Extremal problems of Erdős, Faudree, Schelp and Simonovits on paths and cycles

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Abstract

For positive integers $n > d \geq k$, let $\phi(n, d, k)$ denote the least integer ϕ such that every n-vertex graph with at least ϕ vertices of degree at least d contains a path on $k+1$ vertices. Many years ago, Erdős, Faudree, Schelp and Simonovits proposed the study of the function $\phi(n, d, k)$, and conjectured that for any positive integers $n > d \geq k$, it holds that $\phi(n, d, k) \leq \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{n}{d+1} \rfloor + \epsilon$, where $\epsilon = 1$ if k is odd and $\epsilon = 2$ otherwise. In this paper we determine the value of the function $\phi(n, d, k)$ exactly. This confirms the above conjecture of Erdős et al. for all positive integers $k \neq 4$ and in a corrected form for the case $k = 4$. Our proof utilizes, among others, a lemma of Erdős et al. [3], a theorem of Jackson [6], and a (slight) extension of a very recent theorem of Kostochka, Luo and Zirlin [7], where the latter two results concern maximum cycles in bipartite graphs. Besides, we construct examples to provide answers to two closely related questions raised by Erdős et al.

1 Introduction

We consider the following extremal problem asked by Erdős, Faudree, Schelp and Simonovits in [3]: for given positive integers $n > d \geq k$, what is the minimum value ℓ such that every n-vertex graph with at least ℓ vertices of degree at least d contains a path P_{k+1} on $k+1$ vertices? The goal of the present paper is to provide a complete solution for all positive integers $n > d \geq k$.

One of the best known results in extremal graph theory is the Erdős-Gallai Theorem [4], which states that any n-vertex graph with more than $(k-1)n/2$ edges contains a path on $k+1$ vertices. Since then, there has been many other extremal results on the existence of long paths in graphs with a large number of edges or vertices of high-degree. In this paper we investigate the following function, whose study was proposed by Erdős, Faudree, Schelp and Simonovits [3].

Definition 1.1. *For positive integers $n > d \geq k$, define $\phi(n, d, k)$ to be the smallest integer ϕ such that every n-vertex graph with at least ϕ vertices of degree at least d contains a path P_{k+1} on $k+1$ vertices.*

In this language, the well-known theorem of Dirac [2] asserts that $\phi(n, d, d) \leq n$ and by that, we see that the function $\phi(n, d, k)$ is well-defined if and only if $d \geq k$. A result of Bazgan, Li and Woźniak [1] shows that $\phi(n, d, d) \leq \frac{n}{2}$. For the general case, Erdős, Faudree, Schelp and Simonovits [3] announced that for any k, there exists a constant c such that if n is large enough with respect to k, then $\phi(n, d, k) \leq \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{n}{d+1} \rfloor + c$. They [3] further made the following conjecture (also see [5]).

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Conjecture 1.2 (Erdős, Faudree, Schelp and Simonovits, [3]). *Let n, d and k be any positive integers with $n > d \geq k$. Then $\phi(n, d, k) \leq \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{n}{d+1} \rfloor + \epsilon$, where $\epsilon = 1$ if k is odd and $\epsilon = 2$ otherwise.*

Much attention in [3] was devoted to the special case when $d + 1 \leq n \leq 2d + 1$. In this case, the authors [3] showed that approximately $k/2$ vertices of degree at least d are enough to ensure the existence of P_{k+1}. They also commented that "unfortunately, even for this interval of values we are not able to prove the exact statement of the conjecture." However, as we shall see later, this case (i.e., $d+1 \leq n \leq 2d+1$) is a major difficulty that we face to and where several novel ideas take place in our proof.

Our main result determines the function $\phi(n, d, k)$ completely in the following statement.

Theorem 1.3. *For any positive integers n, d and k with $n > d \geq k$, the followings are true:*
(i) If k is odd, then $\phi(n, d, k) = \frac{k-1}{2} q + 1$, where $n = q(d + 1) + r$ with $0 \leq r \leq d$.
(ii) If k is even, then

(a) for $k = 2$, $\phi(n, d, 2) = 1$;

(b) for $k = 4$, $\phi(n, d, 4) = \begin{cases} 2q + 1, & 0 \leq r \leq d; \\ 2q + 2, & d < r < 2d, \end{cases}$ *where $n = 2qd + r$ with $0 \leq r < 2d$;*

(c) for $k \geq 6$, $\phi(n, d, k) = \begin{cases} \frac{k-2}{2} q + 1, & 0 \leq r \leq d - \frac{k}{2}; \\ \frac{k-2}{2} q + 2, & d - \frac{k}{2} < r \leq d, \end{cases}$ *where $n = q(d + 1) + r$ with $0 \leq r \leq d$.*

We see immediately that $\phi(n, d, k) = \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{n}{d+1} \rfloor + 1$ when k is odd, and $\phi(n, d, k) \leq \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{n}{d+1} \rfloor + 2$ when $k \neq 4$ is even. However, the case $k = 4$ is different. In summary, we have the following.

Corollary 1.4. *Conjecture 1.2 is true for any integer $k \neq 4$ and false for $k = 4$.*

Our proof of Theorem 1.3 is inductive in its nature. For that it is crucial for us to manage the base case when $d + 1 \leq n \leq 2d + 1$. It turns out in the proof of the base case that we make use of results on maximum cycles in bipartite graphs due to Jackson [6] and Kostochka, Luo and Zirlin [7] (see Theorems 3.1 and 3.2, respectively). To be precise, we partition the vertex set into two parts X and Y, where X consists of vertices of degree at least d, and then apply the above results on maximum cycles to find few number of disjoint paths in the bipartite subgraph $G(X, Y)$ to cover all vertices in X; finally, an application of a lemma of Erdős et al. in [3] (see Lemma 3.6) will ensure the desired long path. We would like to point out that it seems to be an incredible coincidence that the bounds we need in this argument are exactly what the recent result of Kostochka, Luo and Zirlin [7] provided. The case when $n \geq 2d + 2$ will be handled differently, which is essentially reduced to the base case.

The remainder of this paper is organized as follows. In Section 2, we construct extremal graphs for the function $\phi(n, d, k)$ and establish the lower bound of Theorem 1.3. In Section 3, we introduce the notation, a lemma of Erdős et al., and results of Jackson and Kostochka, Luo and Zirlin on maximum cycles in bipartite graphs; we also provide some variance and extension of these cycle results for the coming proof. In Section 4, we complete the proof of Theorem 1.3. In Section 5, we give better constructions to answer two questions in [3] which are closely related to Conjecture 1.2.

2 Extremal graphs

In this section, we construct extremal graphs for the function $\phi(n, d, k)$. This will give the matched lower bound of $\phi(n, d, k)$ in Theorem 1.3.

We start with some notation. Let $n > d \geq k$ be positive integers and let G, H be two graphs. By $G + H$ we mean the disjoint union of G and H, and we use $k \cdot G$ to denote the union of k disjoint copies of the same graph G.[1] Let K_n be the n-vertex clique, I_n be the graph induced by an independent set of n vertices, and $K_{1,n}$ be the star with n leaves. We define two special yet important graphs as following (see Figure 1):

- The graph $H_{d,k}$ is obtained from the disjoint union of $K_{\lfloor \frac{k-1}{2} \rfloor}$ and $I_{d+1-\lfloor \frac{k-1}{2} \rfloor}$ by joining every vertex of $K_{\lfloor \frac{k-1}{2} \rfloor}$ to every vertex of $I_{d+1-\lfloor \frac{k-1}{2} \rfloor}$.
- For even integers $k \geq 4$, let $H^*_{d,k}$ be the graph obtained from $H_{d,k}$ by adding a disjoint copy of $I_{d+1-\frac{k}{2}}$ and joining every vertex in $I_{d+1-\frac{k}{2}}$ to a fixed vertex of degree $\frac{k}{2}-1$ in $H_{d,k}$.

Note that $H_{d,k}$ has $d+1$ vertices in total and $\lfloor \frac{k-1}{2} \rfloor$ vertices of degree at least d, while $H^*_{d,k}$ has $2d+2-\frac{k}{2}$ vertices in total and $\frac{k}{2}$ vertices of degree at least d. In particular, $H^*_{d,4}$ is the graph obtained from two disjoint stars on d vertices by joining the two centers (we will also call it a *double-star*).

With the above notation, now we define the extremal graph $G(n,d,k)$ for the function $\phi(n,d,k)$.

Definition 2.1. *For positive integers $n > d \geq k$, we define the graph $G(n,d,k)$ as follows.*

- *For $k \in \{1,2\}$, let $G(n,d,k) = I_n$.*
- *For $k = 4$, write $n = 2qd + r$ with $0 \leq r < 2d$ and let*

$$G(n,d,k) = \begin{cases} q \cdot H^*_{d,4} + I_r, & \text{if } r \leq d; \\ q \cdot H^*_{d,4} + K_{1,d} + I_{r-d-1}, & \text{otherwise.} \end{cases}$$

- *For odd $k \geq 3$, write $n = q(d+1) + r$ with $0 \leq r \leq d$ and let $G(n,d,k) = q \cdot H_{d,k} + I_r$.*
- *For even $k \geq 6$, write $n = q(d+1) + r$ with $0 \leq r \leq d$ and let*

$$G(n,d,k) = \begin{cases} q \cdot H_{d,k} + I_r, & \text{if } r \leq d - \frac{k}{2}; \\ (q-1) \cdot H_{d,k} + H^*_{d,k} + I_{r-d+\frac{k}{2}-1}, & \text{otherwise.} \end{cases}$$

It is straightforward to check the following fact on $G(n,d,k)$.

Lemma 2.2. *For any positive integers $n > d \geq k$, the n-vertex graph $G(n,d,k)$ contains no P_{k+1} and thus the lower bound of Theorem 1.3 holds.*

3 Preliminaries and some results on bipartite graphs

Let G be a graph. For disjoint subsets $X, Y \subseteq V(G)$, we use $G(X,Y)$ to denote the bipartite subgraph of G induced by two parts X and Y. Let P be a path or a cycle. By $|P|$, we mean the number of vertices in P. For $x, y \in V(P)$, let xPy be a subpath of P between x and y. When P is associated with an orientation, the successor and predecessor of x along the direction are denoted by x^+ and x^- (if exist), respectively. We also denote by $x^{++} := (x^+)^+$ and $x^{--} := (x^-)^-$. For a subset $S \subseteq V(G)$, we define $N(S)$ to be the set of all vertices $x \in V(G) \backslash S$ which is adjacent to some vertex in S. If S consists of a single vertex x, then we write $N(S)$ as $N(x)$.

We now introduce two theorems on the existence of maximum cycles in bipartite graphs, which provide crucial tools for the proof of our main result. The first one is due to Jackson [6].

[1]Throughout this paper, the word *disjoint* always mean for *vertex-disjoint* unless otherwise specified.

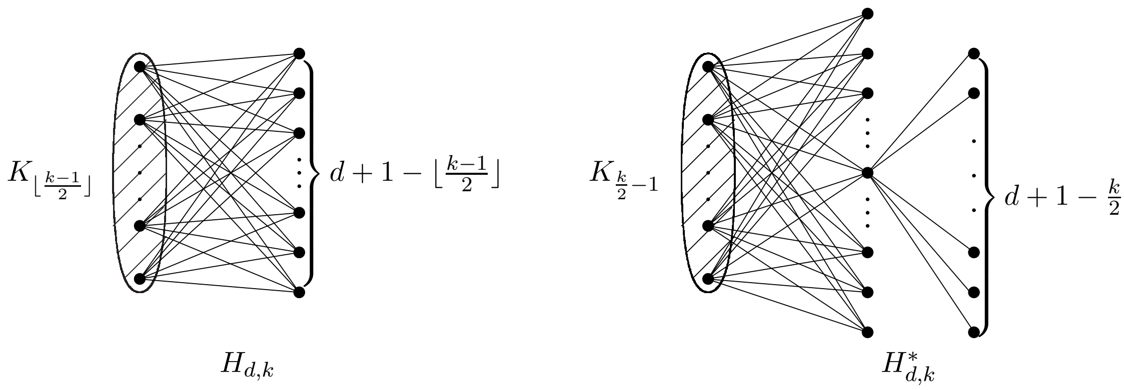

Figure 1: $H_{d,k}$ (for all k) and $H^*_{d,k}$ (for even k)

Theorem 3.1 (Jackson, [6, Theorem 1])**.** *Let G be a bipartite graph with two parts X and Y. If $2 \le |X| \le d$, $|Y| \le 2d-2$, and every vertex in X has degree at least d, then G has a cycle containing all vertices in X.*

The following theorem, conjectured by Jackson [6] and proved by Kostochka, Luo and Zirlin [7] recently, extends the above theorem of Jackson to the setting of 2-connected graphs.

Theorem 3.2 (Kostochka et al., [7, Theorem 1.6])**.** *Let G be a 2-connected bipartite graph with two parts X and Y. If $2 \le |X| \le d$, $|Y| \le 3d-5$, and every vertex in X has degree at least d, then G has a cycle containing all vertices in X.*

Our proof actually needs some intermediate statements from the proof of [7]. Let us give some notation used in [7] first. Let G be bipartite with parts X and Y which is not a forest. For a cycle C and a vertex x in G, we say (C, x) is a *tight pair* if C is a longest cycle in G, $x \in X \backslash V(C)$, and subject to these, $d_C(x) := |N(x) \cap V(C)|$ is maximum. Clearly G has a tight pair if and only if G has no cycle containing all vertices in X. The followings are collected from the proof of Theorem 1.6 in [7].

Lemma 3.3 (Kostochka et al., [7])**.** *Let G be a bipartite graph with two parts X and Y such that $|X| \le d \le \min\{d(x) : x \in X\}$. Let (C, x) be a tight pair in G with $c = |C|/2$. Then the followings hold:*

(i) If $d_C(x) \le 1$ and there is a path connecting two vertices in C and passing through x, then $|Y| \ge 3d-4$ (see Case 1 in the proof of Theorem 1.6 in [7]);

(ii) If $2 \le d_C(x) < c$, then $|Y| \ge 3d-4$ (i.e., Lemma 2.6 in [7]);

(iii) If $d_C(x) = c$, then for each $x_i \in X \cap V(C)$ and each $y \in N_{G-C}(x_i)$, x_i is a cut-vertex separating y from $V(C) - x_i$ (i.e., Lemma 2.7 in [7]).

Let G be a connected graph which is not a forest. We say that G is *essentially-2-connected*, if $G - V_1$ is 2-connected, where V_1 denotes the set of vertices of degree one in G. We need a variance of Theorem 3.2 for essentially-2-connected graphs.

Lemma 3.4. *Let G be an essentially 2-connected bipartite graph with parts X and Y. Suppose that $2 \le |X| \le d-1$, $|Y| \le 3d-5$, and every vertex in X has degree at least d. Then G has a cycle containing all vertices in X.*

Proof. Suppose for a contradiction that G has no cycle containing all vertices in X. Let (C, x) be a tight pair of G and $c = |C|/2$. So $c < |X|$. Since every vertex in X has degree at least $d \geq 3$, we see that $X \cup V(C)$ are contained in the 2-connected subgraph $G - V_1$ and thus there is a path connecting two vertices in C and passing through x. If $d_C(x) < c$, then by Lemma 3.3 (i) and (ii), we get $|Y| \geq 3d-4$, a contradiction. So we assume that $d_C(x) = c$, i.e., x is adjacent to all vertices in $Y \cap V(C)$.

We claim that every vertex $y \in Y \backslash V(C)$ has degree one in G. Suppose otherwise that there exists some $y \in Y \backslash V(C)$ with $d(y) \geq 2$. Since G is essentially 2-connected, there is a path P connecting two vertices in C and passing through y. By Lemma 3.3 (iii), the end-vertices of P are both in $Y \cap V(C)$ and $|P| \geq 5$ as G is bipartite. Let y_1, y_2 be the end-vertices of P. If there exists a subpath Q between y_1 and y_2 in C of length two, then replacing Q with the path P in C, we can get a longer cycle than C, a contradiction. Fix an orientation of C. Then we have that $y_2 \neq y_1^{++}$ (and also $y_1 \neq y_2^{++}$) in C. If $x \in V(P)$, then $y \in V(xPy_i)$ for some $i \in \{1, 2\}$. Without loss of generality, suppose that $y \in V(xPy_1)$. Replacing $y_1y_1^+y_1^{++}$ with the path $y_1Px \cup xy_1^{++}$ in C, again we have a longer cycle than C, a contradiction. So $x \notin V(P)$. Let C' be the cycle obtained from C by deleting the edges in $y_1y_1^+y_1^{++} \cup y_2y_2^+y_2^{++}$ and adding the paths P and $y_1^{++}xy_2^{++}$. Then C' is a longer cycle than C, a contradiction. This proves the claim.

Note that every vertex in X has at least $d - c$ neighbors outside C. By the previous claim, these neighbors all have degree one in G and thus are distinct for different vertices in X. This shows that $|Y| \geq c + |X|(d-c) \geq c + (c+1)(d-c) = c(d-c) + d$. Since $2 \leq c \leq |X| - 1 \leq d - 2$, now we can infer that $|Y| \geq 3d - 4$, a contradiction. This completes the proof of the lemma. □

We remark that the condition $|X| \leq d - 1$ in Lemma 3.4 cannot be relaxed to that $|X| \leq d$ by the following examples. Let $H = H(X, Y_1)$ be a complete bipartite graph with $|X| = d$ and $|Y_1| = d - 1$. Let G be the bipartite graph obtained from H by adding at least one new vertex x' for each vertex $x \in X$ and then adding the edge xx' for every new vertex x'. Let Y be the part of G other than X. Then the size of Y can be any integer at least $2d - 1$, every vertex in X has degree at least d in G, and G is essentially 2-connected but has no cycle containing all vertices in X.

The following lemma will be pivotal for the proof of our main result Theorem 1.3.

Lemma 3.5. *Let G be a bipartite graph with parts X and Y. Suppose every vertex in X has degree at least d. Then the followings are true:*

(i) If $|X| \leq d + 1$ and $|Y| \leq 2d - 1$, then G has a path containing all vertices in X;

(ii) If G is connected, $|X| \leq d$ and $|Y| \leq 3d - 3$, then G has a path containing all vertices in X;

(iii) Let $t \geq 1$ be any integer. If $|X| \leq d + t$ and $|Y| \leq 3d + 2t - 3$, then there exist at most $t + 1$ disjoint paths in G containing all vertices in X.

Proof. (i). It is obvious when $|X| = 1$. So assume $|X| \geq 2$. Let G' be the graph obtained from G by adding a new vertex y and joining y to every vertex in X. Then every vertex in X has degree at least $d + 1$ in G'. Since $|X| \leq d + 1$ and $|Y \cup \{y\}| \leq 2d = 2(d+1) - 2$. By Theorem 3.1, G' has a cycle C containing all vertices in X. The vertex y may be contained in C or not. In either case, one can find a path in G containing all vertices in X (by considering $C - y$). This proves (i).

(ii). Similarly we may assume $|X| \geq 2$. Let G' be obtained from G by adding a new vertex y and joining y to every vertex in X. Let $Y' = Y \cup \{y\}$. Then we have that $2 \leq |X| \leq d$, $|Y'| \leq 3d - 2$, and every vertex in X has degree at least $d + 1$ in G'. We claim that G' is essentially 2-connected. To see this, consider a spanning tree T in G (note that G is connected). Let G'' be obtained from T by adding the vertex y and joining y to every vertex in X. Clearly we have $G'' \subseteq G'$, and by definition,

G'' is essentially 2-connected. This implies that G' is essentially 2-connected. Now applying Lemma 3.4, we can conclude that G' has a cycle C containing all vertices in X. The vertex y lies on C or not. In either case, $C - y$ (and thus G) contains a path containing all vertices in X. This proves (ii).

(iii). Let G' be obtained from G by deleting all isolated vertices in Y, adding t new vertices $y_1, ..., y_t$ and then joining every y_i for $i \in [t]$ to all vertices in X. Let Y_0 be the set of all isolated vertices of G in Y and let $Y' = (Y \backslash Y_0) \cup \{y_1, ..., y_t\}$. So G' is a connected bipartite graph with parts X and Y' such that $|X| \leq d + t$, $|Y'| \leq |Y| + t \leq 3(d + t) - 3$, and every vertex in X has degree at least $d + t$ in G'. By (ii), G' has a path P containing all vertices in X. By deleting the vertices $y_1, ..., y_t$, we can obtain at most $t + 1$ disjoint paths (in G) from P containing all vertices of X. This proves Lemma 3.5. □

Lastly, we need the following convenient lemma due to Erdős, Faudree, Schelp and Siminovits [3]. An analog for cycles can be found in [8].

Lemma 3.6 (Erdős et al., [3, Lemma 1])**.** *Let G be a graph with at most $2d+1$ vertices and $\mathcal{P}$ be any family of disjoint paths P_i, where both end-vertices of each $P_i \in \mathcal{P}$ have degree at least d in G. Then G contains a path Q such that both its end-vertices have degree at least d in G and $\bigcup_{P_i \in \mathcal{P}} V(P_i) \subseteq V(Q)$.*

4 Proof of Theorem 1.3

For given positive integers $n > d \geq k$, let ϕ be the function $\phi(n, d, k)$ defined in Theorem 1.3 throughout this section. To complete the proof of Theorem 1.3, in view of Lemma 2.2, it suffices to prove that

$$\text{any } n\text{-vertex graph with at least } \phi \text{ vertices of degree at least } d \text{ contains a path } P_{k+1}. \tag{1}$$

We will prove this by contradiction. Consider any positive integers $d \geq k$ for which (1) fails. Then there exists a counterexample G to the statement (1) as follows:

(a). G is an n-vertex graph with at least $\phi = \phi(n, d, k)$ vertices of degree at least d,

(b). G does not contain any path P_{k+1} (on $k + 1$ vertices),

(c). subject to (a) and (b), n is minimum, and

(d). subject to (a), (b) and (c), G has the minimum number of edges.

We proceed the proof by proving a sequence of claims. Our first claim is the following. In the rest of the proof, we say a vertex is *high* if it has degree at least d in G and *low* otherwise. A path is a *high-end* path if both its end-vertices are high vertices.

Claim 4.1. *G has no high-end path on at least $k - 1$ vertices.*

Proof. Suppose that P is a high-end path on at least $k - 1$ vertices. Let u, v be end-vertices of P. If $|P| \geq k + 1$, then there is nothing to prove. If $|P| = k$, as $d(u) \geq d \geq k$, u has a neighbor u' outside $V(P)$, and $P \cup uu'$ is a desired path P_{k+1}. So $|P| = k - 1$. Then u has a neighbor u' outside $V(P)$, and v has a neighbor v' outside $V(P) \cup \{u'\}$. Now $u'u \cup P \cup vv'$ is a path P_{k+1}. In any case, we get a contradiction. □

Claim 4.2. *We may assume that $k \geq 5$.*

Proof. We first point out that the case $k \in \{1,2\}$ is trivial: a graph G contains P_2 if and only if G has a vertex of degree at least 1, and G contains P_3 if and only if G has a vertex of degree at least 2.

Now consider the case $k = 3$. So G contains no P_4. Then each component H of G is a K_1, K_2, K_3 or a star $K_{1,s}$ for some $s \geq 2$. Note that only the component $K_{1,s}$ with $s \geq d$ can have one high vertex. It follows that H has at most $\lfloor \frac{|V(H)|}{d+1} \rfloor$ high vertices. So G has at most

$$\sum_{\text{each component } H} \left\lfloor \frac{|V(H)|}{d+1} \right\rfloor \leq \left\lfloor \frac{n}{d+1} \right\rfloor = \phi - 1$$

high vertices, a contradiction.

Finally consider the case $k = 4$. In this case, G contains no P_5. Let H be any component of G. By Claim 4.1, H has no high-end path on 3 vertices. This shows that H has at most two high vertices, and if u, v are the two high vertices of H, then uv must be a cut-edge of H and thus $N(u) \cap N(v) = \emptyset$. It also follows that if H has only one high vertex, then $|V(H)| \geq d+1$ and if H has two high vertices, then $|V(H)| \geq 2d$. So we can conclude that H has at most $\frac{|V(H)|}{d}$ high vertices (and if H has exactly $\frac{|V(H)|}{d}$ high vertices, then $|V(H)| = 2d$). Therefore G has at most

$$\sum_{\text{each component } H} \frac{|V(H)|}{d} \leq \frac{n}{d}$$

high vertices.[2] This gives that the number of high vertices of G is at most $\phi - 1$, a contradiction. □

We now discuss several useful properties that the graph G has.

Claim 4.3. *If u is a low vertex in G, then every neighbor of u has degree exactly d and thus is high.*

Proof. Suppose, otherwise, that there is a vertex $v \in N(u)$ with $d(v) \leq d-1$ or $d(v) \geq d+1$. Then $G' = G - uv$ is a graph satisfying $(a), (b)$ and (c), but having less edges than G. This violates (d) and the choice of G. □

Claim 4.4. *G is connected.*

Proof. Suppose that G is not connected. Then G is a disjoint union of two subgraphs G_1 and G_2. For $i \in \{1,2\}$, let $|V(G_i)| := n_i = q_i(d+1) + r_i$ for $0 \leq r_i \leq d$, and let $n = q(d+1) + r$ for $0 \leq r \leq d$. As $n = n_1 + n_2$, we have that either (1) $q = q_1 + q_2$ and $r = r_1 + r_2 \leq d$, or (2) $q = q_1 + q_2 + 1$ and $0 \leq r = r_1 + r_2 - (d+1) \leq d$. Note that each G_i contains no P_{k+1}.

First consider that k is odd, or $k \geq 6$ is even and $r_i \leq d - \frac{k}{2}$ for both $i \in \{1,2\}$. In this case, since G_i has no P_{k+1}, each G_i has at most $q_i \lfloor \frac{k-1}{2} \rfloor$ high vertices. So G has at most $(q_1+q_2)\lfloor \frac{k-1}{2} \rfloor \leq q \lfloor \frac{k-1}{2} \rfloor \leq \phi - 1$ high vertices, a contradiction.

Now assume that $k \geq 6$ is even and exactly one of r_1 and r_2 is at most $d - \frac{k}{2}$. Without loss of generality, assume that $r_1 \leq d - \frac{k}{2}$ and $r_2 > d - \frac{k}{2}$. Then G_1 has at most $q_1 \frac{k-2}{2}$ high vertices and G_2 has at most $q_2 \frac{k-2}{2} + 1$ high vertices. Thus G has at most $(q_1 + q_2)\frac{k-2}{2} + 1$ high vertices. Note that either $q = q_1 + q_2 + 1$, or $q = q_1 + q_2$ and $d \geq r = r_1 + r_2 > d - \frac{k}{2}$. In both cases, we see that the number of high vertices in G is at most $(q_1 + q_2)\frac{k-2}{2} + 1 \leq \phi - 1$, a contradiction.

Finally, $k \geq 6$ is even and $r_i > d - \frac{k}{2}$ for both $i \in \{1,2\}$. In this case, each G_i has at most $q_i \frac{k-2}{2} + 1$ high vertices, and so G has at most $(q_1+q_2)\frac{k-2}{2} + 2$ high vertices. Since $r_1 + r_2 > 2d - k \geq d$, we must have that $q = q_1 + q_2 + 1$. Thus the number of high vertices in G is at most $(q_1 + q_2)\frac{k-2}{2} + 2 \leq q\frac{k-2}{2} \leq \phi - 1$, where the first inequality holds because $k \geq 6$. This contradiction completes the proof of Claim 4.4. □

[2]We also see that in this case, if G has exactly $\frac{n}{d}$ high vertices, then every component of G forms a double-star $H^*_{d,4}$.

Claim 4.5. *G has exactly ϕ high vertices.*

Proof. Suppose that G has at least $\phi+1$ high vertices. If every vertex is high, then $\delta(G) \geq d \geq k$ and by Erdős-Gallai Theorem [4], there is a path of length $d \geq k$ in G, a contradiction. So there is some low vertex, say u in G. By Claim 4.4, G is connected, so there exists some $v \in N(u)$. By Claim 4.3, v is high. Then $G' = G - uv$ is a graph satisfying $(a), (b)$ and (c), but with less number of edges, contradicting the choice of G. This proves the claim. $\square$

In the rest of the proof, we write $n = q(d+1) + r$ for some integers $q \geq 1$ and $0 \leq r \leq d$. Since $k \geq 5$, we have $\phi = \lfloor \frac{k-1}{2} \rfloor q + 1 + \epsilon$, where $\epsilon = 1$ if k is even and $r > d - \frac{k}{2}$, and $\epsilon = 0$ otherwise.

Claim 4.6. *We have $n \geq 2d+2$ and thus $q \geq 2$.*

Proof. Suppose that $d < n \leq 2d+1$. So $q = \lfloor \frac{n}{d+1} \rfloor = 1$ and $n = d+1+r$. We will reach a contradiction to Claim 4.1 by finding a high-end path in G on at least $k-1$ vertices. Let X be the set of all high vertices in G and $Y = V(G) \backslash X$. Let $G' = G(X, Y)$ be the spanning bipartite subgraph of G with parts X and Y. We have $|X| = \phi$ by Claim 4.5.

Suppose that k is odd. Then $|X| = \phi = \frac{k+1}{2}$, and $|Y| = n - |X| \leq 2d + 1 - \frac{k+1}{2}$. For every $x \in X$, $d_{G'}(x) \geq d_G(x) - (|X| - 1) \geq d - \frac{k-1}{2} := d_1$. Since $d \geq k$, we can derive that $|X| = \frac{k+1}{2} \leq d_1$ and $|Y| \leq 2d + 1 - \frac{k+1}{2} \leq 3d_1 - 1$. By Lemma 3.5 (iii) with $t = 1$, G' has at most two disjoint paths (say P_1, P_2) containing all vertices in X. We may assume that all end-vertices of P_1, P_2 are in X, so P_1, P_2 are high-end paths of G. As $n \leq 2d+1$, by Lemma 3.6, there is a high-end path P in G satisfying $V(P_1) \cup V(P_2) \subseteq V(P)$ and thus $|P| \geq |P_1| + |P_2| = 2|X| - 2 = k - 1$.

Now suppose that k is even and $r \leq d - \frac{k}{2}$. Then $|X| = \phi = \frac{k}{2}$ and $|Y| = n - |X| = (d+1+r) - |X| \leq d + 1 + (d - \frac{k}{2}) - \frac{k}{2} = 2d + 1 - k$. Also for every $x \in X$, $d_{G'}(x) \geq d - \frac{k}{2} + 1 := d_2$. Since $d \geq k$, we see $|X| \leq d_2$ and $|Y| \leq 2d_2 - 1$. By Lemma 3.5 (i), G' has a path P containing all vertices in X. We may view P as a path with both end-vertices in X. So P is a high-end path of G with $|P| = 2|X| - 1 = k - 1$.

It remains to consider the case when k is even and $d \geq r > d - \frac{k}{2}$. In this case, $|X| = \phi = \frac{k}{2} + 1$, $|Y| = n - |X| = (d+1+r) - |X| \leq 2d - \frac{k}{2}$, and for every $x \in X$, $d_{G'}(x) \geq d - \frac{k}{2} := d_3$. Since $d \geq k$, one can deduce that $|X| \leq d_3 + 1$ and $|Y| \leq 3d_3$. By Lemma 3.5 (iii) with $t = 2$, G' has at most three disjoint paths P_1, P_2, P_3 containing all vertices in X, all of which can be viewed as high-end paths of G. Using Lemma 3.6, G has a high-end path P containing all vertices in $P_1 \cup P_2 \cup P_3$. Therefore $|P| \geq |P_1| + |P_2| + |P_3| = 2|X| - 3 = 2\left(\frac{k}{2} + 1\right) - 3 = k - 1$.

In any case, we get a contradiction to Claim 4.1. This finishes the proof of Claim 4.6. $\square$

Claim 4.7. *If T is any set of at least $d+1$ vertices, then $T \cup N(T)$ contains at least $\lfloor \frac{k+1}{2} \rfloor$ high vertices.*

Proof. Suppose that $T \cup N(T)$ contains at most $\lfloor \frac{k-1}{2} \rfloor$ high vertices. Let T' be any subset of T with $|T'| = d+1$, and let G' be obtained from G by deleting all vertices in T'. Then $n' := |V(G')| = n - (d+1) > d$ and G' has at least $\phi - \left\lfloor \frac{k-1}{2} \right\rfloor = \phi(n', d, k)$ high vertices. Thus G' is a counterexample smaller than G (which satisfies (a) and (b) but violates (c)), a contradiction. $\square$

Since G is connected and has $\phi \geq 2$ high vertices, there exist high-end paths in G. Now we choose a high-end path P in G such that the number of high vertices in P is maximum, and subject to this, $|P|$ is maximum. By Claim 4.1, we have $|P| \leq k-2$. Note that as $q \geq 2$, we have $\phi \geq \lfloor \frac{k-1}{2} \rfloor q + 1 \geq k - 1$. So there must be some high vertex outside $V(P)$.

Let u_1, u_2 be the two end-vertices of P. We assign the orientation of P from u_1 to u_2.

Claim 4.8. *Let $S_i = N_{G-P}(u_i)$ for $i \in \{1, 2\}$ and $S = S_1 \cup S_2$. Then any vertex in S is a low vertex, $S_1 \cap S_2 = \emptyset$, and $N_P(u_1) \cup \{u_1\} \subseteq V(P) \backslash (N_P(u_2))^+$. In particular, we have $d_P(u_1) + d_P(u_2) \leq |P| - 1$.*

Proof. First any vertex in S is a low vertex (as, otherwise, say $s \in N_{G-P}(u_1)$ is high, then $P \cup u_1 s$ contradicts the choice of P). If u_1, u_2 has a common neighbor say v outside $V(P)$, then v is a low vertex and let $C = P \cup u_2 v u_1$; if there exists a vertex $v \in N_P(u_1) \cap (N_P(u_2))^+$ such that $u_1 v, u_2 v^- \in E(G)$, then let $C = (P - v^- v) \cup \{u_1 v, u_2 v^-\}$. In any case C is a cycle containing all vertices in P. As we just discussed before this claim, there exists some high vertex x outside $V(P)$ (and thus outside $V(C)$). Now let P' be any path in G from x to C and internally disjoint from C (recall Claim 4.4 that G is connected). Then $C \cup P'$ contains a high-end path containing more high vertices than P, a contradiction to the choice of P. □

In the following claims, we investigate more properties on the sets S_1 and S_2.

Claim 4.9. *Every vertex in $N(S_1) \cup N(S_2)$ is high and lies in P, every vertex in $(N(S_1) \backslash \{u_1\})^- \cup (N(S_2) \backslash \{u_2\})^+$ is low, and moreover, $|S| \geq d + 3$.*

Proof. By Claim 4.3, any neighbor of a low vertex is a high vertex. So for $i \in \{1, 2\}$, every vertex in $N(S_i)$ is high and must lie on P (otherwise, P can be extended to a longer high-end path, which contradicts the choice of P). Consider any $v \in S_1$ and $x \in N(v) \backslash \{u_1\}$. So $x \in V(P) \backslash \{u_1\}$. If x^- is high, then $P' = x^- P u_1 \cup u_1 v x \cup x P u_2$ is a high-end path such that $V(P) \subsetneq V(P')$, a contradiction to the choice of P. Thus we conclude that all vertices in $(N(S_1) \backslash \{u_1\})^-$ are low, and similarly all vertices in $(N(S_2) \backslash \{u_2\})^+$ are low. By Claim 4.8, one can get that $|S| = |S_1| + |S_2| \geq (d(u_1) - d_P(u_1)) + (d(u_2) - d_P(u_2)) \geq 2d - (k - 3) \geq d + 3$. □

Claim 4.10. *There is at most one vertex $v \in V(P)$ such that $v, v^+ \in N(S)$. In particular, we have $v \in N(S_1) \backslash N(S_2)$ and $v^+ \in N(S_2) \backslash N(S_1)$.*

Proof. First, both $v, v^+ \in N(S)$ are high. If $v \in N(S_2)$, then by Claim 4.9, v^+ is low, a contradiction. Thus $v \in N(S_1) \backslash N(S_2)$, and similarly, $v^+ \in N(S_2) \backslash N(S_1)$. Note that possibly $v = u_1$ or $v^+ = u_2$.

Suppose for a contradiction that there are two such vertices, say v_1 and v_2. Assume that $v_1 \in V(u_1 P v_2)$. Recall that $v_1, v_2 \in N(S_1)$ and $v_1^+, v_2^+ \in N(S_2)$. We remark that $v_1^+ \neq v_2$ (because v_2^- is low by Claim 4.9). So v_1, v_1^+, v_2, v_2^+ appear on P in order. For $i \in \{1, 2\}$, let $x_i \in S_1$ with $x_i v_i \in E(G)$ and $y_i \in S_2$ with $y_i v_i^+ \in E(G)$ such that if $v_1 = u_1$ then choose $x_1 = x_2$, and if $v_2^+ = u_2$ then choose $y_1 = y_2$. Now we define a new path P' as follows (see Figure 4):

$$P' = \begin{cases} u_1 P v_1 \cup v_1 x_1 v_2 \cup v_2 P v_1^+ \cup v_1^+ y_1 v_2^+ \cup v_2^+ P u_2, & \text{if } x_1 = x_2 \text{ and } y_1 = y_2; \\ u_1^+ P v_1 \cup v_1 x_1 u_1 x_2 v_2 \cup v_2 P v_1^+ \cup v_1^+ y_1 v_2^+ \cup v_2^+ P u_2, & \text{if } x_1 \neq x_2 \text{ and } y_1 = y_2; \\ u_1 P v_1 \cup v_1 x_1 v_2 \cup v_2 P v_1^+ \cup v_1^+ y_1 u_2 y_2 v_2^+ \cup v_2^+ P u_2^-, & \text{if } x_1 = x_2 \text{ and } y_1 \neq y_2; \\ u_1^+ P v_1 \cup v_1 x_1 u_1 x_2 v_2 \cup v_2 P v_1^+ \cup v_1^+ y_1 u_2 y_2 v_2^+ \cup v_2^+ P u_2^-, & \text{if } x_1 \neq x_2 \text{ and } y_1 \neq y_2. \end{cases}$$

Then $V(P') = V(P) \cup \{x_1, x_2, y_1, y_2\}$, and the possible end-vertices u_1^+ and u_2^- of P' can be low vertices. Let P'' be the path obtained from P' by removing its low end-vertices (which only can be u_1^+ and/or u_2^-). Note that u_1^+ (respectively, u_2^-) is an end-vertex of P' if and only if $x_1 \neq x_2$ (respectively, $y_1 \neq y_2$). By Claim 4.3, P'' is a high-end path and contains exactly the same high vertices as in P. However, P'' is longer than P, as $|P''| \geq |P'| - \mathbf{1}_{x_1 \neq x_2} - \mathbf{1}_{y_1 \neq y_2} \geq |P| + 2$. This contradicts the choice of P. □

Now we can derive from Claims 4.9 and 4.10 that $|N(S)| \leq \frac{|P|+2}{2} \leq \frac{k}{2}$. On the other hand, as $|S| \geq d + 3$ (by Claim 4.9), $S \cup N(S)$ contains at least $\lfloor \frac{k+1}{2} \rfloor$ high vertices (by Claim 4.7). All vertices in S are low (by Claim 4.8), so $|N(S)| \geq \lfloor \frac{k+1}{2} \rfloor$. Combining the above, we have

$$\left\lfloor \frac{k+1}{2} \right\rfloor \leq |N(S)| \leq \frac{|P| + 2}{2} \leq \frac{k}{2}. \tag{2}$$

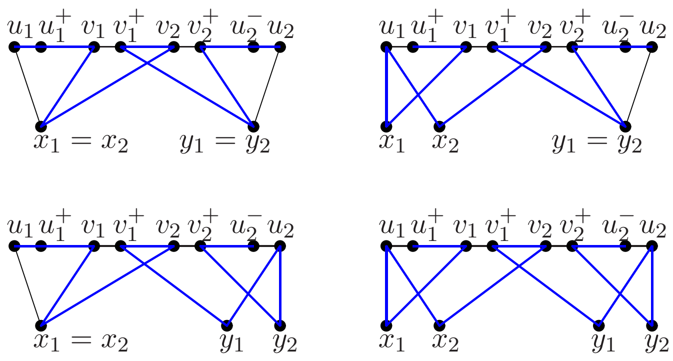

Figure 2: Key steps in the proof of Claim 4.10.

This indicates that k is even, $|P| = k-2$, $|N(S)| = \frac{k}{2}$, and there is exactly one vertex $v \in V(P)$ satisfying Claim 4.10. Furthermore, by letting $k = 2s$, we can express

$$P = a_0 b_1 a_1 \cdots b_j a_j a_{j+1} b_{j+1} \cdots b_{s-2} a_{s-1}$$

for some $0 \le j \le s-2$ such that $a_0 = u_1, a_j = v, a_{j+1} = v^+, a_{s-1} = u_2$ and $N(S) = \{a_0, a_1, ..., a_{s-1}\}$.

Claim 4.11. *We have $N(S_1) = \{a_0, ..., a_j\}$ and $N(S_2) = \{a_{j+1}, ..., a_{s-1}\}$.*

Proof. It suffices to show that $N(S_1) \subseteq \{a_0, ..., a_j\}$ and $N(S_2) \subseteq \{a_{j+1}, ..., a_{s-1}\}$. By symmetry, we will only prove that $N(S_1) \subseteq \{a_0, ..., a_j\}$. Suppose not. Then there exists some $a_\ell \in N(S_1)$ with $\ell \ge j+1$, and we may assume that subject to the condition $\ell \ge j+1$, ℓ is minimal. By Claim 4.9, we see that $(a_\ell)^-$ is a low vertex. This implies that $\ell \ge j+2$ and $b_{\ell-1} = (a_\ell)^-$ is low. Then by the minimality of ℓ, we have $a_{\ell-1} \in N(S_2)$. Let $z_1 \in S_1, z_2 \in S_2$ be two vertices such that $z_1 a_\ell, z_2 a_{\ell-1} \in E(G)$. Then $P' := a_j P u_1 \cup u_1 z_1 a_\ell \cup a_\ell P u_2 \cup u_2 z_2 a_{\ell-1} \cup a_{\ell-1} P a_{j+1}$ is a high-end path such that $V(P') = (V(P) \backslash \{b_{\ell-1}\}) \cup \{z_1, z_2\}$ and $|P'| = |P| + 1$. So P' contains all high vertices of P and is longer than P, a contradiction to the choice of P. This proves Claim 4.11. □

Hence by Claim 4.9, every a_i for $0 \le i \le s-1$ is high and every b_i for $1 \le i \le s-2$ is low.

Claim 4.12. *For any $a_i \notin \{v, v^+\}$, every vertex $z \in N_{G-P}(a_i)$ is a low vertex such that $N(z) \subseteq V(P)$. On the other hand, every b_i satisfies $N(b_i) \subseteq V(P)$.*

Proof. First we consider b_i for any $1 \le i \le s-2$. Without loss of generality, we may assume $1 \le i \le j$. Suppose that b_i has a neighbor z outside $V(P)$. Then by Claim 4.3, z is high. Let $w \in S_1$ be such that $wa_i \in E(G)$. Now $zb_i \cup b_i P u_1 \cup u_1 w a_i \cup a_i P u_2$ is a high-end path containing more high vertices than P, a contradiction. This proves $N(b_i) \subseteq V(P)$. Now consider any $a_i \notin \{v, v^+\}$. We are done by Claims 4.8 and 4.9 if $a_i \in \{u_1, u_2\}$. Hence, without loss of generality assume that $1 \le i \le j-1$. Consider $z \in N_{G-P}(a_i)$. Let $w' \in S_1$ be such that $w' a_{i+1} \in E(G)$ and let $P' := z a_i \cup a_i P u_1 \cup u_1 w' a_{i+1} \cup a_{i+1} P u_2$. If z is high, then P' is a high-end path containing more high vertices than P, a contradiction. So any such z must be low. Suppose for a contradiction that z has a neighbor z' outside $V(P)$. Then by Claim 4.3, z' is high. So $z'z \cup P'$ is a high-end path containing more high vertices than P, again a contradiction. Thus we can conclude that $N(z) \subseteq V(P)$. □

We are ready to complete the proof of Theorem 1.3. Recall that G has a high vertex (say x) outside $V(P)$. Let $U = V(P) \backslash \{v, v^+\}$. We can derive from Claim 4.12 that $\{v, v^+\}$ is a 2-cut of G separating the vertex x from the set $S \cup U$. Let D be the union of (at most two) components in $G - \{v, v^+\}$

containing vertices in $S \cup U$. Claim 4.12 also shows that all high vertices in $D \cup N(D)$ are those in $V(P)$, i.e., vertices in $N(S) = \{a_0, a_1, ..., a_{s-1}\}$. By Claim 4.8, we have

$$|S \cup V(P)| = |S_1|+|S_2|+|P| \geq d_{G-P}(u_1)+d_{G-P}(u_2)+(d_P(u_1)+d_P(u_2))+1 = d(u_1)+d(u_2)+1 \geq 2d+1.$$

Either v or v^+ has some neighbor not in $S \cup V(P)$. Without loss of generality we assume that v has some neighbor not in $S \cup V(P)$. By Claim 4.4, v has degree d. Also note that $vv^+ \in E(G)$. So v has at most $d-2$ neighbors in $S \cup U$. Set $S' = (S \cup U) \backslash N(v)$. Then using $|S \cup V(P)| \geq 2d+1$, we have

$$|S'| \geq |S \cup U| - (d-2) = (|S \cup V(P)| - 2) - (d-2) \geq d+1.$$

By Claim 4.7, $S' \cup N(S')$ contains at least $s = \frac{k}{2}$ high vertices. However, as S' is a subset in D, by definition we see $S' \cup N(S') \subseteq (D \cup N(D)) \backslash \{v\}$. We have pointed out that all high vertices in $D \cup N(D)$ are $a_0, a_1, ..., a_{s-1}$. So $S' \cup N(S')$ contains at most $s-1$ high vertices. This final contradiction finishes the proof of Theorem 1.3. $\square$

5 On two related questions

In this section, we consider two questions related to Conjecture 1.2 that are raised by Erdős, Faudree, Schelp and Simonovits in [3]. We will provide better constructions than the ones given in [3], which give (negative and positive) answers to their questions.

It is natural to consider the analog of Definition 1.1 for long cycles. For integers $n > d \geq k \geq 2$, define $\theta(n, d, k)$ to be the smallest integer θ such that every n-vertex graph with at least θ vertices of degree at least d contains a cycle on at least $k+1$ vertices. In this language, the well-known Dirac's theorem [2] states that $\theta(n, d, d) \leq n$. Improving Dirac's theorem, Woodall [9] proved that $\theta(n, d, d) \leq \frac{(d+2)(n-1)}{2d}$ if d is even and $\theta(n, d, d) \leq \frac{d(n-2)}{2(d-1)}$ otherwise; while for the general case, he [9] showed that $\theta(n, d, k) \leq \frac{(k+3)(n-1)}{2d}$. Recall the graph $H_{d,k+1}$ that it contains no cycles on at least $k+1$ vertices and has $d+1$ vertices in total, where $\lfloor \frac{k}{2} \rfloor$ vertices have degree d (call them *high* vertices) and all other vertices have degree $\lfloor \frac{k}{2} \rfloor$ (call them *low* vertices). By considering the graphs consisting of blocks $H_{d,k+1}$, the authors of [3] raised the following question: Is it possible that $\theta(n, d, k) \leq \lfloor \frac{k}{2} \rfloor \lfloor \frac{n-1}{d} \rfloor + 2$?

In the following proposition, we give a negative answer to the above question.

Proposition 5.1. *For any integer $k \geq 2$, there exist infinitely many integers d such that the following holds. There exists some constant $c = c(d, k) > 0$ such that $\theta(n, d, k) > (\lfloor \frac{k}{2} \rfloor + c) \cdot \frac{n-1}{d}$ for infinite integers n.*

Proof. We show a slightly stronger statement: Let α, β be positive integers such that $d = (1+\alpha)\lfloor \frac{k}{2} \rfloor$ and $n = 1 + d + \alpha\beta d$ satisfy $n > d \geq k \geq 2$. Then $\theta(n, d, k) \geq \frac{n-1}{d} \lfloor \frac{k}{2} \rfloor + \frac{n-(d+1)}{\alpha d} + 1$.

We construct an n-vertex graph G as follows. Let H_0 be a copy of $H_{d,k+1}$ with a low vertex v_0. For each $1 \leq i \leq \beta$, let H_i be obtained from α copies of $H_{d,k+1}$ by identifying one low vertex from each copy of $H_{d,k+1}$ (call the resulting vertex u_i); let v_i be a low vertex in H_i other than u_i. Finally, let G be obtained from $H_0, H_1, \ldots, H_\beta$ by identifying v_{i-1} and u_i for all $1 \leq i \leq \beta$. Since each block of G is a copy of $H_{d,k+1}$, we see that G contains no cycles of at least $k+1$ vertices. However, G has $(1+\alpha\beta)\lfloor \frac{k}{2} \rfloor + \beta = \frac{n-1}{d} \lfloor \frac{k}{2} \rfloor + \frac{n-(d+1)}{\alpha d}$ vertices of degree at least d. This proves the proposition. $\square$

Note that in the above proof, one can take $\alpha \geq 1$ for even k and $\alpha \geq 2$ for odd k. For the cases $d > k$ in Lemma 5.1, the constant c can be taken up to $\frac{1}{2}$ (i.e., when it corresponds to $\alpha = 2$).

Another question concerned in [3] is the restricted version of Conjecture 1.2 when the graph G is assumed to be connected. For positive integers $n > d \geq k$, define $\psi(n, d, k)$ to be the smallest integer ψ

such that every n-vertex connected graph with at least ψ vertices of degree at least d contains a path P_{k+1} on $k+1$ vertices. Erdős et al. [3] observed that the graph, obtained from $\lfloor\frac{n-1}{d}\rfloor$ copies of $H_{d,\lceil\frac{k}{2}\rceil+1}$ by identifying at a fixed high vertex of each $H_{d,\lceil\frac{k}{2}\rceil+1}$, contains no P_{k+1}. This gives that $\psi(n,d,k) \geq \lfloor\frac{k-3}{4}\rfloor\lfloor\frac{n-1}{d}\rfloor + 2$, which is approximately a half of the number of high vertices in Conjecture 1.2 (as $k \to \infty$). They [3] asked if there is a better construction. We show that it is possible to improve the leading coefficient of n by a positive constant factor in the above lower bound of $\psi(n,d,k)$.

Proposition 5.2. *For any integer $k \geq 7$, there exist infinitely many integers d such that the following holds. There exists some constant $c' = c'(d,k) > 0$ such that $\psi(n,d,k) > (\lfloor\frac{k-3}{4}\rfloor + c')\cdot\frac{n-1}{d}$ for infinite integers n.*

Proof. Let $\alpha \geq 2$ and $\beta \geq d$ be any integers. Let $d = 1+\alpha\lfloor\frac{k-3}{4}\rfloor$ and $n = 1+\beta(1+\alpha d)$. Note that the graph $H^* = H_{d,\lceil\frac{k}{2}\rceil-1}$ contains $\lfloor\frac{k-3}{4}\rfloor$ vertices of degree d, and all other vertices have degree $\lfloor\frac{k-3}{4}\rfloor$ (call them *low* vertices). We will construct an n-vertex connected graph G to show that $\psi(n,d,k) \geq \lfloor\frac{k-3}{4}\rfloor\frac{n-1}{d} + \left(1-\frac{1}{d}\lfloor\frac{k-3}{4}\rfloor\right)\frac{n-1}{1+\alpha d} + 2$ as follows. Let H_0 be a star $K_{1,\beta}$ with leaves v_i for $1 \leq i \leq \beta$. For each $1 \leq i \leq \beta$, let H_i be obtained from α copies of H^* by identifying one low vertex from each H^* (call the resulting vertex u_i). Now let G be obtained from $H_0, H_1, ..., H_\beta$ by identifying v_i and u_i for each $1 \leq i \leq \beta$. Then G contains no path P_{k+1} and has $\alpha\beta\lfloor\frac{k-3}{4}\rfloor + \beta + 1 = \left(\frac{n-1}{d} - \frac{\beta}{d}\right)\lfloor\frac{k-3}{4}\rfloor + \beta + 1 = \lfloor\frac{k-3}{4}\rfloor\frac{n-1}{d} + \left(1-\frac{1}{d}\lfloor\frac{k-3}{4}\rfloor\right)\frac{n-1}{1+\alpha d} + 1$ vertices of degree at least d, as desired. □

It would be very interesting to determine the functions $\theta(n,d,k)$ and $\psi(n,d,k)$ exactly.

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