_i, q'_i\}$. Since $G - Z$ is 3-connected, $N_G(M) \cap Q_i - A_i \neq \emptyset$. If $|Q_i| = s+1$ and $Z \neq \emptyset$, then let $\epsilon = 1$; for otherwise, let $\epsilon = 0$. Let G_M be the graph obtained from $G[V(M) \cup Q_i]$ by identifying all vertices in A_i into a vertex u_M, identifying all vertices in $Q_i - A_i$ into a vertex v_M, and deleting all resulting loops and parallel edges. Since $|Q_i| \geq s+1$ and $i = 2$, and since G is of girth four and does not contain a 5-cycle, no vertex of M is adjacent in G to both Z and Q_2, so the minimum degree of (G_M, q_i, v_M) is at least $d-(s-2)-|Z|+\epsilon$ by the definition of Z. So (G_M, q_i, v_M) is a rooted graph of minimum degree at least $d-(s-2)-|Z|+\epsilon$. Since $G-Z$ is 3-connected and M is a component of $G-(Q_i \cup Z)$, we know (G_M, u_M, v_M) is a 2-connected rooted graph of minimum degree at least $d-s+2-|Z|+\epsilon$. By Theorem 3.1, there exist $d-s+1-|Z|+\epsilon$ admissible paths in G_M from u_M to v_M. So there exist $d-s+1-|Z|+\epsilon$ admissible paths $P_{M,1}, P_{M,2}, ..., P_{M,d-s+1-|Z|+\epsilon}$ in $G[V(M) \cup Q_i]$ from A_i to $Q_i - A_i$ internally disjoint from $V(Q) \cup Z$. For each i with $1 \leq i \leq d-s+1-|Z|+\epsilon$, let α_i be the ends of $P_{M,i}$ in A_i and let β_i be the end of $P_{M,i}$ in $Q_i - A_i$. Since $|Q_i| \geq s+1$ and $i=2$ and Q is a complete bipartite graph, there exist $s+|Z|-\epsilon$ admissible paths $Q_{M,1}, ..., Q_{M,s+|Z|-\epsilon}$ in $G[V(Q) \cup Z]$ from α_i to β_i. Then the set $\{P_{M,j} \cup Q_{M,k} : 1 \leq j \leq d-s+1-|Z|+\epsilon, 1 \leq k \leq s+|Z|-\epsilon\}$ contains $(d-s+1-|Z|+\epsilon)+(s+|Z|-\epsilon)-1=d$ admissible cycles, a contradiction.

So for every $i \in \{1,2\}$, every component of $G-(Q_i \cup Z)$ intersects Q_{3-i}. Let G' be the graph obtained from $G-Z$ by identifying all vertices in Q_1 into a vertex u', identifying all vertices in Q_2 into a vertex v', and deleting resulting loops and parallel edges. Since G is of girth four and does not contain a 5-cycle, no vertex of $G-(V(Q) \cup Z)$ is adjacent in G to either both Z and Q_2 or both Q_1 and Q_2, so the minimum degree of (G', u', v') is at least $d-(s-2)-|Z|$ by the definition of Z. Since G is 3-connected, $G-Z$ is 2-connected, so every cut-vertex of G' is u' or v'. Since for every $i \in \{1,2\}$, every component of $G-(Q_i \cup Z)$ intersects Q_{3-i}, we know G' is 2-connected. So (G', u', v') is a 2-connected rooted graph of minimum degree at least $d-s-|Z|+1$. By Theorem 3.1, there exist $d-s-|Z|+1$ admissible paths in G' from u' to v'. So there exist $d-s-|Z|+1$ admissible paths $R_1, R_2, ..., R_{d-s-|Z|+1}$ in $G-Z$ from Q_1 to Q_2 internally disjoint from $V(Q)$. For every i with $1 \leq i \leq d-s-|Z|+1$, let x_i, y_i be the ends of R_i in Q_1, Q_2, respectively. Since Q is a complete bipartite graph, for each i with $1 \leq i \leq d-s-|Z|+1$, $G[V(Q) \cup Z]$ contains $s+|Z|$ admissible paths $R'_1, R'_2, ..., R'_{s+|Z|}$ from x_i to y_i. So the set $\{R_j \cup R'_k : 1 \leq j \leq d-s-|Z|+1, 1 \leq k \leq s+|Z|\}$ contains d admissible cycles, a contradiction. This proves the lemma. ∎

Lemma 5.7. *Let G be a 3-connected bipartite graph. If G does not contain a cycle of length four, then G contains a non-separating induced cycle C such that for every non-cut-vertex v of $G-V(C)$, $|N_G(v) \cap V(C)| \leq 1$.*

Proof. Let C be a cycle of G such that

(i) the largest component of $G-V(C)$ is as large as possible, and

(ii) subject to (i), $|V(C)|$ is as small as possible.

Let M be a component of $G-V(C)$ with $|V(M)|$ maximum.

If there exists a chord e of C, then one of P_e+e and Q_e+e is a cycle shorter than C such that M is a component of the graph obtained from G by deleting this cycle, a contradiction, where P_e, Q_e are the two subpaths of C with ends the same as e. Hence C is an induced cycle.

Suppose there exists a component M' of $G-V(C)$ other than M. Let $A=N_G(M) \cap V(C)$ and $B=N_G(M') \cap V(C)$. Since G is 3-connected, $\min\{|A|,|B|\} \geq 3$. Since $|A| \geq 3$ and $|B| \geq 2$, there exists a subpath Q of C whose ends belong to B such that some internal vertex of Q belongs to A. Since M' is connected, there exists a path Q' from one end of Q to another end of Q such that all internal vertices belong to $V(M')$. Let Q'' be the subpath of C with the same ends as Q but internally disjoint from Q. Then $Q' \cup Q''$ is a cycle in G such that some component of $G-V(Q' \cup Q'')$ contains M and a vertex in A, contradicting (i).

Hence C is a non-separating cycle in G. Suppose that there exists a non-cut-vertex v of $G-V(C)$ such that $|N_G(v) \cap V(C)| \geq 2$. Let x, y be distinct vertices in $N_G(v) \cap V(C)$ such that no internal vertex of R_1 belongs to $N_G(v) \cap V(C)$, where R_1, R_2 are the two subpaths of C with ends x and y. If

$|E(R_1)| \leq 2$, then $R_1 + vx + vy$ is a cycle of length at most four, contradicting that G is a bipartite graph with no 4-cycle. So $|E(R_1)| \geq 3$. Hence $R_2+vx+vy$ is a cycle shorter than C. Since $|E(R_1)| \geq 3$, there exist distinct internal vertices x', y' of R_1. Since C is an induced cycle and every vertex of G has degree at least three, $N_G(x') - V(C) \neq \emptyset \neq N_G(y') - V(C)$. Since C is a non-separating cycle, $N_G(x') \cap V(M) = N_G(x') - V(C) \neq \emptyset \neq N_G(y') - V(C) = N_G(y') \cap V(M)$. Since x', y' are internal vertices of R_1, $\{x', y'\} \cap N_G(v) = \emptyset$. So $N_G(x') \cap V(M) - \{v\} \neq \emptyset$ and $N_G(y') \cap V(M) - \{v\} \neq \emptyset$. Since v is not a cut-vertex of $G - V(C)$, $M - v$ is connected. So some component of $G - V(R_2 + vx + vy)$ contains $(V(M) - \{v\}) \cup \{x', y'\}$, contradicting (i). This proves the lemma. ∎

Lemma 5.8. *Let $d \geq 5$ be an integer. Let G be a 3-connected bipartite graph of minimum degree at least d. If G does not contain cycles of length four, then G contains d admissible cycles.*

Proof. Suppose to the contrary that G does not contain d admissible cycles. By Lemma 5.7, there exist a positive integer s and an induced non-separating cycle $C = v_0v_1 \dots v_{2s-1}v_0$ in G such that for every non-cut-vertex of $G - V(C)$, it is adjacent in G to at most one vertex in $V(C)$. Since G is a bipartite graph with no 4-cycle, $s \geq 3$. For any any $0 \leq i < j \leq 2s-1$, let $Q_{i,j}$ and $Q'_{i,j}$ be the two subpaths of C with ends v_i, v_j.

Suppose $G-V(C)$ is 2-connected. Since C is an induced non-separating cycle and G is of minimum degree at least $d \geq 4$, there exist distinct vertices x, y in $V(G) - V(C)$ such that $\{xv_0, yv_{s-2}\} \in E(G)$. Since $(G-V(C), x, y)$ is a 2-connected rooted graph of minimum degree at least $d-1$, there exist $d-2$ admissible paths $P_1, P_2, ..., P_{d-2}$ in $G - V(C)$ from x to y by Theorem 3.1. Note that $||E(Q_{0,s-2})| - |E(Q'_{0,s+2})|| = 4$. Since $d-2 \geq 2$, the set $\{(P_i \cup Q_{0,s-2}) + xv_0 + yv_{s-2}, (P_i \cup Q'_{0,s-2}) + xv_0 + yv_{s-2} : 1 \leq i \leq d-2\}$ contains d admissible cycles, a contradiction.

So $G-V(C)$ is not 2-connected. In particular, there exist two distinct end-blocks B_1, B_2 of $G-V(C)$. Since G is 3-connected, each B_1, B_2 is 2-connected. For $i \in \{1, 2\}$, let b_i be the cut-vertex of $G - V(C)$ contained in $V(B_i)$. Since G is 2-connected, for each $i \in \{1, 2\}$, there exist an integer r_i with $0 \leq r_i \leq 2s-1$ and a vertex u_i in $V(B_i) - \{b_i\}$ such that $u_iv_{r_i} \in E(G)$. Since G is 3-connected, r_1 and r_2 can be chosen to be distinct. For $i \in \{1, 2\}$, since (B_i, u_i, b_i) is a 2-connected rooted graph of minimum degree at least $d-1$, there exist $d-2$ admissible paths $P_{i,1}, P_{i,2}, ..., P_{i,d-2}$ in B_i from u_i to b_i. Let Q be a path in $G - V(C)$ from b_1 to b_2 internally disjoint from $V(B_1) \cup V(B_2)$. Then the set $\{(P_{1,i} \cup Q \cup P_{2,j} \cup Q_{r_1,r_2}) + u_1v_{r_1} + u_2v_{r_2} : 1 \leq i \leq d-2, 1 \leq j \leq d-2\}$ contains $2(d-2)-1 = 2d-5 \geq d$ admissible cycles, a contradiction. This proves the lemma. ∎

For every graph H, a *1-subdivision* of H is a graph that is obtained from H by subdividing each edge exactly once.

Lemma 5.9. *Let G be a graph of girth at least five. Let H be a subgraph of G that is a 1-subdivision of K_4. If there exists a vertex in $V(G) - V(H)$ adjacent in G to two vertices in $V(H)$, then G contains a cycle of length five or ten.*

Proof. We may assume G is of girth at least six, for otherwise we are done. Let v be a vertex in $V(G) - V(H)$ adjacent in G to two vertices x, y in $V(H)$. Let S be the set of vertices of H of degree three. Since G has girth at least five, at least one of x, y does not belong to S. Then since G has girth at least six, both x, y do not belong to S. So there exist edges e, e' of K_4 such that x and y are obtained by subdividing e and e', respectively. Since G has girth at least five, e and e' form a matching in K_4. Let z be a vertex of H obtained by subdividing an edge other than e, e'. Then $(H - \{z\}) + vx + vy$ has a Hamiltonian cycle of length ten. This proves the lemma. ∎

We say a graph is a *theta graph* is a subdivision of K_4^-. The *branch vertices* of a theta graph are the vertices of degree at least three. A subgraph H of a graph G is *spanning* if $V(H) = V(G)$.

Lemma 5.10. *Let G be a graph of girth at least six that does not contain a cycle of length ten. Let H be a subgraph of G isomorphic to a theta graph such that $|V(H)|$ is minimum. Then the following hold.*

1. *H is an induced subgraph of G.*
2. *There exists at most one vertex of $G - V(H)$ adjacent in G to at least two vertices in $V(H)$.*
3. *If there exists a vertex v of $G - V(H)$ adjacent in G to at least two vertices in $V(H)$, then $G[V(H) \cup \{v\}]$ is isomorphic to a 1-subdivision of K_4.*

Proof. Suppose that H is not induced. Then there exists $e \in E(G) - E(H)$ with both ends in $V(H)$. Since the girth of G is at least six, there exists a subgraph H' of G with $V(H') \subset V(H)$ such that H' is a theta graph, contradicting the minimality of $|V(H)|$.

So H is an induced subgraph. We may assume there exists a vertex v of $G - V(H)$ adjacent in G to at least two vertices in $V(H)$, for otherwise we are done. Let x, y be the branch vertices of H. Let P_1, P_2, P_3 be the three internally disjoint paths in H from x to y.

By the minimality of $|V(H)|$ and the girth condition of G, v is not adjacent to any branch vertices of H. Similarly, for each $i \in \{1, 2, 3\}$, v is adjacent to at most one vertex in $V(P_i)$. So there exist distinct i, j such that v is adjacent to exactly one vertex a in $V(P_i) - \{x, y\}$ and exactly one vertex b in $V(P_j) - \{x, y\}$. By symmetry, we may assume $i = 1$ and $j = 2$. Since $(H - (V(P_3) - \{x, y\})) + av + bv$ is a theta graph, by the minimality of $|V(H)|$, $|V(P_3)| \leq 3$. Let L_1 be the subpath of P_1 from x to a. Since the graph obtained from $H + av + bv$ by deleting all internal vertices of L_1 is a theta graph, L_1 contains at most one internal vertex by the minimality of $|V(H)|$. Similarly, the subpath L_2 of P_2 from x to b contains at most one internal vertex. Since $L_1 \cup L_2 \cup avb$ is a cycle in G and G has girth at least six, both L_1, L_2 contain exactly one internal vertex. Similarly, each of the subpath of P_1 from a to y and the subpath of P_2 from b to y contains exactly one vertex. Since $P_1 \cup P_3$ is a cycle in G, P_3 contains exactly one internal vertex. Hence $G[V(H) \cup \{v\}]$ contains a 1-subdivision of K_4 as a spanning subgraph. Since G is of girth at least six, $G[V(H) \cup \{v\}]$ is isomorphic to a 1-subdivision of K_4.

If there exists a vertex v' of $V(G) - V(H)$ other than v adjacent in G to at least two vertices in $V(H)$, then v' is adjacent in G to at least two vertices in $V(H) \cup \{v\}$ which induces a subgraph isomorphic to a 1-subdivision of K_4, so G contains a cycle of length ten by Lemma 5.9, a contradiction. So v is the only vertex that is adjacent in G to at least two vertices in $V(H)$. This proves the lemma. ∎

Lemma 5.11. *Let $d \geq 5$ be an integer. Let G be a 5-connected graph of girth at least six and of minimum degree at least d that does not contain a cycle of length ten. Let H be an induced subgraph of G isomorphic to a 1-subdivision of K_4. Then G contains d admissible cycles.*

Proof. Suppose to the contrary that G does not contain d admissible cycles. Note that every vertex of $G - V(H)$ is adjacent in G to at most one vertex in $V(H)$ by Lemma 5.9. We say a pair of two distinct vertices x, y of H are *useful* if there exist paths H_1, H_2, H_3 in H from x to y of lengths h_1, h_2, h_3, respectively, such that $(h_1, h_2, h_3) \in \{(1, 3, 5), (2, 4, 6), (1, 5, 7)\}$.

Let M be a component of $G - V(H)$. Since G is 5-connected, there exists a useful pair of vertices x, y such that x is adjacent in G to some vertex x' in $V(M)$ and y is adjacent in G to some vertex y' in $V(M)$. Note that $x' \neq y'$, for otherwise some vertex of M is adjacent in G to two vertices in $V(H)$.

Suppose M is 2-connected. Then (M, x', y') is a 2-connected rooted graph of minimum degree at least $d - 1$. By Theorem 3.1, there exist $d - 2$ admissible paths $P_1, \dots, P_{d-2}$ in G' from x' to y'. Since $d \geq 5$, the set $\{(P_i \cup H_j) + xx' + yy' : 1 \leq i \leq d - 2, 1 \leq j \leq 3\}$ contains d admissible cycles, a contradiction.

So M is not 2-connected. Let B_1, B_2 be two distinct end-blocks of M. Let b_1, b_2 be the cut-vertex of $G - V(H)$ contained in $V(B_1), V(B_2)$, respectively. Since G is 3-connected, some vertex x_1 in

$V(B_1) - \{b_1\}$ is adjacent in G to some vertex u_1 in $V(H)$, and some vertex x_2 in $V(B_2) - \{b_2\}$ is adjacent in G to some vertex u_2 in $V(H)$. For each $i \in \{1,2\}$, since (B_i, x_i, b_i) is a 2-connected rooted graph of minimum degree at least $d-1$, there exist $d-2$ admissible paths $Q_{i,1}, ..., Q_{i,d-2}$ in B_i from x_i to b_i. Let Q be a path in M from b_1 to b_2 internally disjoint from $V(B_1) \cup V(B_2)$, and let Q' be a path in H from u_1 to u_2. Then the set $\{(Q_{1,i} \cup Q \cup Q_{2,j} \cup Q') + x_1u_1 + x_2u_2 : 1 \leq i \leq d-2, 1 \leq j \leq d-2\}$ contains $2(d-2)-1 = 2d-5 \geq d$ admissible cycles, a contradiction. This proves the lemma. ∎

Lemma 5.12. *Let H be a theta graph. Then there exist two distinct vertices x, y and three paths in H from x to y such that the lengths of these three paths modulo 5 are pairwise distinct.*

Proof. Let P_1, P_2, P_3 be the three internally disjoint paths in H between the branch vertices of H. For each $i \in \{1,2,3\}$, we denote P_i by $v_{i,0}v_{i,1}...v_{i,|E(P_i)|}$, where $v_{1,0} = v_{2,0} = v_{3,0}$. For each $i \in \{1,2,3\}$ and each $j \in \{1, ..., |E(P_i)|\}$, let $L_{i,j} = v_{i,0}v_{i,1}...v_{i,j}$ and let $R_{i,j} = v_{i,j}v_{i,j+1}...v_{i,|E(P_i)|}$.

Suppose to the contrary that there do not exist two distinct vertices x, y and three paths from x to y with pairwise distinct lengths modulo 5. So the lengths of P_1, P_2, P_3 modulo 5 are not pairwise distinct. Hence, by symmetry, there exists $t \in \{0,1,2,3,4\}$ such that $|E(P_1)|$ and $|E(P_2)|$ equal t modulo 5.

Suppose that $|E(P_3)| = t$ modulo 5. By symmetry, we may assume that $|E(P_3)| \leq |E(P_i)|$ for every $i \in \{1,2\}$. So $\min\{|E(P_1)|, |E(P_2)|\} \geq 2$. Note that the paths $R_{1,1} \cup R_{2,2}, L_{1,1} \cup P_3 \cup R_{2,2}, R_{1,1} \cup P_3 \cup L_{2,2}$ are three paths from $v_{1,1}$ to $v_{2,2}$ with lengths $2t-3, 2t-1, 2t+1$ modulo 5, respectively, a contradiction.

Hence there exists $s \in \{0,1,2,3,4\} - \{t\}$ such that $|E(P_3)| = s$ modulo 5. By symmetry, we may assume that $|E(P_1)| > 1$. For every $r \in \{1,2\}$, the paths $R_{1,r}, L_{1,r} \cup P_2, L_{1,r} \cup P_3$ are three paths from $v_{1,r}$ to $v_{1,|E(P_1)|}$ of lengths $t-r, t+r, s+r$ modulo 5, respectively, so $t-r = s+r$ modulo 5. That is, $t-1 = s+1$ modulo 5 and $t-2 = s+2$ modulo 5, a contradiction. This proves the lemma. ∎

Lemma 5.13. *Let a and d be integers such that $d \in \{1,2\}$. Let B be a subset of $\{0,1,2,3,4\}$ of size three. Then the set $\{a+id+b : 0 \leq i \leq 2, b \in B\}$ contains a multiple of 5.*

Proof. Let $X = \{a+id+b : 0 \leq i \leq 2, b \in B\}$. If there exists an integer s such that the three elements of B are either $s, s+1, s+2$ modulo 5 or $s, s+2, s+4$ modulo 5, then X contains a multiple of 5. So by shifting, we may without loss of generality assume that $B = \{0,1,3\}$. If $d = 1$, then $X \supseteq \{a, a+1, a+2, a+3, a+4\}$, so X contains a multiple of 5. If $d = 2$, then $X \supseteq \{a, a+1, a+2, a+3, a+4\}$, so X contains a multiple of 5. This proves the lemma. ∎

Lemma 5.14. *If G is a 5-connected graph of girth at least five, then G contains a cycle of length 0 modulo 5.*

Proof. Suppose to the contrary that G does not contain a cycle of length 0 modulo 5. In particular, G does not contain a 5-cycle and a 10-cycle, and G does not contain five admissible cycles. So the girth of G is at least six, and G does not contain a cycle of length ten.

Let H be a subgraph of G isomorphic to a theta graph with $|V(H)|$ minimum. By Lemma 5.10, H satisfies the following.

- H is an induced subgraph of G.
- There exists at most one vertex of $G - V(H)$ adjacent in G to at least two vertices in $V(H)$.
- If there exists a vertex v of $G - V(H)$ adjacent in G to at least two vertices in $V(H)$, then $G[V(H) \cup \{v\}]$ is isomorphic to a 1-subdivision of K_4.

If there exists a vertex of $G-V(H)$ adjacent in G to at least two vertices of $V(H)$, then there exists an induced subgraph H' isomorphic to an induced 1-subdivision of K_4, so by Lemma 5.11, G contains five admissible cycles, a contradiction.

So every vertex of $G-V(H)$ is adjacent in G to at most one vertex in $V(H)$. Let $G' = G-V(H)$. Let $d = 5$.

Suppose that there exists a component M of G' such that M is not 2-connected. Let B_1, B_2 be distinct end-blocks of M. Since G is 3-connected and every vertex in $V(M)$ is adjacent in G to at most one vertex in $V(H)$, B_1 and B_2 are 2-connected. For each $i \in \{1,2\}$, let b_i be the cut-vertex of M contained in $V(B_i)$. Since G is 3-connected, for each $i \in \{1,2\}$, there exists $x_i \in V(B_i)-\{b_i\}$ such that x_i is adjacent in G to some vertex y_i in $V(H)$. For each $i \in \{1,2\}$, since (B_i, x_i, b_i) is a 2-connected rooted graph of minimum degree at least $d-1$, there exist $d-2$ admissible paths $P_{i,1}, ..., P_{i,d-2}$ in B_i from x_i to b_i by Theorem 3.1. Let Q be a path in M from b_1 to b_2 internally disjoint from $V(B_1) \cup V(B_2)$. Let Q' be a path in H from y_1 to y_2. Then the set $\{(P_{1,i} \cup Q \cup P_{2,j} \cup Q') + x_1y_1 + x_2y_2 : 1 \leq i \leq d-2, 1 \leq j \leq d-2\}$ contains $2(d-2)-1 \geq d = 5$ admissible paths, a contradiction.

So every component of G' is 2-connected. Suppose that G' is not connected. Let M_1, M_2 be two distinct components of G'. For each $i \in \{1,2\}$, since G is 4-connected, there exist distinct vertices $x_{i,1}$ and $x_{i,2}$ in $V(M_i)$ such that $x_{i,1}$ is adjacent in G to a vertex $y_{i,1}$ in $V(H)$ and $x_{i,2}$ is adjacent in G to a vertex $y_{i,2}$ in $V(H)$. For each $i \in \{1,2\}$, since $(M_i, x_{i,1}, x_{i,2})$ is a 2-connected rooted graph of minimum degree at least $d-1$, there exist $d-2$ admissible paths $R_{i,1}, ..., R_{i,d-2}$ in M_i from $x_{i,1}$ to $x_{i,2}$ by Theorem 3.1. Since H is 2-connected, there exist two disjoint paths Q_1, Q_2 in H from $\{y_{1,1}, y_{1,2}\}$ to $\{y_{2,1}, y_{2,2}\}$. Then the set $\{(R_{1,i} \cup Q_1 \cup R_{2,j} \cup Q_2) + x_{1,1}y_{1,1} + x_{1,2}y_{1,2} + x_{2,1}y_{2,1} + x_{2,2}y_{2,2} : 1 \leq i \leq d-2, 1 \leq j \leq d-2\}$ contains $2(d-2)-1 \geq 5$ admissible cycles, a contradiction.

So G' is 2-connected. By Lemma 5.12, there exist two distinct vertices x, y in H such that there exist three paths A_1, A_2, A_3 in H from x to y with pairwise distinct lengths modulo 5. Since G is 5-connected and H is an induced subgraph, there exist distinct vertices x', y' in $V(G')$ such that $\{xx', yy'\} \subseteq E(G)$. Since (G', x', y') is a 2-connected rooted graph of minimum degree at least $d-1=4$, by Theorem 3.1, there exist three admissible paths Z_1, Z_2, Z_3 in G' from x' to y'. By Lemma 5.13, the set $\{(Z_i \cup A_j) + xx' + yy' : 1 \leq i \leq 3, 1 \leq j \leq 3\}$ contains a cycle of length 0 modulo 5, a contradiction. This proves the lemma. ∎

Now we are ready to prove Theorem 1.9. The following is a restatement of Theorem 1.9.

Theorem 5.15. *For $d \geq 3$, every d-connected graph contains a cycle of length zero modulo d.*

Proof. By [3, Theorem 1] and [6, Theorem 1.2], the theorem is true for $d \in \{3,4\}$. So we may assume that $d \geq 5$. Suppose to the contrary that G does not contain a cycle of length 0 modulo d. So G does not contain a K_d' subgraph and does not contain a $K_{d,d}^-$ subgraph. In addition, G does not contain d cycles of consecutive length, and when d is odd or G is bipartite, G does not contain d admissible cycles.

Since G is $(d-1)$-connected and does not contain a K_d' subgraph, G does not contain a K_4^- subgraph by Lemma 5.1. Since G does not contain a K_4^- subgraph, by Lemma 5.2, G does not contain a K_3 subgraph. Since G does not contain a K_3 subgraph, by Lemma 5.5, either $d=5$ or G is bipartite. Since G does not contain a K_3 subgraph and a $K_{d,d}^-$ subgraph, by Lemma 5.6, either G does not contain a 4-cycle, or G contains a cycle of length four and a cycle of length five, or d is even.

Suppose that G does not contain a 4-cycle. Then G is not bipartite by Lemma 5.8. So $d=5$. Since G does not contain a K_3 subgraph and a 4-cycle, G is of girth at least five. So G contains a cycle of length 0 modulo $5 = d$ by Lemma 5.14, a contradiction.

So either G contains a 4-cycle and a 5-cycle, or d is even. Note that either case implies $d \neq 5$. So G is bipartite, contradicting that G contains a 5-cycle. This proves the theorem. ∎

When $d \geq 6$, we can strengthen the conclusion of Theorem 5.15.

Theorem 5.16. *For integers $d \geq 6$ and t satisfying $2t \neq 2$ modulo d, every d-connected graph contains a cycle of length $2t$ modulo d.*

Proof. Suppose to the contrary that there exist integers $d \geq 6$ and t with $2t \neq 2$ modulo d such that there exists a d-connected graph G that does not contain a cycle of length $2t$ modulo d. In particular, G does not contain a K'_{d+1} subgraph and does not contain a $K^-_{d,d}$ subgraph. In addition, G does not contain d cycles of consecutive length, and when d is odd or G is bipartite, G does not contain d admissible cycles.

By Lemma 5.1, G does not contain a K_4^- subgraph. By Lemma 5.2, G does not contain a K_3 subgraph. By Lemma 5.5, G is bipartite. Since G is bipartite, by Lemma 5.6, G does not contain a 4-cycle. By Lemma 5.8, G contains d admissible cycles. But G is bipartite, a contradiction. ∎

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