On the rainbow matching conjecture for 3-uniform hypergraphs

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Abstract

Aharoni and Howard, and, independently, Huang, Loh, and Sudakov proposed the following rainbow version of Erdős matching conjecture: For positive integers n, k, m with $n \geq km$, if each of the families $F_1, \ldots, F_m \subseteq \binom{[n]}{k}$ has size more than $\max\{\binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k}\}$, then there exist pairwise disjoint subsets $e_1, \ldots, e_m$ such that $e_i \in F_i$ for all $i \in [m]$. We prove that there exists an absolute constant n_0 such that this rainbow version holds for $k = 3$ and $n \geq n_0$. We convert this rainbow matching problem to a matching problem on a special hypergraph H. We then combine several existing techniques on matchings in uniform hypergraphs: find an absorbing matching M in H; use a randomization process of Alon et al to find an almost regular subgraph of $H - V(M)$; and find an almost perfect matching in $H - V(M)$. To complete the process, we also need to prove a new result on matchings in 3-uniform hypergraphs, which can be viewed as a stability version of a result of Łuczak and Mieczkowska and might be of independent interest.

Key words: Rainbow matching conjecture, Erdős matching conjecture, Stability

MSC2010: 05C65, 05D05.

1 Introduction

For a positive integer k and a set V, let $[k] := \{1, ..., k\}$ and $\binom{V}{k} := \{A \subseteq V : |A| = k\}$. A hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq 2^{V(H)}$. A hypergraph H is *k-uniform* if all its edges have size k and we call it a *k-graph* for short. Throughout this paper, we often identify $E(H)$ with H when there is no confusion and, in particular, denote by $|H|$ the number of edges in H. Given a set T of edges in H, we use $V(T)$ to denote $\bigcup_{e \in T} e$. Given a vertex subset $S \subseteq V(H)$ in H, we use $H[S]$ to denote the subgraph of H induced on S, and let $H - S = H[V(H) \setminus S]$.

A *matching* in a hypergraph H is a set of pairwise disjoint edges in H. We use $\nu(H)$ to denote the maximum size of a matching in H. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of hypergraphs on the same vertex set. A set of m pairwise disjoint edges is called a *rainbow matching* for $\mathcal{F}$ if each edge is from a different F_i. If such a matching exists, then we also say that *$\mathcal{F}$ admits a rainbow matching.*

A classical problem in extremal set theory asks for the maximum number of edges in n-vertex k-graphs H with $\nu(H) < m$. Let n, k, m be positive integers with $n \geq km$. The k-graphs $S(n, m, k) := \binom{[n]}{k} \setminus \binom{[n]\setminus[m-1]}{k}$ and $D(n, m, k) := \binom{[km-1]}{k}$ on the same vertex set $[n]$ do not have matchings of size m. Erdős [6] conjectured in 1965 that among all k-graphs with no matching of size m, $S(n, m, k)$ or $D(n, m, k)$ has the maximum number of edges: Any n-vertex k-graph H with $\nu(H) < m$ contains at most

$$f(n, m, k) := \max\left\{\binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k}\right\}$$

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edges. This is often referred to as the *Erdős matching conjecture* in the literature, and there has been extensive research on this conjecture, see, for instance, [3, 5, 8, 9, 10, 11, 12, 22]. In particular, the special case for $k = 3$ was settled for large n by Łuczak and Mieczkowska [22] and completely resolved by Frankl [9].

The following analogous conjecture, known as the *rainbow matching conjecture*, was made by Aharoni and Howard [1] and, independently, by Huang, Loh, and Sudakov [15]. For related topics on rainbow type problems, we refer the interested reader to [16, 18, 20, 23].

Conjecture 1.1 ([1, 15])**.** *Let n, k, m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of k-graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, k)$ for all $i \in [m]$. Then $\mathcal{F}$ admits a rainbow matching.*

The case $k = 2$ of this conjecture is in fact a direct consequence of an earlier result of Akiyama and Frankl [2] (which was restated [7]). The following was obtained by Huang, Loh, and Sudakov [15].

Theorem 1.2 ([15], Theorem 3.3)**.** *Conjecture 1.1 holds when $n > 3k^2m$.*

Keller and Lifshitz [17] proved that Conjecture 1.1 holds when $n \geq f(m)k$ for some large constant $f(m)$ which only depends on m, and this was further improved to $n = \Omega(m \log m)k$ by Frankl and Kupavskii [13]. Both proofs use the junta method. Very recently, Lu, Wang, and Yu [21] showed that Conjecture 1.1 holds when $n \geq 2km$ and n is sufficiently large.

The following is our main result, which proves Conjecture 1.1 for $k = 3$ and sufficiently large n.

Theorem 1.3. *There exists an absolute constant n_0 such that the following holds for all $n \geq n_0$. For any positive integers n, m with $n \geq 3m$, let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of 3-graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for all $i \in [m]$. Then $\mathcal{F}$ admits a rainbow matching.*

Our proof of Theorem 1.3 uses some new ideas and combines different techniques from Alon-Frankl-Huang-Rődl-Rucinski-Sudakov [3], Łuczak-Mieczkowska [22], and Lu-Yu-Yuan [19]. (For a high level description of our proof, we refer the reader to Section 2 and/or Section 7.) In the process, we prove a stability result on 3-graphs (see Lemma 4.2) that plays a crucial role in our proof and might be of independent interest: If the number of edges in an n-vertex 3-graph H with $\nu(H) < m$ is close to $f(n, m, 3)$, then H must be close to $S(n, m, 3)$ or $D(n, m, 3)$.

The rest of the paper is organized as follows. In Section 2, we introduce additional notation, and state and/or prove a few lemmas for later use. In Section 3, we deal with the families $\mathcal{F}$ in which most 3-graphs are close to the same 3-graph that is $S(n, m, 3)$ or $D(n, m, 3)$. To deal with the remaining families, we need the above mentioned stability result for matchings in 3-graphs, which is done in Section 4. In Section 5, we show that there exists an absolute constant $c > 0$ such that Theorem 1.3 holds for $m > (1 - c)n/3$. The proof of Theorem 1.3 for $m \leq (1 - c)n/3$ is completed in Section 6. Finally, we complete the proof of Theorem 1.3 in Section 7.

2 Previous results and lemmas

In this section, we define saturated families and stable hypergraphs, and state several lemmas that we will use frequently. We begin with some notation. Suppose that H is a hypergraph and U, T are subsets of $V(H)$. Let $N_H(T) := \{A : A \subseteq V(H) \setminus T \text{ and } A \cup T \in E(H)\}$ be the *neighborhood* of T in H, and let $d_H(T) := |N_H(T)|$. We write $d_H(v)$ for $d_H(\{v\})$. Let $\Delta(H) := \max_{v \in V(H)} d_H(v)$ and $\Delta_2(H) := \max_{T \in \binom{V(H)}{2}} d_H(T)$. In case $T \subseteq U$, we often identify $d_{H[U]}(T)$ with $d_U(T)$ when there is no confusion.

It will be helpful to consider "maximal" counterexamples to Conjecture 1.1. Let n, k, m be positive integers with $n \geq km$ and let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of k-graphs on the same vertex set $[n]$. We say that $\mathcal{F}$ is **saturated**, if $\mathcal{F}$ does not admit a rainbow matching, but for every $F \in \mathcal{F}$ and $e \notin F$, the new family $\mathcal{F}(e, F) := (\mathcal{F} \backslash \{F\}) \cup \{F \cup \{e\}\}$ admits a rainbow matching. The following lemma says that the vertex degrees of every k-graph in a saturated family are typically small.

Lemma 2.1. *Let n, k, m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a saturated family of k-graphs on the same vertex set $[n]$. Then for each $v \in [n]$ and each $i \in [m]$, $d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}$ or $d_{F_i}(v) = \binom{n-1}{k-1}$.*

Proof. Suppose $d_{F_i}(v) < \binom{n-1}{k-1}$, where $v \in [n]$ and $i \in [m]$. Then there exists $e \in \binom{[n]}{k} \setminus F_i$ such that $v \in e$. Since $\mathcal{F}$ is saturated, the family $\mathcal{F}(e, F_i)$ admits a rainbow matching, say $M \cup \{e\}$, with M being a rainbow matching for the family $\mathcal{F} \setminus \{F_i\}$.

If $d_{\mathcal{F}_i}(v) > \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1} = \left|\binom{[n]\setminus\{v\}}{k-1} \setminus \binom{[n]\setminus(\{v\}\cup V(M))}{k-1}\right|$, then there exists an edge $f \in F_i$ such that $v \in f$ and $f \cap V(M) = \emptyset$. Now $M \cup \{f\}$ is a rainbow matching for $\mathcal{F}$, a contradiction. So $d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}$. □

We will be removing vertices of degree $\binom{n-1}{k-1}$ and use Lemma 2.1 to produce saturated family $\mathcal{F} = \{F_1, ..., F_m\}$ of k-graphs such that for each $v \in V(F_i)$ and each $i \in [m]$, $d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-1-k(m-1)}{k-1}$.

Next we define stable hypergraphs. Let n, k be positive integers with $n \geq k$. Let $e = \{a_1, ..., a_k\}$ and $f = \{b_1, ..., b_k\}$ be members of $\binom{[n]}{k}$ with $a_1 < a_2 < ... < a_k$ and $b_1 < b_2 < ... < b_k$. We write $e \leq f$ if $a_i \leq b_i$ for all $1 \leq i \leq k$, and $e < f$ if $e \leq f$ and $e \neq f$.

A k-graph $F \subseteq \binom{[n]}{k}$ is said to be **stable** if $e < f \in F$ implies $e \in F$. A family $\mathcal{F}$ of k-graphs on the same vertex set $[n]$ is **stable** if each k-graph in $\mathcal{F}$ is stable.

The following result of Huang, Loh, and Sudakov [15] will be used frequently, which enables us to work with stable families when proving Conjecture 1.1.

Lemma 2.2 ([15], Lemma 2.1)**.** *Let n, k, m be positive integers with $n \geq km$. If the family $\{F_1, ..., F_m\}$ of k-graphs with $V(F_i) = [n]$ for all $i \in [m]$ has the property that it does not admit a rainbow matching, then there exists a stable family $\{F'_1, ..., F'_m\}$ of k-graphs with $|F_i| = |F'_i|$ and $V(F'_i) = [n]$ for all $i \in [m]$ which still preserves this property.*

Corollary 2.3. *Let n, k, m be positive integers with $n \geq km$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the vertex set $[n]$ that does not admit a rainbow matching. Then there exists a family $\mathcal{F}' = \{F'_1, \ldots, F'_m\}$ of k-graphs on the same vertex set $[n]$ such that $\mathcal{F}'$ is both stable and saturated and $|F'_i| \geq |F_i|$ for $i \in [m]$.*

Proof. Let $\mathcal{F}^* = \{F^*_1, \ldots, F^*_m\}$ be a family of k-graphs on the same vertex set $[n]$ such that $\mathcal{F}^*$ admits no rainbow matching, $|F^*_i| \geq |F_i|$ for $i \in [m]$, and, subject to these, $\sum_{i\in[m]} |F^*_i|$ is maximum.

Then $\mathcal{F}^*$ is saturated. Now apply Lemma 2.2 to $\mathcal{F}^*$ we obtain a stable family $\mathcal{F}' = \{F'_1, \ldots, F'_m\}$ of k-graphs on the vertex set $[n]$ such that $\mathcal{F}'$ admits no rainbow matching, and $|F'_i| = |F^*_i|$ for $i \in [m]$. By the choice of $\mathcal{F}^*$, we see that $\mathcal{F}'$ is also saturated. □

We now describe an operation that converts a rainbow matching problem to a matching problem on a single hypergraph. Let n, k, m, r be non-negative integers, with $r = \lfloor n/k \rfloor - m$ and $m \geq 1$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a family of k-graphs on the same vertex set $[n]$, and let $\mathcal{V} = \{v_1, ..., v_m\}$ and $\mathcal{U} = \{u_1, ..., u_r\}$ be two disjoint sets such that $(\mathcal{V} \cup \mathcal{U}) \cap [n] = \emptyset$. We use $H(\mathcal{F})$ to denote the $(k+1)$-graph with vertex set $[n] \cup \mathcal{V}$ and edge set $\bigcup_{i=1}^m \{e \cup \{v_i\} : e \in F_i\}$, and use $H^*(\mathcal{F})$ to denote the $(k+1)$-graph with the vertex set $[n] \cup \mathcal{V} \cup \mathcal{U}$ and the edge set $E(H(\mathcal{F})) \cup \bigcup_{i=1}^r \{e \cup \{u_i\} : e \in \binom{[n]}{k}\}$. If $F_1 = ... = F_m = S(n, m, k)$ (respectively, $F_1 = ... = F_m = D(n, m, k)$), then we write $H(\mathcal{F})$ as $H_S(n, m, k)$ (respectively, $H_D(n, m, k)$).

It is easy to see that $\mathcal{F}$ admits a rainbow matching if and only if $H(\mathcal{F})$ has a matching of size m, which is also if and only if $H^*(\mathcal{F})$ has a matching of size $m + r$. This allows us to access existing approaches and tools invented for matching problems. For instance, we take the approach by considering whether or not the hypergraphs $H(\mathcal{F})$ in question are close to the extremal configurations $H_S(n, m, k)$ and $H_D(n, m, k)$. We will see in Section 3 that if $H(\mathcal{F})$ is close to $H_D(n, m, k)$ and $\mathcal{F}$ is stable, then $\mathcal{F}$ admits a rainbow matching.

Here we give an easy lemma concerning a case when $H(\mathcal{F})$ is not close to $H_S(n, m, k)$, which will be used along with Lemma 2.1. Let H_1 and H_2 be two k-graphs on the same vertex set V and let ϵ be some positive real; we say that H_2 is **ϵ-close** to H_1 if $|E(H_1) \setminus E(H_2)| \leq \epsilon |V|^k$.

Lemma 2.4. *For any given integer $k \geq 3$, let ϵ, c be reals such that $0 < \epsilon \ll c \ll 1$.*[1] *Let n, m be integers such that $n/3k^2 \leq m \leq (1-c)n/k$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of k-graphs on vertex set $[n]$. If for every $i \in [m]$ and $v \in [n]$, $d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1}$, then $H(\mathcal{F})$ is not ϵ-close to $H_S(n, m, k)$.*

Proof. We note that $S(n, m, k)$ has $m-1$ vertices of degree $\binom{n-1}{k-1}$. Since for every $i \in [m]$ and $v \in [n]$, $d_{\mathcal{F}_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1}$, we have

$$|E(H_S(n,m,k)) \setminus E(H(\mathcal{F}))| \geq m \cdot (m-1) \cdot \binom{n-k(m-1)-1}{k-1} \cdot \frac{1}{k} > \frac{n^2}{10k^5}\binom{cn}{k-1} > \epsilon n^{k+1},$$

where the second inequality is due to $n/3k^2 \leq m \leq (1-c)n/k$ and the third inequality follows from $\epsilon \ll c$. This shows that $H(\mathcal{F})$ is not ϵ-close to $H_S(n, m, k)$. □

To deal with the case when $H(\mathcal{F})$ is not close to $H_D(n, m, 3)$, we first find a small matching M in $H^*(\mathcal{F})$ such that M can "absorb" small vertex sets and $H^*(\mathcal{F}) - V(M)$ has an almost perfect matching. When $\mathcal{F}$ is stable, the matching M can be found very easily by the following lemma and its proof.

Lemma 2.5. *Let k be a fixed positive integer and let $0 < \gamma' \ll \gamma \ll c \ll 1$ be reals. Let n, m be positive integers with $n/3k^2 \leq m \leq (1-c)n/k$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a stable family of k-graphs such that $V(F_i) = [n]$ and $|F_i| > f(n, m, k)$ for all $i \in [m]$. Then for sufficiently large n, $H^*(\mathcal{F})$ has a matching M with $|M| \leq \gamma n$ such that for any set $S \subseteq V(H^*(\mathcal{F})) \setminus V(M)$ with $|S| \leq \gamma' n$ and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$, $H^*(\mathcal{F})[V(M) \cup S]$ has a perfect matching.*

Proof. Recall that $\mathcal{V} = \{v_1, ..., v_m\}$ and $\mathcal{U} = \{u_1, ..., u_r\}$, where $r = \lfloor n/k \rfloor - m$. Fix an integer t satisfying $\gamma' n < t < \gamma n$. Then $t < \gamma n \leq \lfloor cn/k \rfloor \leq \lfloor n/k \rfloor - m = r$. Let $s = \lceil n/3k^2 \rceil - 1$.

By Theorem 1.2 (viewing all k-graphs as the same k-graph), since $|F_i| > f(n, m, k) \geq f(n, s, k)$ for all $i \in [m]$, every F_i has a matching of size s. Since F_i is stable, $F_i[[s]]$ is a complete k-graph. Hence,

(i) for any $i_1, i_2, ..., i_k \leq kt \leq k\gamma n < s$ and $j \in [m]$, we have $\{v_j, i_1, i_2, ..., i_k\} \in H^*(\mathcal{F})$.

From the definition of $H^*(\mathcal{F})$, we have

(ii) for any $i_1, i_2, ..., i_k \in [n]$ and $j \in [r]$, $\{u_j, i_1, i_2, ..., i_k\} \in H^*(\mathcal{F})$.

Since $t < r$, we may choose a matching M of size t in $H^*(\mathcal{F})$ with $V(M) = \{u_1, ..., u_t\} \cup [kt]$. Note that $|M| = t \leq \gamma n$. We claim that this M is the desired matching. To see this, consider any subset S with $S \cap V(M) = \emptyset$, $|S| \leq \gamma' n$, and $k|S \cap (\mathcal{V} \cup \mathcal{U})| = |S \cap [n]|$. Let $t' = |S \cap (\mathcal{V} \cup \mathcal{U})|$. So $t' \leq \gamma' n < t$. Then by (i) and (ii), there is a perfect matching M_1 in $H^*(\mathcal{F})[S \cap (\mathcal{V} \cup \mathcal{U})) \cup [kt']]$. By (ii), there exists a perfect matching M_2 in $H^*(\mathcal{F})[(V(M) \cup S) \setminus V(M_1)]$. So $M_1 \cup M_2$ is a perfect matching in $H^*(\mathcal{F})[V(M) \cup S]$. □

For the "absorbing" matching M in $H^*(\mathcal{F})$ in Lemma 2.5, we also want $H^*(\mathcal{F}) - V(M)$ to have an almost perfect matching. For this we need to use the following result of Frankl and Rödl [14].

Theorem 2.6 ([14]). *For every integer $k \geq 2$ and any real $\sigma > 0$, there exist $\tau = \tau(k, \sigma)$ and $d_0 = d_0(k, \sigma)$ such that for every integer $n \geq D \geq d_0$ the following holds: Every n-vertex k-graph H with $(1-\tau)D < \Delta_1(H) < (1+\tau)D$ and $\Delta_2(H) < \tau D$ contains a matching covering all but at most σn vertices.*

In order to obtain a k-graph H satisfying Theorem 2.6, we use the approach from [3] by conducting two rounds of randomization on $H^*(\mathcal{F}) - V(M)$. We summarize part of the proof in [3] (more precisely, the proof of Claim 4.1) as a lemma. A *fractional matching* in a k-graph H is a function $w : E(H) \to [0, 1]$ such that for any $v \in V(H)$, $\sum_{\{e \in E(H): v \in e\}} w(e) \leq 1$. A fractional matching is called *perfect* if $\sum_{e \in E(H)} w(e) = |V(H)|/k$.

[1]Here and throughout the rest of the paper, the notation $a \ll b$ means that a is sufficiently small compared with b which need to satisfy finitely many inequalities in the proof.

Lemma 2.7 ([3], retained from the proof of Claim 4.1)**.** *Let $k \geq 3$ and H be a k-graph on at most $2n$ vertices. Suppose that there are subsets $R^i \subseteq V(H)$ for $i = 1, ..., n^{1.1}$ satisfying the following:*

(a). every vertex $v \in V(H)$ satisfies that $|\{i : v \in R^i\}| = (1 + o(1))n^{0.2}$,

(b). every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i,

(c). every edge $e \in H$ is contained in at most one set R^i, and

(d). for every $i = 1, ..., n^{1.1}$, R^i has a perfect fractional matching w^i.

Then H has a spanning subgraph H' such that $d_{H'}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H')$ and $\Delta_2(H') \leq n^{0.1}$.

We will also need to control the indepence number of random subgraphs of $H^*(\mathcal{F}) - V(M)$. The intuition is that when $H(\mathcal{F})$ is not close to $H_D(n, m, k)$ or $H_S(n, m, k)$, $H^*(\mathcal{F}) - V(M)$ does not have very large independence number. The following lemma in [19] was proved by Lu, Yu, and Yuan using the container method.

Lemma 2.8 ([19], Lemma 5.4)**.** *Let d, ϵ', α be positive reals and let k, n be positive integers. Let H be an n-vertex k-graph such that $e(H) \geq dn^k$ and $e(H[S]) \geq \epsilon' e(H)$ for all $S \subseteq V(H)$ with $|S| > \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of H uniformly at random with probability $n^{-0.9}$. Then for any positive real $\gamma \ll \alpha$, the size of maximum independent sets in $H[R]$ is at most $(\alpha+\gamma)n^{0.1}$ with probability at least $1 - (n^{O(1)}e^{-\Omega(n^{0.1})})$*

We need an inequality on the function $f(n, m, k)$ proved by Frankl in [9].

Lemma 2.9 ([9], Proposition 5.1)**.** *Let n, m, k be positive integers with $n \geq km-1$. Then $f(n, m, k) \geq f(n - 1, m - 1, k) + \binom{n-1}{k-1}$.*

We conclude this section with the well known Chernoff inequality.

Lemma 2.10 (Chernoff Inequality, see [4])**.** *Suppose $X_1, ..., X_n$ are independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}(X)$. Then for any $0 < \delta \leq 1$,*

$$\mathbb{P}[X \geq (1 + \delta)u] \leq e^{-\delta^2 u/3} \text{ and } \mathbb{P}[X \leq (1 - \delta)u] \leq e^{-\delta^2 u/3}. \tag{1}$$

In particular, if $X \sim \text{Bin}(n, p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}([|X - np|] \geq \lambda) \leq e^{-\Omega(\lambda^2/np)}. \tag{2}$$

3 Extremal configuration $H_D(n, m, 3)$

From Lemma 2.1 and Lemma 2.4, we see that if $\mathcal{F}$ is a saturated family of k-graphs on vertex set $[n]$ and $H(\mathcal{F})$ is close to the extremal configuration $H_S(n, m, k)$ then there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n-1}{k-1}$. Such vertices v can be removed from all k-graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$, so that $\mathcal{F}'$ admits a rainbow matching if and only if $\mathcal{F}$ admits a rainbow matching.

In this section, we consider the case when $H(\mathcal{F})$ is close to $H_D(n, m, 3)$ and $\mathcal{F}$ is stable.

Lemma 3.1. *Let ϵ, c be reals such that $0 < \epsilon \ll c \ll 1$. Let n, m be positive integers such that $n/27 \leq m \leq (1 - c)n/3$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a stable family of 3-graphs on vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for all $i \in [m]$. If $H(\mathcal{F})$ is ϵ-close to $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. Let $b = 6\epsilon^{1/6}n$. If F_i is $\sqrt{\epsilon}$-close to $D(n,m,3)$, then F_i contains a complete subgraph of size $3m-b$; for, otherwise, as F_i is stable, we have $|E(D(n,m,3)) \setminus E(F_i)| \geq \binom{b}{3} > \sqrt{\epsilon}n^3$, a contradiction.

We claim that for any $i \in [m]$ and $j \in \{0,...,b\}$, $\{2j+1, 2j+2, 3m-j\} \in F_i$. To prove this claim we fix $i \in [m]$. Suppose for a contradiction that there exists an integer $0 \leq t \leq b$ such that $\{2t+1, 2t+2, 3m-t\} \notin F_i$. Since $|F_i| > \binom{3m-1}{3}$ and F_i is stable, we have $\{1,2,3m\} \in F_i$. So $t \geq 1$. We now count the edges in F_i: Let q_1 be the number of edges of F_i in $[3m-1]$, and q_2 be the number of edges of F_i not contained in $[3m-1]$. Since F_i is stable and $\{2t+1, 2t+2, 3m-t\} \notin F_i$, we see that $\{a,b,c\} \notin F_i$ when $2t+2 \leq a < b < 3m-t \leq c \leq 3m-1$. So $q_1 \leq \binom{3m-1}{3} - t\binom{3m-3t-3}{2}$. Since $\{2t+1, 2t+2, 3m-t\} \notin F_i$, we have, for any $e \in F_i$ with $e \cap ([n] \setminus [3m-1]) \neq \emptyset$, $e \cap [2t] \neq \emptyset$. This shows $q_2 \leq 2t(n-3m+1)n$. First suppose that $n \leq \frac{7}{2}m$. Then we have

$$\begin{aligned} |F_i| &\leq \binom{3m-1}{3} - t\binom{3m-3t-3}{2} + 2tn(n-3m+1) \\ &\leq \binom{3m-1}{3} - t\left[\binom{3m-3t-3}{2} - 7m(m/2+1)\right] < \binom{3m-1}{3}, \end{aligned}$$

where the second inequality holds since $n \leq \frac{7}{2}m$, and the last inequality holds since $t \leq b = 6\epsilon^{1/6}n \ll m$, a contradiction. So we may assume $n > \frac{7}{2}m$. Let $m = \alpha n$; then $1/27 \leq \alpha < 2/7$. We assert that $\binom{n}{3} - \binom{n-m+1}{3} > \binom{3m-1}{3} + 2tn^2$. To see this, let $f(x) = 1-(1-x)^3-(3x)^3$, so

$$\frac{6}{n^3}\left(\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3}\right) = f(\alpha) + o(1).$$

Since $f'(x) = 3(1-2x-26x^2)$ is decreasing in $[1/27, 2/7]$ with $f'(1/27) > 0$ and $f'(2/7) < 0$, we have $f(\alpha) \geq \min\{f(1/27), f(2/7)\} = f(2/7) = \frac{2}{343}$ for $1/27 \leq \alpha < 2/7$. This shows that $\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} = \frac{f(\alpha)}{6}n^3 + o(n^3) \geq 2tn^2$, as asserted. Then it follows that

$$|F_i| \leq \binom{3m-1}{3} - t\binom{3m-3t-3}{2} + 2tn(n-3m+1) < \binom{3m-1}{3} + 2tn^2 < \binom{n}{3} - \binom{n-m+1}{3},$$

a contradiction as $|F_i| > f(n,m,3)$. This finishes the proof of Claim. □

Recall $\mathcal{V} = \{v_1,...,v_m\}$ from the definition of $H(\mathcal{F})$. By the above claim, $M_1 := \{\{v_i, 2i-1, 2i, 3m-i+1\} : i \in [b]\}$ is a matching in $H(\mathcal{F})$. Without loss of generality, let $F_1,...,F_a$ be all k-graphs in $\mathcal{F}$ which are not $\sqrt{\epsilon}$-close to $D(n,m,3)$. Since $H(\mathcal{F})$ is ϵ-close to $H_D(n,m,3)$, we have $a \leq \sqrt{\epsilon}n < b$. Then for any $j \in [m]\backslash[b]$, since F_j is $\sqrt{\epsilon}$-close to $D(n,m,3)$, F_j contains a complete subgraph with size at least $3m-b$. Hence we have $\{2j-1, 2j, 3m-j+1\} \in F_j$. So $M_2 := \{\{v_j, 2j-1, 2j, 3m-j+1\} : b < j \leq m\}$ is a matching in $H(\mathcal{F})$ which is disjoint from M_1. Then $M_1 \cup M_2$ forms a matching of size m in $H(\mathcal{F})$. So $\mathcal{F}$ admits a rainbow matching, completing the proof of Lemma 3.1. □

4 A stability lemma

In this section, we prove a result for stable 3-graphs, which may be viewed as a stability version of the following result of Łuczak and Mieczkowska proved in [22].

Theorem 4.1 ([22]). *There exists positive integer n_1 such that for integers m,n with $n \geq n_1$ and $1 \leq m \leq n/3$, if H is an n-vertex 3-graph with $e(H) > f(n,m,3)$, then $\nu(H) \geq m$.*

Building on the proof in [22], we prove the following.

Lemma 4.2. *For any real $\epsilon > 0$, there exists positive integer $n_1(\epsilon)$ such that the following holds. Let m,n be integers with $n \geq n_1(\epsilon)$ and $1 \leq m \leq n/3$, and let H be a stable 3-graph on the vertex set $[n]$. If $e(H) > f(n,m,3) - \epsilon^4 n^3$ and $\nu(H) < m$, then H is ϵ-close to $S(n,m,3)$ or $D(n,m,3)$.*

Proof. Suppose that $e(H) > f(n, m, 3) - \epsilon^4 n^3$ and $s := \nu(H) < m$. Let $M = \{(i_\ell, j_\ell, k_\ell) : \ell \in [s]\}$ be a largest matching in H and partition $V(M) = I \cup J \cup K$ such that every edge $(i, j, k) \in E(M)$ with $i < j < k$ satisfies $i \in I, j \in J$ and $k \in K$. Since H is stable, we may choose $V(M)$ to be $[3s]$.

Let $V' = [n]\backslash[3s]$. For $x \in [3s]$, let $e(x)$ denote the edge in M containing x. Let $F_1 = \{\{v\} \in \binom{[3s]}{1} : d_{V'}(v) \geq 20n\}$, $F_2 = \{\{v, w\} \in \binom{[3s]}{2} : e(v) \neq e(w) \text{ and } d_{V'}(v, w) \geq 20\}$, and $F_3 = \{\{u, v, w\} \in \binom{[3s]}{3} : e(u), e(v) \text{ and } e(w) \text{ are pairwise distinct}\}$. Let $H^* = ([3s], F)$ be the hypergraph with vertex set $[3s]$ and edge set $F = M \cup F_1 \cup F_2 \cup F_3$.

Call an edge $e \in H$ *traceable* if $e \cap [3s] \in F$, and *untraceable* otherwise. Since M is a maximum matching in H, V' is independent in H. So the number of untraceable edges of H is bounded from above by

$$\binom{3s}{1} \cdot 20n + \left(\binom{s}{2}\binom{3}{1}\binom{3}{1} \times 19 + \binom{s}{1}\binom{3}{2}n\right) + \binom{s}{1}\binom{3}{2}\binom{3s-3}{1} \leq 32n^2 = o(n^3),$$

where we use $s < m \leq n/3$. We point out that those edges (there being $o(n^3)$ of them) will be negligible in the following proof.

Let T be a triple of edges from M. We say T is *bad* if $V(T)$ contains three pairwise disjoint edges of H^* whose union intersects I in at most 2 vertices, and *good* otherwise. For each $i \in [3]$, let $f_i(T)$ denote the number of edges of F_i contained in $V(T)$. Note that $f_3(T) \leq 27$. The following two claims are explicit in [22].

Claim 1. *There exist no three pairwise disjoint bad triples (of edges in M). Hence, there exist at most six edges in M such that each bad triple contains one of these edges.*

Claim 2. *Let T be a good triple.*
(i) If $f_3(T) \geq 24$, then $f_1(T) = f_2(T) = 0$.
(ii) If $f_3(T) = 20$, then $f_1(T) \leq 1$ and $f_2(T) \leq 12$.
(iii) If $f_3(T) \leq 19$, then $f_1(T) \leq 3$ and $f_2(T) \leq 15$. Moreover, the only triples T for which $f_3(T) = 19$, $f_2(T) = 15$, and $f_1(T) = 3$, are those in which each edge of H^* contained in $V(T)$ intersects I.
(iv) If $f_3(T) = 21$, then $f_1(T) \leq 1$ and $f_2(T) \leq 10$
(v) If $22 \leq f_3(T) \leq 23$, then $f_1(T) = 0$ and $f_2(T) \leq 7$

We remove exactly six edges from M such that the resulting matching M' only contains good triples. Since H has at most $18n^2$ edges intersecting $V(M\backslash M')$ and $32n^2$ untraceable edges, we have

$$e(H) \leq |F_1|\binom{n-3s}{2} + |F_2|(n-3s) + |F_3| + 50n^2.$$

To bound $|F_i|$, let us consider the summation of $f_i(T)$ over all $T \in \binom{M'}{3}$. Since each edge from F_i is counted exactly $\binom{(s-6)-i}{3-i}$ times in this sum, we have $|F_i|\binom{(s-6)-i}{3-i} = \sum_{T\in\binom{M'}{3}} f_i(T)$. Therefore,

$$\begin{aligned} e(H) &\leq \sum_{T\in\binom{M'}{3}} \left(f_1(T)\frac{\binom{n-3s}{2}}{\binom{s-7}{2}} + f_2(T)\frac{n-3s}{s-8} + f_3(T)\right) + 50n^2 \\ &\leq \sum_{T\in\binom{M'}{3}} \left(f_1(T)\frac{(n-3s)^2}{s^2} + f_2(T)\frac{n-3s}{s} + f_3(T)\right) + O(n^2). \end{aligned}$$

Here, the last inequality is trivial when $s \leq 15$, and it holds when $s > 15$ because the difference between the above two summations is at most

$$\sum_{T\in\binom{M'}{3}} \left(f_1(T)\frac{15(n-3s)^2}{s(s^2-15s)} + f_2(T)\frac{8(n-3s)}{s(s-8)}\right) \leq \binom{s-6}{3}\left(\frac{45(n-3s)^2}{s(s^2-15s)} + \frac{120(n-3s)}{s(s-8)}\right) = O(n^2),$$

where $3s < n$, $f_1(T) \le 3$, and $f_2(T) \le 15$ (from Claim 2).

To further bound $e(H)$, we partition good triples T depending on $f_3(T)$ and $f_1(T)$. Let $T_i = \{T \in \binom{M'}{3} : f_3(T) = i\}$ for $i \in [27]$ and $X = \{T \in \binom{M'}{3} : f_1(T) = 3\}$. Consider any $T \in X$; so T is a good triple.[2] Since $f_1(T) = 3$, the three edges of F_1 contained in $V(T)$ are precisely the three vertices in $V(T) \cap I$, and each edge of H^* contained in $V(T)$ intersects I. Since H is stable and $V(M) = [3s]$, using the definition of F_1, it is not hard to see that $X \subseteq T_{19}$.

Define $x_1 = \sum_{i=1}^{18} |T_i| + |T_{19} \setminus X|$, $x_2 = |T_{20}|$, $x_3 = |T_{21}|$, $x_4 = |T_{22}| + |T_{23}|$, $x_5 = \sum_{i=24}^{26} |T_i|$, $x = |X|$, and $y = |T_{27}|$. So $\sum_{i=1}^{5} x_i + x + y = \binom{s-6}{3}$. From now on, we let $t = (n-3s)/s$. By Claim 2 and the fact $X \subseteq T_{19}$, we can derive from the above upper bound on $e(H)$ that

$$\begin{aligned} e(H) \le\ & (3x + 2x_1 + x_2 + x_3)t^2 + (15x + 15x_1 + 12x_2 + 10x_3 + 7x_4)t \\ & + (19x + 19x_1 + 20x_2 + 21x_3 + 23x_4 + 26x_5 + 27y) + O(n^2). \end{aligned}$$

For convenience, we write

$$f_t(x_1, x_2, x_3, x_4, x_5, x, y) = \sum_{i=1}^{5} \alpha_i(t) \cdot x_i + \beta_1(t) \cdot x + \beta_2(t) \cdot y,$$

$$\begin{aligned} \text{where} \quad & \alpha_1(t) = 2t^2 + 15t + 19, \quad \alpha_2(t) = t^2 + 12t + 20, \quad \alpha_3(t) = t^2 + 10t + 21 \\ & \alpha_4(t) = 7t + 23, \quad \alpha_5(t) = 26, \quad \beta_1(t) = 3t^2 + 15t + 19, \quad \text{and} \quad \beta_2(t) = 27. \end{aligned}$$

Then it follows that

$$e(H) \le f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2).$$

Next, we derive properties of the functions $\alpha_i(t)$ and $\beta_j(t)$.

Claim 3. *For any $t \ge 0$, $\max\{\beta_1(t), \beta_2(t)\} \ge \max\{\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)\} + 0.2$.*

Proof. We have $\beta_2(t) = 27$. It is easy to see that for each $i \in [5]$, the functions $\alpha_i(t), \beta_1(t) - \alpha_i(t)$ and $\beta_1(t)$ are increasing for $t \ge 0$. Note that $\beta_1(0.5) = 27.25$, $\alpha_2(0.5) = 26.25$, $\alpha_3(0.5) = 26.25$ and $\alpha_4(0.5) = 26.5$; so $\max\{\beta_1(t), 27\} \ge \alpha_i(t) + 0.2$ for $t \ge 0$ and $i = 2, 3, 4$. Since $\beta_1(t) - \alpha_1(t) = t^2$, and $\alpha_1(\sqrt{0.2}) < 27 - 0.2$, we see $\max\{\beta_1(t), 27\} \ge \alpha_1(t) + 0.2$ for all $t \ge 0$. $\square$

Since $\beta_1(t)\binom{s-6}{3} \le \frac{1}{2}(n-3s)^2 s + \frac{5}{2}(n-3s)s^2 + \frac{19}{6}s^3 = \frac{1}{6}n^3 - \frac{1}{6}(n-s)^3$, we see $\max\{\beta_1(t), \beta_2(t)\}\binom{s-6}{3} \le \max\left\{\binom{n}{3} - \binom{n-s+1}{3}, \binom{3s-1}{3}\right\} + O(n^2) = f(n, s, 3) + O(n^2)$. By Claim 3 and because $\sum_{i=1}^{5} x_i + x + y = \binom{s-6}{3}$, we have

$$\begin{aligned} f_t(x_1, x_2, x_3, x_4, x_5, x, y) &\le \left(\max\{\beta_1(t), \beta_2(t)\} - 0.2\right) \sum_{i=1}^{5} x_i + \beta_1(t)x + \beta_2(t)y \\ &\le \max\{\beta_1(t), \beta_2(t)\}\binom{s-6}{3} - 0.2\sum_{i=1}^{5} x_i \le f(n, s, 3) - 0.2\sum_{i=1}^{5} x_i + O(n^2). \end{aligned} \tag{3}$$

Let $\cup X$ (respectively, $\cup T_{27}$) denote the set of edges each of which belongs to some triple in X (respectively, in T_{27}). Now we show the following claim.

Claim 4. *$s > m - \epsilon n/4$, and $x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}$ or $y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}$.*

Proof. If $s \le m - \epsilon n/4$, then by (3) we have

$$\begin{aligned} e(H) &\le f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \le f(n, s, 3) + O(n^2) \\ &\le f(n, m, 3) - \binom{\epsilon/4n}{3} + O(n^2) \le f(n, m, 3) - \epsilon^4 n^3, \end{aligned}$$

[2] Since T is good, the union of any three disjoint edges of H^* in $V(T)$ must contain the three vertices in $V(T) \cap I$.

a contradiction. So $s > m - \epsilon n/4$. First we see that $x + y > \binom{s-6}{3} - 10\epsilon^4 n^3$; for, otherwise, $\sum_{i=1}^{5} x_i \geq 10\epsilon^4 n^3$, which together with (3) implies

$$e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leq f(n, m, 3) - 2\epsilon^4 n^3 + O(n^2) \leq f(n, m, 3) - \epsilon^4 n^3,$$

a contradiction. Now suppose that $x > \binom{\epsilon n/12}{3}$ and $y > \binom{\epsilon n/24}{3}$. Then $|\cup X| > \epsilon n/12$ and $|\cup T_{27}| > \epsilon n/24$. For any edge $e = (i, j, k) \in \cup X$ with $i < j < k$, by the previous discussion, we have $i \in F_1$. For any edge $e = (i, j, k) \in \cup T_{27}$ with $i < j < k$, by Claim 2 we see $i \notin F_1$. Thus $(\cup X) \cap (\cup T_{27}) = \emptyset$. The triples $T = \{e_1, e_2, e_3\}$ with $e_1 \in \cup X$ and $e_2, e_3 \in \cup T_{27}$ cannot satisfy both $f_3(T) = 27$ and $f_1(T) = 3$. This shows $x+y < \binom{s-6}{3} - |\cup X|\binom{|\cup T_{27}|}{2} \leq \binom{s-6}{3} - \frac{\epsilon n}{12}\binom{\epsilon n/24}{2}$, contradicting that $x+y > \binom{s-6}{3} - 10\epsilon^4 n^3$. Hence, we have that either $x \leq \binom{\epsilon n/12}{3}$ or $y \leq \binom{\epsilon n/24}{3}$. □

Suppose $x > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/24}{3}$. So $x > \binom{s-6}{3} - \binom{\epsilon n/12}{3}$ and thus $|\cup X| > s - 6 - \epsilon n/12$. Recall that for any $T \in X$, T is a good triple and, hence, each edge of H^* contained in $V(T)$ intersects I. Hence any traceable edge which intersects $V(\cup X)$ must also intersect I. Thus, the number of edges of H not intersecting I is at most $|V(M') \setminus V(\cup X)|\binom{n}{2} + 50n^2 \leq \frac{\epsilon n}{4}\binom{n}{2} + 50n^2 \leq \frac{\epsilon}{4}n^3$. As $|I| = s \leq m-1$,

$$|E(S(n, m, 3)) \setminus E(H)| = |E(H) \setminus E(S(n, m, 3))| + e(S(n, m, 3)) - e(H) \leq \frac{\epsilon}{4}n^3 + \epsilon^4 n^3 < \epsilon n^3.$$

So in this case we see that H is ϵ-close to $S(n, m, 3)$.

By Claim 4, it remains to consider $y > \binom{s-6}{3} - 10\epsilon^4 n^3 - \binom{\epsilon n/12}{3}$. We claim that there exists a complete 3-graph K on more than $3m - 3\epsilon n/2$ vertices and $V(K) \subseteq V(M')$. Suppose to contrary that $V(M')$ does not contain such a complete 3-graph K. Since $|V(M')| - (3m - 3\epsilon n/2) = 3(s-6) - 3m + 3\epsilon n/2 > \frac{\epsilon n}{2}$ and H is stable, $V(M')$ contains an independent set of size $\frac{\epsilon n}{2}$, say A. Note that if $T = \{e_1, e_2, e_3\}$ with $e_i \cap A \neq \emptyset$ for all $i \in [3]$, then $f_3(T) < 27$. Since there are at least $|A|/3 \geq \epsilon n/6$ edges in M' which intersect with A, we see that $y \leq \binom{s-6}{3} - \binom{\epsilon n/6}{3}$, a contradiction.

Then $|E(D(n, m, 3) \setminus E(H)| \leq |E(D(n, m, 3) \setminus E(K)| \leq \frac{3}{2}\epsilon n\binom{n}{2} < \epsilon n^3$, i.e., H is ϵ-close to $D(n, m, 3)$. This finishes the proof of Lemma 4.2. □

5 Almost perfect rainbow matchings

In this section, we prove a lemma about almost perfect rainbow matchings that we will need. In fact, this result holds for families of k-graphs, for any $k \geq 3$.

Lemma 5.1. *For any given integer $k \geq 3$, there exist positive reals c and n_2 such that the following holds. Let n, m be integers with $n \geq km$ and $n \geq n_2$, and let $\mathcal{F} = \{F_1, ..., F_m\}$ be a stable family of k-graphs on the same vertex set $[n]$ such that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$. If $m > (1-c)n/k$, then $\mathcal{F}$ admits a rainbow matching.*

Proof. We choose $c' = c'(k)$ and $c = c(k)$ small enough such that $0 < c \ll c' \ll 1$. Let n be sufficiently large and $n/k \geq m > (1-c)n/k$. Suppose to the contrary that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$ and $\mathcal{F}$ does not admit a rainbow matching.

By Corollary 2.3, we may additionally assume $\mathcal{F}$ is saturated. Let U_i be the vertex set of a largest complete k-graph in F_i for $i \in [m]$. Since F_i is stable, we may choose $U_i = [|U_i|]$ such that $[n] \setminus U_i$ is an independent set in F_i. For each $i \in [m]$, we have $|U_i| > (1-c')km$, for, otherwise, we have the following contradiction for some $i \in [m]$:

$$|F_i| \leq \binom{n}{k} - \binom{c'km}{k} \leq \binom{n}{k} - (cn+1)\binom{n-1}{k-1} \leq \binom{n}{k} - (n-km+1)\binom{n-1}{k-1} < \binom{km-1}{k},$$

where the second inequality holds since $c \ll c' \ll 1$ and $m > (1-c)n/k)$, the third inequality holds since $n - km < cn$, and the last inequality holds since $\binom{n}{k} - \binom{km-1}{k} = \sum_{i=1}^{n-km+1}\binom{n-i}{k-1} < (n-km+1)\binom{n-1}{k-1}$.

Let $U = \bigcap_{i=1}^{m} U_i$. By the above paragraph, we see that $|U| \geq (1-c')km$. If $|U| \geq km$, then it is clear that $\mathcal{F}$ admits a rainbow matching. So we may assume that $U_m = U \subseteq [km-1]$. Because U_m

is the vertex set of a largest complete k-subgraph of F_m and since F_m is stable and $|F_m| > \binom{km-1}{k}$, there exists some k-set $e \notin F_m$ such that $|e \cap U| = k-1$ and $km \in e$. Since $\mathcal{F}$ is saturated, there exists a rainbow matching M in $\mathcal{F} \setminus F_m$ such that $M \cup \{e\}$ is a rainbow matching in $\mathcal{F}(e, F_m)$. Since F_i is stable for each $i \in [m]$, we may assume that $V(M) \cup e = [km]$. Let $M' = \{e' \in M : e' \not\subseteq U\}$.

Claim. (a) $|M'| < c'km$,
(b) Each edge of F_m is contained in U or intersects an edge of $V(M')$, and
(c) For any $v \in V(M) \setminus U$, $d_{F_m[U]}(v) \le c'k^2m\binom{|U|}{k-2}$.

Proof. To prove (a), just observe that $|M'| \le |V(M) \setminus U| = (km-1) - |U| < c'km$.

Suppose (b) fails. That is, there exists an edge $f \in F_m$ such that $f \setminus U \ne \emptyset$ and $f \cap V(M') = \emptyset$. Note that $f \cap (U \setminus V(M')) \ne \emptyset$, as $[n] \setminus U$ is independent in F_m. In particular, $|f \cap (U \setminus V(M'))| \le k-1$. Let $|M'| = m-t$ for some $t \ge 1$. Recall that $U \cup V(M') = V(M) = [km-1]$. Hence $|U \setminus V(M')| = kt-1$ and, thus, $U \setminus (V(M') \cup f)$ induces a common complete k-graph of size at least $k(t-1)$ in all F_i. Then we see that $M' \cup \{f\}$ together with a matching of size $t-1$ in $U \setminus (V(M') \cup f)$ form a rainbow matching for $\mathcal{F}$. So we may assume that (b) holds.

Now we prove (c). For any $v \in V(M) \setminus U \subseteq [km]$, by the maximality of U, there exists $f \in \binom{[n]}{k} \setminus F_m$ such that $v \in f$ and $|f \cap U| = k-1$. So there exists a rainbow matching N in $\mathcal{F} \setminus F_m$ such that $N \cup \{f\}$ is a rainbow matching in $\mathcal{F}'(f, F_m)$. Since F_i is stable for $i \in [m]$, we may assume that $V(N) \cup f = [km]$. Let $N' = \{e' \in N : e' \not\subseteq U\}$. By applying (b) to N', every edge of F_m containing v intersects $V(N')$. Since $V(N') \le k|N'| \le k(km - |U|) \le c'k^2m$, there are at most $c'k^2m\binom{|U|}{k-2}$ edges e' in F_m containing v such that $e' \subseteq U \cup \{v\}$. Hence (c) holds. This proves the claim. $\square$

Note that $|e \cap U| = k-1$ and $V(M) \cup U = [km-1]$. Let q_1 be the number of edges of F_m contained in $[km-1]$, and q_2 be the number of edges of F_m with at least one vertex in $[n] \setminus [km-1]$. By (c), we have

$$q_1 \le \binom{km-1}{k} - |V(M) \setminus U|\binom{|U|}{k-1} + |V(M) \setminus U| \cdot c'k^2m\binom{|U|}{k-2}.$$

By (b), we see $q_2 \le |V(M')| \cdot (n-km+1)\binom{n-2}{k-2}$. So we have

$$\begin{aligned}
|F_m| &\le \binom{km-1}{k} - |V(M) \setminus U|\left[\binom{|U|}{k-1} + c'k^2m\binom{|U|}{k-2}\right] + \left|V(M')\right|(n-km+1)\binom{n-2}{k-2} \\
&\le \binom{km-1}{k} - |V(M) \setminus U|\left[\binom{|U|}{k-1} + c'k^2m\binom{|U|}{k-2}\right] + k|V(M) \setminus U|(cn+1)\binom{n-2}{k-2} \\
&= \binom{km-1}{k} - |V(M) \setminus U| \cdot \left[\binom{|U|}{k-1} - c'k^2m\binom{|U|}{k-2} - k(cn+1)\binom{n-2}{k-2}\right] \\
&< \binom{km-1}{k},
\end{aligned}$$

where the second inequality holds since $n - km < cn$ and $|M'| \le |V(M) \setminus U|$, and the last inequality holds since c', c are small enough and $|U| > (1-c')km > (1-c')(1-c)n$. This is a contradiction, finishing the proof of Lemma 5.1. $\square$

6 Non-extremal configurations

Note that if there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n-1}{k-1}$ then v can be removed from all k-graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$ so that $\mathcal{F}'$ admits a rainbow matching if and only if $\mathcal{F}$ admits a rainbow matching. Hence, if such vertex does not exist in a saturated family $\mathcal{F}$, then from Lemma 2.1, we see that $d_F(v) \le \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1}$ for all $v \in F$ and $F \in \mathcal{F}$. This leads us to the following result.

Lemma 6.1. *Given reals $0 < \epsilon \ll c \ll 1$, let $n \ge n(\epsilon, c)$ be a sufficiently large integer and m be an integer such that $n/27 < m < (1-c)n/3$. Let $\mathcal{F} = \{F_1, ..., F_m\}$ be a stable family of 3-graphs on* *vertex set* $[n]$ *such that for every* $i \in [m]$, $|F_i| > f(n, m, 3)$ *and* $d_{F_i}(v) \le \binom{n-1}{2} - \binom{n-3(m-1)-1}{2}$ *for each* $v \in [n]$. *If* $H(\mathcal{F})$ *is* ϵ*-close to neither* $H_S(n, m, 3)$ *nor* $H_D(n, m, 3)$, *then* $\mathcal{F}$ *admits a rainbow matching.*

Proof. Given $0 < \epsilon \ll c \ll 1$, let n', m' be integers such that n' is sufficiently large and $n'/27 < m' < (1-c)n'/3$. Let $\mathcal{F} = \{F_1, ..., F_{m'}\}$ be a family of 3-graphs on the vertex set $[n']$ such that $|F_i| > f(n', m', 3)$ and $d_{F_i}(v) \le \binom{n'-1}{2} - \binom{n'-1-3(m'-1)}{2}$ for $i \in [m']$ and $v \in [n']$. Suppose that $H(\mathcal{F})$ is not ϵ-close to $H_S(n', m', 3)$ or $H_D(n', m', 3)$. Our ultimate goal is to find a rainbow matching in $\mathcal{F}$.

Let $n' = 3m' + 3r' + s$ where $0 \le s < 3$. Recall the definitions of $H(\mathcal{F})$ and $H^*(\mathcal{F})$ such that $V(H(\mathcal{F})) = [n'] \cup \mathcal{V}'$ and $V(H^*(\mathcal{F})) = [n'] \cup \mathcal{V}' \cup \mathcal{U}'$, where $|\mathcal{V}'| = m'$ and $|\mathcal{U}'| = r'$. By Lemma 2.5, for $0 < \gamma' \ll \gamma \ll \epsilon \ll c \ll 1$, there exists a matching M_a in $H^*(\mathcal{F})$ with $|M_a| \le \gamma n'$ such that for any $S \subseteq V(H^*(\mathcal{F}))\backslash V(M_a)$ with $|S| \le \gamma' n'$ and $3|S \cap (\mathcal{V}' \cup \mathcal{U}')| = |S \cap [n']|$, $H^*(\mathcal{F})[V(M_a) \cup S]$ has a perfect matching. In the rest of the proof, without loss of generality, we use the following notation:

$$H = H^*(\mathcal{F}) - V(M_a), [n] = [n']\backslash V(M_a), \mathcal{V} = \mathcal{V}'\backslash V(M_a) = \{v_1, ..., v_m\}, \mathcal{U} = \mathcal{U}'\backslash V(M_a) = \{u_1, ..., u_r\}.$$

Then $n = 3m + 3r + s$. Using the above property of the matching M_a, it now suffices for us to find an almost perfect matching in H. To find this almost perfect matching, our plan is to show that there exists an almost regular subgraph of H with bounded maximum co-degree so that Theorem 2.6 can be applied. To that end, in what follows we will use the two-round randomization technique developed in [3].

Let R be chosen from $V(H)$ by taking each vertex independently with probability $n^{-0.9}$. We take $n^{1.1}$ independent copies of R and denote them by R^i for $1 \le i \le n^{1.1}$. For $S \subseteq V(H)$, denote $Y_S = |\{i : S \subseteq R^i\}|$. First we have the following claim.

Claim A. With probability $1 - o(1)$, the following hold:
(i) for every $v \in V(H)$, $Y_{\{v\}} = (1 + o(1))n^{0.2}$,
(ii) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets R^i, and
(iii) every edge $e \in H$ is contained in at most one set R^i.

Proof. Note that $Y_S \sim \text{Bin}(n^{1.1}, n^{-0.9|S|})$ for any $S \subseteq V(H)$. Thus, $\mathbb{E}[Y_{\{v\}}] = n^{0.2}$ for every $v \in V(H)$. By Lemma 2.10 (2), we have $P(|Y_{\{v\}} - n^{0.2}| > n^{0.15}) \le e^{-\Omega(n^{0.1})}$. By union bound, we see (i) holds. To prove (ii) and (iii), let

$$Z_2 = \left|\left\{\{u, v\} \in \binom{V(H)}{2} : Y_{\{u,v\}} \ge 3\right\}\right| \text{ and } Z_3 = \left|\left\{S \in \binom{V(H)}{3} : Y_S \ge 2\right\}\right|.$$

Then $\mathbb{E}[Z_2] = \binom{|V(H)|}{2} P(Y_{\{u,v\}} \ge 3) \le \binom{n}{2}(n^{1.1})^3(n^{-1.8})^3 \le 4n^{-0.1}$ and $\mathbb{E}[Z_3] \le \binom{n}{3}(n^{1.1})^2(n^{-2.7})^2 \le 8n^{-0.2}$. By Markov's inequality, we have

$$\mathbb{P}(Z_2 = 0) > 1 - 4n^{-0.1} \quad \text{and} \quad \mathbb{P}(Z_3 = 0) > 1 - 8n^{-0.2}.$$

That implies that (ii) and (iii) hold with probability at least $1 - 4n^{-0.1}$ and $1 - 8n^{-0.2}$, respectively. □

Next we want to prove that there exists a perfect (or, rather, maximum) fractional matching in each $H[R^i]$. To do so, we define a maximal subset $R'^i \subseteq R^i$ that satisfies $R'^i \cap [n] = 3|R'^i \cap (\mathcal{V} \cup \mathcal{U})|$ as follows. If $|R^i \cap [n]| \ge 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denote by R'^i, which is chosen from R^i by deleting $|R^i \cap [n]| - 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$ vertices in $R^i \cap [n]$ independently and uniformly at random. Otherwise $|R^i \cap [n]| < 3|R^i \cap (\mathcal{V} \cup \mathcal{U})|$, we take a subset of R^i denote by R'^i by the following two step: First we delete at most 3 vertices (chosen independently and uniformly at random) in $R^i \cap [n]$ so that the number ℓ of the remaining vertices is a multiple of 3; then we delete $|R^i \cap (\mathcal{V} \cup \mathcal{U})| - \ell/3$ vertices in $R^i \cap (\mathcal{V} \cup \mathcal{U})$ independently and uniformly at random.

For $S \subseteq V(H)$, define $Y'_S = |\{i : S \subseteq R'^i\}|$. Note that $\mathbb{E}(R^i \cap [n]) = n^{0.1}$, $\mathbb{E}(R^i \cap \mathcal{V} \cup \mathcal{U}) = n^{0.1}/3$, and $\mathbb{E}(R^i \cap \mathcal{V}) = n^{-0.9}m$. For each i, let A_i be the event $\left||R^i \cap [n]| - n^{0.1}\right| < n^{0.095}$, B_i be the event $\left||R^i \cap (\mathcal{V} \cup \mathcal{U})| - n^{0.1}/3\right| < n^{0.095}$, and C_i be the event $\left||R^i \cap \mathcal{V}| - n^{-0.9}m\right| < n^{0.095}$.

Claim B. With probability $1-o(1)$, the following hold:
(i) $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ holds,
(ii) for every $v \in V(H)$, $Y'_{\{v\}} = (1+o(1))n^{0.2}$,
(iii) every pair $\{u,v\} \subseteq V(H)$ is contained in at most two sets R'^i, and
(iv) every edge $e \in H$ is contained in at most one set R'^i.

Proof. Since $R'^i \subseteq R^i$, it is clear from Claim A that (iii) and (iv) hold with probability $1-o(1)$. Next we consider (i). By Lemma 2.10 (2) (with $\lambda = n^{0.095}$), for each $1 \le i \le n^{1.1}$, we have

$$\mathbb{P}(\overline{A_i}) \le e^{-\Omega(n^{0.09})}, \ \mathbb{P}(\overline{B_i}) \le e^{-\Omega(3n^{0.09})} = e^{-\Omega(n^{0.09})} \ \text{ and } \ \mathbb{P}(\overline{C_i}) \le e^{-\Omega(\frac{n}{m}n^{0.09})} = e^{-\Omega(n^{0.09})}.$$

Thus by union bound, $\mathbb{P}(\bigwedge_i (A_i \wedge B_i \wedge C_i)) = 1-o(1)$, proving (i).

Assuming $A_i \wedge B_i \wedge C_i$, we see $|R^i \setminus R'^i| < 2n^{0.095}$. Then by the choice of R'^i, for all $v \in V(H)$, the probability $\mathbb{P}(\{v \in R^i \setminus R'^i | (A_i \wedge B_i \wedge C_i) \wedge (v \in R^i)\})$ is at most

$$\max\left\{\frac{|R^i \setminus R'^i|}{|R^i \cap [n]|}, \frac{|R^i \setminus R'^i|}{|R^i \cap (\mathcal{V} \cup \mathcal{U}))|}\right\} \le \frac{|R^i \setminus R'^i|}{|R^i|/4} < \frac{2n^{0.095}}{(n^{0.1}-n^{0.095})/3} < 7n^{-0.005}.$$

Using coupling and applying Lemma 2.10 (2) to $\mathrm{Bin}(|Y_v|, 7n^{-0.005})$ with $\lambda = 3n^{0.195}$, we have

$$\mathbb{P}\left(\left\{Y_{\{v\}} - Y'_{\{v\}} > 10n^{0.195} \,\middle|\, \bigwedge_i (A_i \wedge B_i \wedge C_i) \wedge \left(Y_{\{v\}} = (1+o(1))n^{0.2}\right)\right\}\right) \le e^{-\Omega(n^{0.195})}.$$

Note that with probability $1-o(1)$, $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ and $Y_{\{v\}} = (1+o(1))n^{0.2}$ hold for all $v \in V(H)$. By union bound, we can derive that $0 \le Y_{\{v\}} - Y'_{\{v\}} \le 10n^{0.195} = o(n^{0.2})$ for all $v \in V(H)$ with probability $1-o(1)$. Hence (ii) holds with probability $1-o(1)$. This proves Claim B. □

Let $n_i = |R'^i \cap [n]|$ and $m_i = |R'^i \cap \mathcal{V}|$. Using (i) of Claim B, we see that with probability $1-o(1)$, $m_i = (1+o(1))mn^{-0.9} = \Theta(n^{0.1}) = \Theta(n_i)$ for all $1 \le i \le n^{1.1}$.

Claim C. With probability $1-o(1)$, the following hold for all $1 \le i \le n^{1.1}$:
(a) $H[R'^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$-close to $H_S(n_i, m_i, 3)$ or $H_D(n_i, m_i, 3)$, and
(b) there exists a perfect fractional matching in $H[R'^i]$.

Proof. For each $T \in \binom{V(H)}{\le 2}$, let $\mathrm{Deg}^i(T) := |N_H(T) \cap \binom{R'^i}{4-|T|}|$. By definition of H, we have that

- for any $v_j \in \mathcal{V}$, $d_H(v_j) \ge f(n', m', 3) - (\gamma n')\binom{n'}{2} \ge f(n,m,3) - \gamma n^3$, and
- for any $T = \{v_j, u\}$ with $v_j \in \mathcal{V}$ and $u \in [n]$,

$$d_H(T) = d_{F_j}(u) \le \binom{n'-1}{2} - \binom{n'-1-3(m'-1)}{2} \le \binom{n-1}{2} - \binom{n-1-3(m-1)}{2} + \gamma n^2.$$

Assume that $\bigwedge_i (A_i \wedge B_i \wedge C_i)$ holds. Then $n_i = (1+o(1))n^{0.1}$ and $m_i = (1+o(1))mn^{-0.9}$. Since $R^i \setminus R'^i = o(n_i)$, for each $T \in \binom{V(R'^i)}{t}$ with $t \in [2]$, we have

$$\mathbb{E}[\mathrm{Deg}^i(T)] = (1+o(1))d_H(T)(n^{-0.9})^{4-t}.$$

Thus, for any $v \in \mathcal{V} \cap R^i$,

$$\mathbb{E}[\mathrm{Deg}^i(v)] \ge (1+o(1))(f(n,m,3) - \gamma n^3)(n^{-0.9})^3 \ge f(n_i, m_i, 3) - 2\gamma n_i^3,$$

and, for any $T = \{u,v\}$ with $v \in \mathcal{V}$ and $u \in [n]$, $\mathbb{E}[\mathrm{Deg}^i(T)]$ is at most

$$(1+o(1))\left[\binom{n-1}{2} - \binom{n-1-3(m-1)}{2} + \gamma n^2\right](n^{-0.9})^2 \le \binom{n_i-1}{2} - \binom{n_i-1-3(m_i-1)}{2} + 2\gamma n_i^2.$$

We apply Janson's Inequality (Theorem 8.7.2 in [4]) to bound the deviation of $\mathrm{Deg}^i(T)$ for $|T| \leq 2$. Write $\mathrm{Deg}^i(T) = \sum_{e \in N_H(T)} X_e$, where $X_e = 1$ if $e \subseteq R'^i$ and $X_e = 0$ otherwise. Let $t = |T| \in \{1, 2\}$ and $p = n^{-0.9}$. Then

$$\Delta^* = \sum_{e_i \cap e_j \neq \emptyset,\ e_i, e_j \in \binom{V(H)}{4-t}} \mathbb{P}(X_{e_i} = X_{e_j} = 1) \leq \sum_{\ell=1}^{4-t} p^{2(4-t)-\ell} \binom{n-t}{4-t} \binom{4-t}{\ell} \binom{n-4}{4-t-\ell} = O(n^{0.1(2(4-t)-1)}).$$

By Janson's inequality, for $v \in \mathcal{V} \cap R^i$,

$$\mathbb{P}(\mathrm{Deg}^i(v) \leq (1-\gamma)\mathbb{E}[\mathrm{Deg}^i(v)]) \leq e^{-\gamma^2 \mathbb{E}[\mathrm{Deg}^i(v)]/(2+\Delta^*/\mathbb{E}[\mathrm{Deg}^i(v)])} \leq e^{-\Omega(n^{0.1})},$$

and, for the pair $\{v, u\}$ with $v \in \mathcal{V}$ and $u \in [n]$ (by considering the complement of H), we can have

$$\mathbb{P}(\mathrm{Deg}^i(\{v, u\}) \geq (1+\gamma)\mathbb{E}[\mathrm{Deg}^i(\{v, u\})]) \leq e^{-\Omega(n^{0.1})}$$

By union bound, with probability $1 - o(1)$ we derive from above that for all $1 \leq i \leq n^{1.1}$

1). for any $v \in \mathcal{V} \cap R^i$, $\mathrm{Deg}^i(v) \geq (1-\gamma)\mathbb{E}[\mathrm{Deg}^i(v)]) \geq f(n_i, m_i, 3) - 3\gamma n_i^3$, and

2). for any pair $\{u, v_j\} \subseteq R'^i$ with $v_j \in \mathcal{V}$ and $u \in [n]$,

$$\mathrm{Deg}^i(\{u, v\}) \leq \binom{n_i - 1}{2} - \binom{n_i - 1 - 3(m_i - 1)}{2} + 3\gamma n_i^2 \leq \binom{n_i - 1}{2} - \Omega(n_i^2),$$

which implies that $F_j[R'^i \cap [n]]$ is not $\epsilon^3/2$-close to $S(n_i, m_i, 3)$, since $m_i = (1+o(1))mn^{0.9}$ and $m < (1-c)n/3$.

This shows that $H[R'^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$-close to $H_S(n_i, m_i, 3)$, where $\gamma \ll \epsilon$.

Let $\mathcal{V}_0 := \{v_j \in \mathcal{V} :\ F_i[[n]]$ is not ϵ-close to $D(n, m, 3)\}$. We claim that $|\mathcal{V}_0| > \epsilon n$. Otherwise $|\mathcal{V}_0| \leq \epsilon n$, then we have

$$|E(H_D(n', m', 3)) \setminus E(H(\mathcal{F}))| \leq \epsilon n \binom{n}{3} + (m - \epsilon n)\epsilon n^3 + \gamma (n')^4 \leq \epsilon (n')^4,$$

a contradiction as $H(\mathcal{F})$ is not ϵ-close to $H_D(n', m', 3))$. As $|\mathcal{V}_0| > \epsilon n$, with probability $1 - o(1)$ we have (using Lemma 2.10) that

3). $|R'^i \cap \mathcal{V}_0| \geq \frac{\epsilon n_i}{2}$ for all $1 \leq i \leq n^{1.1}$.

For $v_j \in R'^i \cap \mathcal{V}_0$, we consider $F_j[[n]]$. Let G be the complement of $F_j[[n]]$. Then for any $S \subseteq V(G)$ with $|S| > 3m - \epsilon n$, we have $e(G[S]) \geq \epsilon e(G)$. This is because otherwise $|E(D(n, m, 3)) \setminus E(F_j[[n]])| \leq \epsilon n \binom{n}{2} + \epsilon e(G) < \epsilon n^3$, contradicting $v_j \in \mathcal{V}_0$. By Lemma 2.8, the maximum size of the complete 3-graph in $F_j[R^i \cap [n]]$ is no more than $(3m/n - \epsilon + \gamma)n^{0.1} \leq 3m_i - \epsilon n_i/2$ with probability at least $1 - (n^{O(1)} e^{-\Omega(n^{0.1})})$. Assuming $\bigwedge_i (A_i \wedge B_i \wedge C_i)$, this implies that $F_j[R'^i \cap [n]]$ is not $\epsilon^3/2$-close to $D(n_i, m_i, 3)$. By union bound, with probability $1 - o(1)$, we have

4). for all $1 \leq i \leq n^{1.1}$ and $v_j \in R'^i \cap \mathcal{V}_0$, $F_j[R'^i \cap [n]]$ is not $\epsilon^3/2$-closed to $D(n_i, m_i, 3)$.

By 3) and 4), we see that, with probability $1 - o(1)$, $H[R'^i \setminus \mathcal{U}]$ is not $\epsilon^4/4$-close to $H_D(n_i, m_i, 3)$, proving part (a) of Claim C.

It remains to show part (b) of Claim C, that is, to construct a perfect fractional matching w_i in $H[R'^i]$ for each $1 \leq i \leq n^{1.1}$. Our main tool is the stability result, Lemma 4.2.

Fix some $1 \leq i \leq n^{1.1}$. We write $R'^i \cap [n] = \{x_1^i, ..., x_{n_i}^i\}$ with $x_1^i < x_2^i < ... < x_{n_i}^i$ and define $[d]_i := \{x_1^i, x_2^i, ..., x_d^i\}$ for any integer d. We now state two simple inequalities for later use:

$$f(x, y, 3) \geq f(x, y-a, 3) + \binom{a}{3} \quad \text{and} \quad f(x, y, 3) \geq f(x, y+a, 3) - 3ax^2 \tag{4}$$

hold for any positive integers x, y, a with $a < y$.

To construct a perfect fractional matching w_i in $H[R'^i]$, first we consider $v_j \in R'^i \cap \mathcal{V}_0$ and assign weights to the edges of $H[R'^i]$ containing v_j. Using 1), and by (4) and the fact that $\gamma \ll \epsilon \ll 1$,

$$|F_j[R'^i \cap [n]]| = \mathrm{Deg}^i(v_j) \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i + \epsilon^{20} n_i, 3) - \epsilon^{16} n_i^3.$$

By 2) and 4), $F_j[R'^i \cap [n]]$ is not $\epsilon^3/2$-close to $S(n_i, m_i, 3)$ or $D(n_i, m_i, 3)$. Since $|E(S(n_i, m_i + \epsilon^{20} n_i, 3)) \setminus E(S(n_i, m_i, 3))| \leq \epsilon^{20} n_i^3$ and $|E(D(n_i, m_i + \epsilon^{20} n_i, 3)) \setminus E(D(n_i, m_i, 3))| \leq 3\epsilon^{20} n_i^3$, we see that $F_j[R'^i \cap [n]]$ is not ϵ^4-close to $S(n_i, m_i + \epsilon^{20} n_i, 3)$ or $D(n_i, m_i + \epsilon^{20} n_i, 3)$. Then by Lemma 4.2 and the fact that F_j is stable, $F_j[R'^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i + 3\epsilon^{20} n_i]_i$. Now we assign weights $w_i(e)$ to all edges e of $H[R'^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i + \epsilon^{20} n_i}$, and otherwise let $w_i(e) = 0$.

Next we consider $v_j \in R'^i \cap (\mathcal{V} \setminus \mathcal{V}_0)$. By 1) and (4), we have

$$|F_j[R'^i \cap [n]]| \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i - 6\gamma^{\frac{1}{3}} n_i, 3).$$

By Theorem 4.1 and the fact that F_j is stable, $F_j[R'^i \cap [n]]$ contains a matching M_j with $V(M_j) = [3m_i - 18\gamma^{\frac{1}{3}} n_i]_i$. Then we assign weights $w_i(e)$ to all edges e of $H[R'^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i - 6\gamma^{1/3} n_i}$; and otherwise let $w_i(e) = 0$.

Note that for every $v_j \in R'^i \cap \mathcal{V}$, we have defined weights $w_i(e)$ for all edges $e \in H[R'^i]$ with $v_j \in e$, whose total weights equal one. In the remaining proof, we want to extend this function w_i to entire $H[R'^i]$ to form a perfect fractional matching. We complete this in two steps.

First, we define a perfect fractional matching w (as the *projection* of w_i) in the complete 3-graph K on vertex set $R'^i \cap [n]$. Note that a function $w : E(K) \to [0, 1]$ is a perfect fractional matching if and only if $w(v) := \sum_{v \in f \in K} w(f) = 1$ holds for every $v \in V(K)$. Initially, we define a function $w' : E(K) \to [0, 1]$ such that, for each $f \in E(K)$, $w'(f) := \sum_e w_i(e)$ over all edges $e \in H[R'^i]$ with $f \subseteq e$ and $|e \cap \mathcal{V}| = 1$. Since $|\mathcal{V}_0| > \epsilon n$ and $\gamma \ll \epsilon$, it follows from the above definitions on w_i that for any $v \in R'^i \cap [n]$,

$$w'(v) := \sum_{v \in f \in K} w'(f) \leq \frac{|\mathcal{V}_0|}{m_i + \epsilon^{20} n_i} + \frac{m_i - |\mathcal{V}_0|}{m_i - 6\gamma^{\frac{1}{3}} n_i} \leq \frac{\epsilon n_i}{m_i + \epsilon^{20} n_i} + \frac{m_i - \epsilon n_i}{m_i - 6\gamma^{\frac{1}{3}} n_i} < 1.$$

Since $\epsilon \ll c$, we have $3m_i + 3\epsilon^{20} n_i < n_i - 4$. So there exists a vertex set $\{a_1, a_2, a_3, a_4\}$ in K such that $w'(a_i) = 0$, for $i \in [4]$. Let K' be the 3-graph obtained from K by deleting vertices a_1, a_2, a_3, a_4. Starting with $w := w'$, we increase w using the following iterations: (i) pick a vertex v in $V(K')$ with maximum $w(v)$;[3] (ii) pick any edge $f \in K'$ containing v and update $w(f) \leftarrow w(f) + 1 - w(v)$; (iii) delete all vertices $u \in V(K')$ with $w(u) = 1$ (which must include the vertex v) from K'; (iv) if $|V(K')| \leq 2$, then terminate; otherwise go to (i) again. This must terminate in finitely many iterations and when it terminates, we obtain a fractional matching w in K such that $w(a_i) = 0$ for $i \in [4]$ and $|K'| \leq 2$. So there exist two vertices b_1, b_2 in $V(K) \setminus \{a_1, a_2, a_3, a_4\}$ such that for any vertex v in $V(K) \setminus \{a_1, a_2, a_3, a_4, b_1, b_2\}$, $w(v) = 1$. We may suppose $1 \geq w(b_1) \geq w(b_2)$. Let $w(a_1, a_2, b_1) = 1 - w(b_1)$, $w(a_1, a_2, b_2) = \frac{w(b_1) - w(b_2)}{2}$, $w(a_3, a_4, b_2) = 1 - w(b_1) + \frac{w(b_1) - w(b_2)}{2}$, and $w(a_1, a_2, a_3) = w(a_1, a_2, a_4) = w(a_1, a_3, a_4) = w(a_2, a_3, a_4) = \frac{w(b_1) + w(b_2)}{6}$. It is easy to check that w is a perfect fractional matching in K.

Now we notice that $\sum_{f \in K} w'(f) = \sum_{\{e \in H[R'^i] : |e \cap \mathcal{V}| = 1\}} w_i(e) = |R'^i \cap \mathcal{V}|$ and, $\sum_{f \in K} w(f) = \frac{|R'^i \cap [n]|}{3} = |R'^i \cap (\mathcal{V} \cup \mathcal{U})|$. Moreover, the neighborhood of any $u_j \in R'^i \cap \mathcal{U}$ in $H[R'^i]$ is the complete 3-graph K. So we can partition the total weight $\sum_{f \in K} (w(f) - w'(f)) = |R'^i \cap \mathcal{U}|$ into $|R'^i \cap \mathcal{U}|$ copies of 1's (say each is represented by a set E_j of edges in K), and then for each $u_j \in R'^i \cap \mathcal{U}$, we assign the weight of each $f \in E_j$ to be $w_i(f \cup \{u_j\})$. One can easily check that we obtain a perfect fractional matching w_i in $H[R'^i]$. This completes the proof of Claim C. □

[3]Note that this maximum $w(v)$ is strictly less than 1.

From Claims B and C, we see that the sets R'^i for $1 \leq i \leq n^{1.1}$ satisfy (a)-(d) in Lemma 2.7. Then by Lemma 2.7, there exists a spanning subgraph H' of H such that for each $v \in V(H)$, $d_{H'}(v) = (1 + o(1))n^{0.2}$, and $\Delta_2(H') \leq n^{0.1}$. By Theorem 2.6, H contains a matching M_b such that $S = V(H)\backslash V(M_b)$ contains at most $\gamma' n'$ vertices. Since $|S \cup M_a \cup M_b| = n' = 3r' + 3m' + s$ where $0 \leq s \leq 2$, we can delete at most s elements from S to get a subset S' such that $3|S' \cap (\mathcal{V}' \cup \mathcal{U}')| = |S' \cap [n']|$. By the setting at the beginning of the proof, Lemma 2.5 assures that $H^*(\mathcal{F})[V(M_a) \cup S']$ has a perfect matching, which together with M_b form a matching in $H^*(\mathcal{F})$ of size $r' + m'$. Equivalently, this says that $\mathcal{F}$ admits a rainbow matching, finishing the proof of Lemma 6.1. □

7 Proof of Theorem 1.3

Let n be a sufficiently large integer. Let m be a positive integer with $n \geq 3m$ and let $\mathcal{F} = \{F_1, ..., F_m\}$ be a family of 3-graphs on the same vertex set $[n]$, such that $|F_i| > f(n, m, 3)$ for each $i \in [m]$. Suppose to the contrary that $\mathcal{F}$ does not admit a rainbow matching. In view of Lemma 2.2, we may assume that $\mathcal{F}$ is stable. Then by Lemma 5.1, there exists an absolute constant $c = c(3) > 0$ such that $m \leq (1 - c)n/3$. By Theorem 1.2, $m \geq n/27$. Hence,

$$n/27 \leq m \leq (1 - c)n/3. \tag{5}$$

We now apply the following algorithm. Initially, let $\mathcal{F}_0 = \mathcal{F}$, $n_0 = n$ and $m_0 = m$. We repeat the following iterations. Suppose that we have defined $\mathcal{F}_i$, which contains m_i many 3-graphs on the same vertex set $[n_i]$.

- Step 1: Apply Corollary 2.3 to $\mathcal{F}_i$, we obtain a family $\mathcal{F}_{i+1}$ of 3-graphs on the vertex set $[n_i]$ that is both stable and saturated, and set $n_{i+1} = n_i$ and $m_{i+1} = m_i$.
- Step 2: If for any $F \in \mathcal{F}_{i+1}$ and any $v \in [n_{i+1}]$, $d_F(v) < \binom{n_{i+1}-1}{2}$, then set $t := i + 1$ and output $\mathcal{F}_t, n_t, m_t$.
- Step 3: If there exist $F \in \mathcal{F}_{i+1}$ and $v \in [n_{i+1}]$ such that $d_F(v) = \binom{n_{i+1}-1}{2}$, then set $n'_{i+1} = n_{i+1} - 1$, $m'_{i+1} = m_{i+1} - 1$, and $\mathcal{F}'_{i+1} := \{F' - v : F' \in \mathcal{F}_i \setminus \{F\}\}$. Relabel the vertices if necessary so that all 3-graphs in $\mathcal{F}'_{i+1}$ have the same vertex set $[n'_{i+1}]$. Set $\mathcal{F}_i := \mathcal{F}'_{i+1}, n_i := n'_{i+1}, m_i := m'_{i+1}$ and go to Step 1.

Let $\mathcal{F}_t$ be the resulting family of 3-graphs, which contains m_t 3-graphs on the same vertex set $[n_t]$ and admits no rainbow matching. By (5), we see that $n_t \geq n - m > cn$ is sufficiently large. We also see from Lemma 2.9 that

$$|F| > f(n_t, m_t, 3) \text{ holds for any } F \in \mathcal{F}_t.$$

By definition, we see that $\mathcal{F}_t$ is stable and saturated such that for any $F \in \mathcal{F}_t$ and $v \in V_t$, $d_F(v) < \binom{n_t-1}{2}$. On the other hand, by Lemma 2.1, it further holds that

$$d_F(v) \leq \binom{n_t - 1}{2} - \binom{n_t - 1 - 3(m_t - 1)}{2} \text{ for any } F \in \mathcal{F}_t \text{ and } v \in V_t.$$

Since n_t is sufficiently large, using Lemma 5.1 and Theorem 1.2 again, we may assume that

$$n_t/27 \leq m_t \leq (1 - c)n_t/3.$$

Now we choose $0 < \epsilon \ll c$. Since $\mathcal{F}_t$ satisfies the above properties, by applying Lemmas 2.4, 3.1 and 6.1, we can conclude that $\mathcal{F}_t$ admits a rainbow matching. This is a contradiction, completing the proof of Theorem 1.3. □

Acknowledgement. The authors would like to thank Peter Frankl for bringing the references [2, 7].

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