arXiv:2109.01311v3 [math.CO] 30 Sep 2021

Bipartite-ness under smooth conditions

Tao Jiang * Sean Longbrake† Jie Ma ‡

Abstract

Given a family $\mathcal{F}$ of bipartite graphs, the *Zarankiewicz number* $z(m, n, \mathcal{F})$ is the maximum number of edges in an m by n bipartite graph G that does not contain any member of $\mathcal{F}$ as a subgraph (such G is called $\mathcal{F}$*-free*). For $1 \leq \beta < \alpha < 2$, a family $\mathcal{F}$ of bipartite graphs is (α, β)*-smooth* if for some $\rho > 0$ and every $m \leq n$, $z(m, n, \mathcal{F}) = \rho m n^{\alpha-1} + O(n^{\beta})$. Motivated by their work on a conjecture of Erdős and Simonovits on compactness and a classic result of Andrásfai, Erdős and Sós, in [1] Allen, Keevash, Sudakov and Verstraëte proved that for any (α, β)-smooth family $\mathcal{F}$, there exists k_0 such that for all odd $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\rho(\frac{2n}{5} + o(n))^{\alpha-1}$ is bipartite.

In this paper, we strengthen their result by showing that for every real $\delta > 0$, there exists k_0 such that for all odd $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\delta n^{\alpha-1}$ is bipartite. Furthermore, our result holds under a more relaxed notion of smoothness, which include the families $\mathcal{F}$ consisting of the single graph $K_{s,t}$ when $t \gg s$. We also prove an analogous result for $C_{2\ell}$-free graphs for every $\ell \geq 2$, which complements a result of Keevash, Sudakov and Verstraëte in [20]. We will discuss the relations between our results and the conjecture of Erdős and Simonovits on compactness in the concluding remarks.

1 Introduction

Given a family $\mathcal{F}$ of graphs, a graph G is called $\mathcal{F}$*-free* if G does not contain any member of $\mathcal{F}$ as a subgraph. If $\mathcal{F}$ consists of a single graph F then we simply say that G is F*-free*. The *Turán number* of $\mathcal{F}$, denoted by $\mathrm{ex}(n, \mathcal{F})$, is the maximum possible number of edges in an n-vertex $\mathcal{F}$-free graph. As is well known, this function is well-understood when $\mathcal{F}$ consists only of non-bipartite graphs due to the celebrated Erdős-Stone-Simonovits theorem [10, 12] but is generally open when $\mathcal{F}$ contains bipartite graphs. For a family of graphs $\mathcal{F}$, a closely related notion is the so-called

*Department of Mathematics, Miami University, Oxford, OH 45056, USA. Email: `jiangt@miamioh.edu`. Research supported by National Science Foundation grant DMS-1855542.

†Department of Mathematics, Miami University, Oxford, OH 45056, USA. Email: `longbrsa@miamioh.edu`. Research supported by National Science Foundation grant DMS-1855542.

‡School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China. Email: `jiema@ustc.edu.cn`. Research supported by the National Key R and D Program of China 2020YFA0713100, National Natural Science Foundation of China grant 12125106, and Anhui Initiative in Quantum Information Technologies grant AHY150200.

2010 Mathematics Subject Classifications: 05C35, 05C38. *Key Words*: Turán numbers, cycles, bipartite graphs

Zarankiewicz number $z(n, \mathcal{F})$, which is defined to be the maximum number of edges in an n-vertex $\mathcal{F}$-free bipartite graph. More generally, we denote by $z(m, n, \mathcal{F})$ the maximum number of edges in an m by n bipartite graph that is $\mathcal{F}$-free. In a seminal paper [11], Erdős and Simonovits raised a number of intriguing conjectures on Turán numbers for bipartite graphs. One of them is the following (Conjecture 3 in [11]). Given a positive odd integer k, let $\mathcal{C}_k$ denote the family of all odd cycles of length at most k. Throughout this paper, we write $f(n) \sim g(n)$ for two functions $f, g : \mathbb{N} \to \mathbb{R}$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.

Conjecture 1.1 (Erdős-Simonovits [11]). *Given any finite family $\mathcal{F}$ of graphs, there exists an odd integer k such that as $n \to \infty$*

$$\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim z(n, \mathcal{F}).$$

Erdős and Simonovits [11] verified the conjecture for $\mathcal{F} = \{C_4\}$ by showing that $\mathrm{ex}(n, \{C_4, C_5\}) \sim z(n, C_4) \sim (\frac{n}{2})^{\frac{3}{2}}$. Keevash, Sudakov and Verstraëte [20] further confirmed this conjecture for $\mathcal{F}_\ell := \{C_4, C_6, \ldots, C_{2\ell}\}$ where $\ell \in \{2, 3, 5\}$ in stronger forms and proved a related result for the chromatic number of $\mathcal{F}_\ell \cup \{C_k\}$-free graphs of minimum degree $\Omega(n^{1/\ell})$. In a subsequent paper [1], Allen, Keevash, Sudakov and Verstraëte provided a general approach to Conjecture 1.1 (using Scott's sparse regularity lemma [26]), which works for the following families of bipartite graphs.

Definition 1.2. Let α, β be reals with $2 > \alpha > \beta \geq 1$. Let $\mathcal{F}$ be a family of bipartite graphs. If there exists some $\rho > 0$ such that for every $m \leq n$,

$$z(m, n, \mathcal{F}) = \rho m n^{\alpha-1} + O(n^\beta)$$

holds, then we say that $\mathcal{F}$ is (α, β)*-smooth* with relative density ρ. We call a bipartite family $\mathcal{F}$ *smooth* if it is (α, β)-smooth for some α and β.

It is evident to note that for any (α, β)-smooth family $\mathcal{F}$, we have $z(n, \mathcal{F}) = \rho(n/2)^\alpha + O(n^\beta)$.

Before we mention the results of [1], let us discuss some known examples of smooth families. Improving results of Kövári-Sós-Turán [23], Füredi [14] showed that if $m \leq n$ and $s, t \in \mathbb{N}$ then

$$z(m, n, K_{s,t}) \leq (t - s + 1)^{1/s} m n^{1-1/s} + sm + sn^{2-2/s}. \tag{1}$$

This together with the constructions of Brown [6] and Füredi [15] shows that $K_{2,t}$ and $K_{3,3}$ are smooth families (see [1]). Allen et al. [1] also showed that $\{K_{2,t}, B_t\}$ is smooth, where B_t consists of t copies of C_4 sharing an edge (and no other vertices). However, it is not known if $K_{s,t}$ is smooth for any $s \geq 3$ and $t \geq 4$ and if $C_{2\ell}$ is smooth for any $\ell \geq 3$, due to a lack of constructions that asymptotically match upper bounds on Zarankiewicz numbers. We would like to point out that not all families of bipartite graphs are smooth – in the concluding remarks we provide an example of bipartite graphs which are not smooth.

The main result of Allen et al. [1] is as follows. A family $\mathcal{G}$ of graphs is *near-bipartite* if every graph $G \in \mathcal{G}$ has a bipartite subgraph H such that $e(G) \sim e(H)$ as $|V(G)| \to \infty$.

Theorem 1.3 (Allen-Keevash-Sudakov-Verstraëte [1]). *Let $\mathcal{F}$ be an (α, β)-smooth family with $2 > \alpha > \beta \geq 1$. There exists k_0 such that if $k \geq k_0 \in \mathbb{N}$ is odd, then the family of all extremal $\mathcal{F} \cup \{C_k\}$-free graphs is near-bipartite and, in particular,* $\mathrm{ex}(n, \mathcal{F} \cup \{C_k\}) \sim z(n, \mathcal{F})$.

Allen et al. [1] also raised a question whether the extremal n-vertex $\mathcal{F} \cup \{C_k\}$-free graph in Theorem 1.3 is exactly bipartite when n is sufficiently large. Motivated by the classic result of Andrásfai, Erdős and Sós [4] stating that any n-vertex triangle-free graph with minimum degree more than $2n/5$ must be bipartite, Allen et al. [1] proved the following theorem, which answers their question for extremal graphs satisfying appropriate minimum degree condition.

Theorem 1.4 (Allen-Keevash-Sudakov-Verstraëte [1]). *Let $\mathcal{F}$ be an (α, β)-smooth family with relative density ρ and $2 > \alpha > \beta \geq 1$. Then there exists k_0 such that for any odd $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\rho(\frac{2n}{5} + o(n))^{\alpha-1}$ is bipartite.*

In this paper, we strengthen Theorem 1.4 by showing that the minimum degree condition can be lowered to $\delta n^{\alpha-1}$ for any given real $\delta > 0$ and furthermore, the condition on smoothness can be relaxed to the following notion.

Definition 1.5. Let α, β be reals with $2 > \alpha > \beta \geq 1$. Let $\mathcal{F}$ be a family of bipartite graphs. We say that $\mathcal{F}$ is (α, β)-*quasi-smooth* with upper density ρ and lower density ρ_0, if there exist constants $\rho, \rho_0 > 0$ and C such that for all positive integers $m \leq n$,

$$z(m, n, \mathcal{F}) \leq \rho m n^{\alpha-1} + C n^{\beta} \text{ and } \mathrm{ex}(n, \mathcal{F}) \geq \rho_0 n^{\alpha}.$$

If $\mathcal{F}$ consists of a single graph F, then we just say that F is (α, β)-quasi-smooth.

The collection of (α, β)-quasi-smooth families is conceivably broader than the one of (α, β)-smooth families. For instance, it is proved that $\mathrm{ex}(n, K_{s,t}) = \Omega(n^{2-1/s})$ for $t \geq (s-1)! + 1$ in [2, 21] and for $t \geq C^s$ in a very recent paper of Bukh [7] (where C is a constant). Hence in view of (1), we know that $K_{s,t}$ is quasi-smooth under these conditions. The following is our main result in this paper.

Theorem 1.6. *Let $\mathcal{F}$ be an (α, β)-quasi-smooth family with $2 > \alpha > \beta \geq 1$. For any real $\delta > 0$, there exists a positive integer k_0 such that for any odd integer $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\delta n^{\alpha-1}$ is bipartite.*

The proof of Theorem 1.6 uses expansion properties and a robust reachability lemma that is in part inspired by a lemma in a recent paper by Letzter [24] on the Turán number of tight cycles. Theorem 1.6 also yields a strengthening of Theorem 1.3, which we will discuss in the concluding remarks (Theorem 5.3).

We also prove an analogous result for $C_{2\ell}$-free graphs, which complements the following result in Keevash-Sudakov-Verstraëte [20]: For any integer $\ell \geq 2$, odd integer $k \geq 4\ell + 1$ and any real $\delta > 0$, the chromatic number of any n-vertex $\{C_4, C_6, ..., C_{2\ell}, C_k\}$-free graph with minimum degree at least $\delta n^{1/\ell}$ is less than $(4k)^{\ell+1}/\delta^{\ell}$.

Theorem 1.7. *Let $\ell \geq 2$ be an integer. For any real $\delta > 0$, let $k_0 = 3\ell(8\ell/\delta)^{\ell} + 2\ell + 2$. Then for any odd integer $k \geq k_0$ and sufficiently large n, any n-vertex $\{C_{2\ell}, C_k\}$-free graph with minimum degree at least $\delta n^{1/\ell}$ is bipartite.*

This proof follows the same line as that of Theorem 1.6, except that we will use a more efficient robust reachability lemma for $C_{2\ell}$-free graphs and as a result get better control on k_0.

We should point out that the existence of such graphs in Theorem 1.7 is known only for $\ell \in \{2,3,5\}$ (see [16]). Also note that this result is not covered by Theorem 1.6, since $C_{2\ell}$ is not known to be (α,β)-quasi-smooth for any $\ell \geq 3$. In the concluding remarks, we will mention that Theorems 1.6 and 1.7 can be extended to a slightly broader family of bipartite graphs that include both (α,β)-quasi-smooth graphs and $C_{2\ell}$'s.

The rest of the paper is organized as follows. In Section 2, we develop some useful lemmas. In Section 3, we develop a lemma for $C_{2\ell}$-free graphs. In Section 4, we prove Theorem 1.6 and Theorem 1.7, respectively. In Section 5, we give some concluding remarks. Throughout this paper, we denote $[k]$ by the set $\{1,2,...,k\}$ for positive integers k.

2 Some general lemmas

The main content of this section is to present key lemmas for our main result Theorem 1.6.

Definition 2.1. Let α,β be reals with $2>\alpha>\beta\geq 1$. Let $\ell_0(\alpha,\beta)$ be defined as follows:

$$\ell_0 = \left\lfloor \log_\beta \frac{(2-\beta)(\alpha-1)}{\alpha-\beta} \right\rfloor + 2, \text{ for } \beta>1 \quad \text{ and } \quad \ell_0 = \lfloor 1/(\alpha-1)\rfloor + 1, \text{ for } \beta = 1.$$

Lemma 2.2. *Let α,β be reals with $2>\alpha>\beta\geq 1$ and $\ell_0=\ell_0(\alpha,\beta)$ be defined as in Definition 2.1. Let $\mathcal{F}$ be an (α,β)-quasi-smooth family of bipartite graphs that satisfies $z(m,n,\mathcal{F}) \leq \rho m n^{\alpha-1} + Cn^\beta$ for all $m\leq n$. For any $\delta>0$, there exists a positive real $\mu=\mu(\mathcal{F},\delta)$ such that for all sufficiently large n the following is true. Let G be an $\mathcal{F}$-free bipartite graph with at most n vertices and minimum degree at least $\delta n^{\alpha-1}$. Let $u\in V(G)$. For each $i\in\mathbb{N}$, let $N_i(u)$ denote the set of vertices at distance i from u. Then for some $j_0\leq \ell_0$ we have $\min\{|N_{j_0}(u)|,|N_{j_0+1}(u)|\}\geq \mu n$.*

Proof. For each $i\in\mathbb{N}$, let B_i denote the set of vertices at distance at most i from u. Let

$$\gamma = (\delta/12\rho)^{1/(\alpha-1)} \text{ and } \mu = \min\left\{(1/2)(\delta/2\rho)^{1/(\alpha-1)}, (\delta/4\rho)\gamma^{2-\alpha}, \gamma/\ell_0\right\}. \tag{2}$$

First, we show that $|B_{\ell_0}|\geq \gamma n$. Suppose for a contradiction that $|B_{\ell_0}|<\gamma n$. Let $i\in[\ell_0-1]$. Then clearly $|B_i|<\gamma n$, and since G has minimum degree at least $\delta n^{\alpha-1}$, we have

$$\sum_{v\in B_i} d(v) \geq \delta n^{\alpha-1}|B_i|. \tag{3}$$

On the other hand, $\sum_{v\in B_i} d(v) = 2e(B_i)+e(B_i,B_{i+1}\setminus B_i)$. Since G is bipartite and $\mathcal{F}$-free, $e(B_i)\leq \max_{(a,b)}\{\rho a b^{\alpha-1}+Cb^\beta\}$ over all pairs of positive integers $a\leq b$ with $a+b=|B_i|$. Hence, $e(B_i)\leq \rho|B_i|^\alpha + C|B_i|^\beta \leq 2\rho|B_i|^\alpha$, when n is sufficiently large. With some generosity, we can upper bound $e(B_i,B_{i+1}\setminus B_i)$ by $z(|B_i|,|B_{i+1}|,\mathcal{F})$ to get $e(B_i,B_{i+1}\setminus B_i)\leq \rho|B_i||B_{i+1}|^{\alpha-1}+C|B_{i+1}|^\beta$. Putting the above estimations all together, we get

$$\sum_{v\in B_i} d(v) \leq 4\rho|B_i|^\alpha + \rho|B_i||B_{i+1}|^{\alpha-1} + C|B_{i+1}|^\beta. \tag{4}$$

Combining (3) and (4), we get

$$\delta n^{\alpha-1}|B_i| \le \sum_{v\in B_i} d(v) \le 4\rho|B_i|^\alpha + \rho|B_i||B_{i+1}|^{\alpha-1} + C|B_{i+1}|^\beta. \tag{5}$$

If the first term on the right-hand side of (5) is the largest term, then we get $\delta n^{\alpha-1}|B_i| \le 12\rho|B_i|^\alpha$, from which we get $|B_i| \ge (\delta/12\rho)^{1/(\alpha-1)}n \ge \gamma n$, contradicting our assumption. If the second term on the right-hand side of (5) is the largest term, then we get $\delta n^{\alpha-1}|B_i| \le 3\rho|B_i||B_{i+1}|^{\alpha-1}$, from which we get $|B_{i+1}| \ge (\delta/3\rho)^{1/(\alpha-1)}n \ge \gamma n$ and hence $|B_{\ell_0}| \ge |B_{i+1}| \ge \gamma n$, contradicting our assumption. Hence we may assume that for each $i \in [\ell_0 - 1]$, we have

$$\delta n^{\alpha-1}|B_i| \le 3C|B_{i+1}|^\beta,$$

which yields that for each $i \in [\ell_0 - 1]$,

$$|B_{i+1}| \ge (\delta/3C)^{1/\beta} n^{(\alpha-1)/\beta}|B_i|^{1/\beta}. \tag{6}$$

Let $\{b_i\}$ be a sequence recursively defined by letting $b_1 = \alpha - 1$ and $b_{i+1} = (1/\beta)b_i + (\alpha-1)/\beta$ for each $i \ge 1$. If $\beta = 1$ then a closed form formula for b_i is $b_i = (\alpha-1)i$. If $\beta > 1$ then a closed form formula for b_i is $b_i = \frac{\alpha-1}{\beta-1} + (\frac{1}{\beta})^{i-1}(\alpha - 1 - \frac{\alpha-1}{\beta-1})$. Note that we may assume $C \ge 1$, so $|B_1| \ge \delta n^{\alpha-1} \ge (\delta/3C)n^{b_1}$. Then it follows by (6) and induction that $|B_i| \ge (\delta/3C)^i n^{b_i}$ for each $i \in [\ell_0]$. However, using the definition of ℓ_0 we get $b_{\ell_0} > 1$, which yield $|B_{\ell_0}| > n$ as n is sufficiently large. This is a contradiction and thus proves that $|B_{\ell_0}| \ge \gamma n$.

Let $j \in [\ell_0]$ be the smallest index such that $|N_j| \ge (\gamma/\ell_0)n$. By the pigeonhole principle, such j exists. By (2), $|N_j| \ge \mu n$. Let $U = N_j$ and $V = N_{j-1} \cup N_{j+1}$. We show that $|V| \ge 2\mu n$, from which it follows that either $|N_{j-1}| \ge \mu n$ or $|N_{j+1}| \ge \mu n$ and thus the lemma holds with $j_0 = j$ or $j_0 = j-1$. Since all the edges of G that are incident to U are between U and V, we have $e(G[U,V]) \ge \delta n^{\alpha-1}|U|$. On the other hand, $G[U,V]$ is $\mathcal{F}$-free. If $|V| \ge |U|$, then we have $e(G[U,V]) \le \rho|U||V|^{\alpha-1} + C|V|^\beta \le 2\rho|U||V|^{\alpha-1}$, where the last inequality holds as $|U| = |N_j| \ge \mu n$ and n is sufficiently large. Combining the two inequalities and solving for $|V|$, we get $|V| \ge (\delta/2\rho)^{1/(\alpha-1)}n \ge 2\mu n$, as desired. Otherwise, we have $|V| \le |U|$. Then $\delta n^{\alpha-1}|U| \le e(G[U,V]) \le \rho|V||U|^{\alpha-1} + C|U|^\beta$. Since $C|U|^\beta \ll \delta n^{\alpha-1}|U|$ for sufficiently large n, we can derive from the above that $\delta n^{\alpha-1}|U| \le 2\rho|V||U|^{\alpha-1}$. Solving for $|V|$, we have $|V| \ge (\delta/2\rho)n^{\alpha-1}|U|^{2-\alpha} \ge (\delta/2\rho)n^{\alpha-1}(\gamma n)^{2-\alpha} = (\delta/2\rho)\gamma^{2-\alpha}n \ge 2\mu n$, where the last inequality holds by (2), as desired. □

The following lemma, which we call *robust reachability lemma* is key to our proof of the main results. It is inspired by a lemma used in a recent paper of Letzter [24] on the Turán number of tight cycles in hypergraphs.

Lemma 2.3. *Let $\mathcal{F}$ be an (α,β)-quasi-smooth family of bipartite graphs with $2 > \alpha > \beta \ge 1$. Let $\ell_0 = \ell_0(\alpha,\beta)$ be defined as in Definition 2.1. For any real $\delta > 0$, the following holds for all sufficiently large n. Let G be an $\mathcal{F}$-free bipartite graph with at most n verticers and minimum degree at least $\delta n^{\alpha-1}$. Let $u \in V(G)$. Then there exists a set S of at least $\mu(\mathcal{F}, \delta/2)n$ vertices, and a family $\mathcal{P} = \{P_v : v \in S\}$, where for each $v \in S$, P_v is a u,v-path of length at most ℓ_0, such that no vertex except u is used on more than $\frac{n}{\log n}$ of the paths in $\mathcal{P}$.*

Proof. Let S be a maximum set of vertices such that there is an associated family $\mathcal{P} = \{P_v : v \in S\}$, where for each $v \in S$, P_v is a path of length at most ℓ_0 such that no vertex is on more than $\frac{n}{\log n}$ of the paths. Let W denote the set of vertices in G (other than u) that lie on exactly $\frac{n}{\log n}$ of the paths P_v in $\mathcal{P}$. Then $|W|\frac{n}{\log n} \leq |S|\ell_0 \leq n\ell_0$ and thus $|W| \leq \ell_0 \log n < (\delta/2)n^{\alpha-1}$ for sufficiently large n. Hence, $G - W$ has minimum degree at least $(\delta/2)n^{\alpha-1}$. If $|S| < \mu(\mathcal{F}, \delta/2)n$, then by Lemma 2.2, there exists a vertex $z \notin S$ and a u,z-path P_z of length at most ℓ_0 in $G - W$ that we can add to S to contradict our choice of S. Hence $|S| \geq \mu(\mathcal{F}, \delta/2)n$. □

The next folklore lemma will be used a few times and we include a proof for completeness. We would like to mention that it might be easy for one to overlook the connectedness of H statement in the conclusion. But this condition will play important role in the main proofs.

Lemma 2.4. *Let G be a connected graph. Let H be a maximum spanning bipartite subgraph of G. Then H is connected and for each $v \in V(G), d_H(v) \geq (1/2)d_G(v)$.*

Proof. Let (X, Y) denote a bipartition of H. Suppose for contradiction that H is disconnected and F is a component of H. Since G is connected, it contains an edge e joining $V(F)$ to $V(G) \setminus V(F)$. But then $H \cup e$ is still bipartite, since adding e does not create a new cycle. Furthermore, $H \cup e$ has more edges than H, contradicting our choice of H.

Next, let v be any vertex in H. Without loss of generality, suppose $v \in X$. Suppose $d_H(v) < (1/2)d_G(v)$. Then from H by deleting the edges incident to x and adding the edges in G from v to X, we obtained a bipartite subgraph of G that has more edges than H, a contradiction. Hence $\forall v \in V(G), d_H(v) \geq (1/2)d_G(v)$. □

We conclude this section with the following lemma about the diameter. The *diameter* of a graph G is the least integer k such that there exists a path of length at most k between any two vertices in G.

Lemma 2.5. *Let G be an n-vertex connected graph with minimum degree at least D. Then G has diameter at most $3n/D$.*

Proof. Let x, y be two vertices at maximum distance in G. Let $v_0v_1 \cdots v_\ell$ be a shortest x,y-path in G where $v_0 = x$ and $v_\ell = y$. Let $q = \lfloor \ell/3 \rfloor$. Note that $N(v_0), N(v_3), N(v_6), \cdots, N(v_{3q})$ are pairwise disjoint (or else we can find a shorter x,y-path, a contradiction). Hence $n \geq \sum_{i=0}^{q} |N(v_{3i})| \geq (q+1)D$. This implies that $(q+1) \leq n/D$ and hence $\ell \leq 3(q+1) \leq 3n/D$. □

3 An efficient robust reachability lemma for $C_{2\ell}$-free graphs

In this section, we develop a more efficient robust reachability lemma than Lemma 2.3 for $C_{2\ell}$-free graphs, which may be of independent interest. We need the following lemma from [27].

Lemma 3.1 (Verstraëte [27])**.** *Let $\ell \geq 2$ be an integer and H a bipartite graph of average degree at least 4ℓ and girth g. Then there exist cycles of at least $(g/2-1)\ell \geq \ell$ consecutive even lengths in H. Moreover, the shortest of these cycles has length at most twice the radius of H.*

Our lemma is as follows.

Lemma 3.2. *Let $\ell \geq 2$ and d be positive integers. Let H be a bipartite $C_{2\ell}$-free graph with minimum degree at least d. Let u be any vertex in H. Then the following items hold.*

(1). The number of vertices that are at distance at most ℓ from u is at least $(d/4\ell)^\ell$.

(2). Suppose H has at most n vertices and $d \geq 15\ell \log n$, where n is sufficiently large. Then there is a set S of at least $(1/2)(d/8\ell^2)^\ell$ vertices together with a family $\mathcal{P} = \{P_v : v \in S\}$, where for each $v \in S$, P_v is a u,v-path of exactly length ℓ, such that no vertex of H except u lies on more than $d^{\ell-1}$ of these paths and each vertex v in S lies only on P_v.

Proof. First we prove the first part (1) of the theorem. Let $B_0 = \{u\}$. Consider any $i \in [\ell]$. Let B_i denote the set of vertices at distance at most i from u in H and H_i the subgraph of H induced by B_i. If $H[B_i]$ has average degree at least 4ℓ, then by Lemma 3.1, G_i contains cycles of ℓ consecutive even lengths the shortest of which has length at most $2i \leq 2\ell$ and hence it contains $C_{2\ell}$, contradicting G being $C_{2\ell}$-free. So for each $i \in [\ell]$, we have $d(H_i) < 4\ell$, which implies that $e(H_i) < 2\ell|B_i|$. On the other hand, H_i contains all the edges of G that are incident to B_{i-1}. So $e(H_i) \geq d|B_{i-1}|/2$. Combining these two inequalities, we get $2\ell|B_i| > d|B_{i-1}|/2$. Hence, $|B_i| > (d/4\ell)|B_{i-1}|$ for each $i \in [\ell]$. Thus, $|B_\ell| \geq (d/4\ell)^\ell$, as desired.

Next, we prove the second part (2). Let us randomly split the vertices of G into ℓ parts $V_1, \ldots, V_\ell$. For each vertex x, and each $i \in [\ell]$, the degree $d_i(x)$ of x in V_i has a binomial distribution $\mathrm{Bin}(d(x), 1/\ell)$. Hence, using Chernoff's inequality (see [3] or [17] Corollary 2.3), we have

$$\mathbb{P}[d_i(x) < (1/2\ell)d(x)] \leq \mathbb{P}[|d_i(x) - (1/\ell)d(x)| > (1/2\ell)d(x)] \leq 2e^{-d(x)/12\ell} = o(n^{-1}),$$

since $d(x) \geq 15\ell \log n$. Hence, for sufficiently large n, there exists a splitting of $V(H)$ such that for each $x \in V(H)$ and for each $i \in [\ell]$, $d_i(x) \geq (1/2\ell)d(x) \geq d/2\ell$. Now, we form a subgraph H' of H as follows. First, we include exactly $d/2\ell$ of the edges from u to V_1. Denote the set of reached vertices in V_1 by S_1. Then for each vertex in S_1 including exactly $d/2\ell$ edges from it to V_2. Denote the set of reached vertices in V_2 by S_2. We continue like this till we define S_ℓ. Let $B_0 = S_0 = \{u\}$. For each $i \in [\ell]$, let $B_i = \bigcup_{j=0}^{i} S_i$. and H_i the subgraph of H' induced by B_i. Note that H_i has radius i. As in the proof of the first part of the lemma, since H_i is $C_{2\ell}$-free, $e(H_i) < 2\ell|B_i|$. On the other hand, H_i contains all the edges of H' that are incident to B_{i-1}. So $e(H_i) \geq (1/2)(d/2\ell)|B_{i-1}|$. Combining these two inequalities, we get $2\ell|B_i| > (d/4\ell)|B_{i-1}|$. Hence, $|B_i| > (d/8\ell^2)|B_{i-1}|$ for each $i \in [\ell]$. Thus, $|B_\ell| \geq (d/8\ell^2)^\ell$.

It is easy to see that $\sum_{i=0}^{\ell-1} |S_i| \leq \sum_{i=0}^{\ell-1} (d/2\ell)^i \leq 2(d/2\ell)^{\ell-1}$, when n is sufficiently large. Hence

$$|S_\ell| = |B_\ell \backslash \cup_{i=0}^{\ell-1} S_i| \geq (d/8\ell^2)^\ell - 2(d/2\ell)^{\ell-1} > (1/2)(d/8\ell^2)^\ell,$$

where n (and thus d) is sufficiently large. By the definition of H', for each $v \in S_\ell$, there is a path of length ℓ from u to v that intersects each of $V_1, V_2, \ldots, V_\ell$. From the union of these paths one can find a tree T of height ℓ rooted at u, in which all the vertices in S_ℓ are at distance ℓ from u. Furthermore, by the definition of H', T has maximum degree at most $(d/2\ell) + 1$. For each $v \in S_\ell$, let P_v be the unique u,v-path in T. If x is any vertex in T other than u, then clearly x lies on at most $(d/2\ell)^{\ell-1}$ of the paths P_v. Furthermore, each $v \in S_\ell$ doesn't lie on any P_w for $w \in S_\ell \backslash \{v\}$. $\square$

4 Proof of Theorem 1.6 and Theorem 1.7

Even though the proofs of Theorem 1.6 and Theorem 1.7 are essentially the same, there are sufficiently different choices of parameters that we will prove them separately.

Proof Theorem 1.6: Let $\mathcal{F}$ be an (α, β)-quasi-smooth family with $2 > \alpha > \beta \geq 1$. Given any real $\delta > 0$, we first define k_0 as following. Let $\ell_0 = \ell_0(\alpha, \beta)$ and $\mu(\mathcal{F}, \delta/2)$ be defined as in Definition 2.1 and Lemma 2.2, respectively. Define

$$L := \left\lfloor \frac{3}{\mu(\mathcal{F}, \delta/2)} \right\rfloor \cdot \ell_0 \quad \text{and} \quad k_0 := 2\ell_0 + L + 2. \tag{7}$$

Let $k \geq k_0$ be odd. Let n be sufficiently large so that all subsequent inequalities involving n hold. Let G be an n-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\delta n^{\alpha-1}$. We may assume that G is connected. Let H be a maximum bipartite spanning subgraph of G. By Lemma 2.4, H is connected and has minimum degree at least $(\delta/2)n^{\alpha-1}$.

Since H is $\mathcal{F}$-free, by Lemma 2.2, for each vertex x, the set of vertices that are at distance at most ℓ_0 from x is at least $\mu(\mathcal{F}, \delta/2)n$. Hence by Lemma 2.5 (applied to the ℓ_0-th power H^{ℓ_0} of H), H^{ℓ_0} has diameter at most $\lfloor \frac{3n}{\mu(\mathcal{F},\delta/2)n} \rfloor = \lfloor \frac{3}{\mu(\mathcal{F},\delta/2)} \rfloor$ and hence H has diameter at most $\lfloor \frac{3}{\mu(\mathcal{F},\delta/2)} \rfloor \cdot \ell_0 = L$, as defined in (7).

Let (X, Y) be the unique bipartition of H. We show that G is also bipartite with (X, Y) being a bipartition of it. Suppose otherwise. We may assume, without loss of generality, that there exist two vertices $u, v \in X$ such that $uv \in E(G)$. We will derive a contradiction by finding a copy of C_k in G that contains uv.

Let us randomly split $V(H)$ into two subsets A, B. For each vertex x of degree $d(x)$ in H, let $d_A(x)$ and $d_B(x)$ denote the degree of x in A and B, respectively. Then both $d_A(x)$ and $d_B(x)$ satisfy the binomial distribution $\mathrm{Bin}(d(x), 1/2)$. Hence, by Chernoff's inequality, we have

$$\mathbb{P}(d_A(x) < (1/4)d(x)) \leq \mathbb{P}(|d_A(x) - d(x)/2| \geq d(x)/4) \leq 2e^{-d(x)/24} = o(n^{-1}),$$

since $d(x) \geq (\delta/2)n^{\alpha-1}$ and n is sufficiently large. Hence with positive probability we can ensure that for any $x \in V(H)$, $\min\{d_A(x), d_B(x)\} \geq (1/4)d(x) \geq (\delta/8)n^{\alpha-1}$. Let us fix such a partition A, B of $V(H)$.

Without loss of generality, suppose that the vertex u is in A. Let $H[A]$ denote the subgraph of H induced by A. By our discussion above, $H[A]$ has minimum degree at least $(\delta/8)n^{\alpha-1}$. By Lemma 2.3, there exists a set U of at least $\mu(\mathcal{F}, \delta/16)n$ vertices and a family $\mathcal{P} = \{P_z : z \in U\}$, where for each $z \in U$, P_z is a u,z-path in $H[A]$ of length at most ℓ_0. By the pigeonhole principle, there exists a value $p \in [\ell_0]$ and a subset $S \subseteq U$ of size

$$|S| \geq |U|/\ell_0 \geq (\mu(\mathcal{F}, \delta/16)/\ell_0)n$$

such that for each $z \in S$, P_z has length p and no vertex in H lies on more than $n/\log n$ of these paths P_z. Now, let us randomly color the vertices in $H[A]$ with colors 1 and 2. For each $z \in S$, the path P_z is *good* if z is colored 2 and the other p vertices on P_z are colored 1. The probability that P_z is good is $1/2^{p+1}$. Hence, for some coloring, there are at least $|S|/2^{p+1}$ good paths. Let S' denote the subset of vertices $z \in S$ such that P_z is good and let $\mathcal{P}' = \{P_z : z \in S'\}$. By our discussion,

$$|\mathcal{P}'| = |S'| \geq \mu(\mathcal{F}, \delta/16)/(2^{\ell_0+1}\ell_0).$$

Note that by our definition of $\mathcal{P}'$, no vertex in S' is used as an internal vertex of any path in $\mathcal{P}'$. Let H' denote the bipartite subgraph of H induced by the two parts S' and B. By our earlier discussion, each vertex in S' has at least $(\delta/8)n^{\alpha-1}$ neighbors in B. Hence,

$$e(H') \geq |S'|(\delta/8)n^{\alpha-1} \geq \left(\mu(\mathcal{F}, \delta/16)\delta/(2^{\ell_0+4}\ell_0)\right) n^{\alpha} = \gamma n^{\alpha},$$

where $\gamma := \gamma(\mathcal{F}, \delta) = \mu(\mathcal{F}, \delta/16)\delta/(2^{\ell_0+4}\ell_0)$. So H' has average degree at least $2\gamma n^{\alpha-1}$. By a well-known fact, H' contains a subgraph H'' of minimum degree at least $\gamma n^{\alpha-1}$. Let $S'' = V(H'') \cap S'$.

Let Q be a shortest path in H from the vertex v to $V(H'')$. Let y be the endpoint of Q in $V(H'')$. By our choice of Q, y is the only vertex in $V(Q) \cap V(H'')$ (note that it is possible that $y = v$). Let q denote the length of Q. Since H has diameter at most L, we have $q \leq L$.

By Lemma 2.2, for some $j_0 \leq \ell_0$, inside the graph H'' we have $\min\{|N_{j_0}(y)|, |N_{j_0+1}(y)|\} \geq \mu(\mathcal{F}, \gamma)n$. Note that one of $N_{j_0}(y)$ and $N_{j_0+1}(y)$ lies completely inside $V(H'') \cap A$. Denote this set by W. Let

$$W_0 = \{w \in W : P_w \cap V(Q) \neq \emptyset\}.$$

Since Q contains at most $L+1$ vertices, each of which lies on at most $n/\log n$ of the members of $\mathcal{P}$, we see $|W_0| \leq (L+1)(n/\log n)$. Since n is sufficiently large, we have

$$|W \setminus W_0| \geq \mu(\mathcal{F}, \gamma)n - (L+1)n/\log n \geq (1/2)\mu(\mathcal{F}, \gamma)n.$$

Let $H[W \setminus W_0, B]$ denote the subgraph of H consisting of edges that have one endpoint in $W \setminus W_0$ and the other endpoint in B. Since each vertex in $W \setminus W_0$ has at least $(\delta/8)n^{\alpha-1}$ neighbors in B,

$$e(H[W \setminus W_0, B]) \geq |W \setminus W_0|(\delta/8)n^{\alpha-1} \geq (1/16)\mu(\mathcal{F}, \gamma)\delta n^{\alpha}.$$

Hence $H[W \setminus W_0, B]$ contains a subgraph H^* with minimum degree at least $(1/16)\mu(\mathcal{F}, \gamma)\delta n^{\alpha-1} \geq k$, for sufficiently large n. Let w be any vertex in $V(H^*) \cap (W \setminus W_0)$. By our definition of W, there is a path R_w in H'' of length $r \leq j_0 + 1 \leq \ell_0 + 1$ from y to w. Let $t = k - 1 - q - r - p$. Since $q \leq L, p \leq \ell_0, r \leq \ell_0 + 1$ and $k \geq k_0 = 2\ell_0 + L + 2$, we get $t \geq 0$. Since there is a path in H of length p from w to u (by the definition of $S'' \subseteq W$) and $R_w \cup Q$ is a path in H of length $q + r$ from w to v and $u, v \in X$, p and $q + r$ have the same parity. Since k is odd, we see that t is even. Since H^* has minimum (much) larger than k, greedily we can build a path T of length t in H^* from w to some vertex w^* in $V(H^*) \cap (W \setminus W_0)$ such that T intersects $Q \cup R_w$ only in w. Now, let

$$C := uv \cup Q \cup R_w \cup T \cup P_{w^*}$$

By our definitions, $Q \cup R_w \cup T$ is a path. Also, since $w^* \in W \setminus W_0$, P_{w^*} is vertex disjoint from Q. Finally, by our definition of $\mathcal{P}'$, $V(P_{w^*}) \setminus \{w^*\}$ is disjoint from S' and hence from S''. It is certainly also disjoint from B and hence is vertex disjoint from $R_w \cup T$. So, C is a cycle in G of length $1 + q + r + t + p = k$, a contradiction. □

Proof of Theorem 1.7: Let $\ell \geq 2$ be an integer. Let $\delta > 0$ be a real. Define

$$L := 3\ell(8\ell/\delta)^\ell \quad \text{and} \quad k_0 := 2\ell + L + 2. \tag{8}$$

Let $k \geq k_0$ be an odd integer. Let n be sufficiently large so that all subsequent inequalities involving n hold. Let G be an n-vertex $C_{2\ell}$-free graph with minimum degree at least $\delta n^{1/\ell}$. We may assume that G is connected. Let H be a maximum spanning subgraph of G. By Lemma 2.4, H is connected with minimum degree at least $(\delta/2)n^{1/\ell}$. Since H is $C_{2\ell}$-free, by Lemma 3.2, the ℓ-th power H^ℓ of H has minimum degree at least $(\delta n^{1/\ell}/8\ell)^\ell = (\delta/8\ell)^\ell n$. Hence, by Lemma 2.5, H^ℓ has diameter at most $3n/[(\delta/8\ell)^\ell n] = 3(8\ell/\delta)^\ell$. Therefore, H has diameter at most $3\ell(8\ell/\delta)^\ell = L$, as defined in (8).

Let (X, Y) the unique bipartition of H. We show that G is also bipartite with (X, Y) being a bipartition of it. Suppose otherwise. Then without loss of generality, we may assume that there exist two vertices $u, v \in X$ such that $uv \in E(G)$. We will derive a contradiction by finding a copy of C_k in G that contains uv. As in the proof of Theorem 1.6, we can split $V(H)$ into two subsets A, B such that for each vertex $x \in V(H)$, we have $d_A(x), d_B(x) \geq (1/4)d(x) \geq (\delta/8)n^{1/\ell}$.

Without loss of generality, suppose $u \in A$. Let $H[A]$ denote the subgraph of H induced by A. By our discussion above, $H[A]$ has minimum degree at least $(\delta/8)n^{1/\ell}$. By Lemma 3.2 (with $d = (\delta/8)n^{1/\ell}$), there exists a set S of size at least $(\delta/64\ell^2)^\ell(n/2)$ and a family $\mathcal{P} = \{P_z : z \in S\}$, where for each $z \in S$, P_z is a u,z-path in $H[A]$ of length ℓ, such that no vertex other than u in H lies on more than $(\delta n^{1/\ell}/8)^{\ell-1} = (\delta/8)^{\ell-1}n^{1-1/\ell}$ of these paths. Furthermore, for each $z \in S$, z lies only on P_z.

Let $H[S, B]$ denote the bipartite subgraph of H induced by the two parts S and B. By our earlier discussion, each vertex in S has at least $(\delta/8)n^{1/\ell}$ neighbors in B. Hence,

$$e(H) \geq |S|(\delta/8)n^{1/\ell} \geq (\delta^{\ell+1}/2^{6\ell+4}\ell^{2\ell})n^{1+1/\ell} = \gamma n^{1+1\ell},$$

where $\gamma := \delta^{\ell+1}/2^{6\ell+4}\ell^{2\ell}$. Then $H[S, B]$ has average degree at least $2\gamma n^{1/\ell}$ and thus contains a subgraph H' of minimum degree at least $\gamma n^{1/\ell}$. Let $S' = V(H') \cap S$ and $B' = V(H') \cap B$.

If ℓ is even, then $S' \subseteq X$ and let Q be a shortest path in H from v to S'. If ℓ is odd, then $B' \subseteq X$ and let Q be a shortest path in H from v to B'. Let y denote the endpoint of Q opposing v (it is possible that $y = v$). Let q denote the length of Q. So q is even and $y \in X$. In either case, it is easy to see that $q \leq L + 1$ and that $V(H')$ contains y and at most one other vertex of Q. Hence $H' - (V(Q) \setminus \{y\})$ has minimum degree at least $\gamma n^{1/\ell} - 1 \geq (\gamma/2)n^{1/\ell}$. By Lemma 3.2 (with $d = (\gamma/2)n^{1/\ell}$), inside the graph $H' - (V(Q) \setminus \{y\})$ there is a set W of size at least $(1/2)(\gamma/16\ell^2)^\ell n$ such that for each $w \in W$ there is a path R_w of length ℓ from y to w in $H' - (V(Q) \setminus \{y\})$. Furthermore, by our definition of Q, we can get $W \subseteq S'$. Recall the paths P_w in $\mathcal{P}$. Let

$$W_0 = \{w \in W : P_w \cap V(Q) \neq \emptyset\}.$$

Since Q contains at most $L+1$ vertices each of which lies on at most $(\delta/8)^{\ell-1}n^{1-1/\ell}$ of the members of $\mathcal{P}$, we see $|W_0| \leq (L+1)(\delta/8)^{\ell-1}n^{1-1/\ell}$. Since n is sufficiently large, we have

$$|W \setminus W_0| \geq (1/2)(\gamma/16\ell^2)^\ell n - (L+1)(\delta/8)^{\ell-1}n^{1-1/\ell} \geq (1/4)(\gamma/16\ell^2)^\ell n.$$

Since each vertex in $W \setminus W_0$ has at least $(\delta/8)n^\ell$ neighbors in B,

$$e(H[W \setminus W_0, B]) \geq |W \setminus W_0|(\delta/8)n^{1/\ell} \geq (\delta \cdot \gamma^\ell)/(2^{4\ell+5} \cdot \ell^{2\ell})n^{1+1/\ell}.$$

Hence $H[W \setminus W_0, B]$ contains a subgraph H^* with minimum degree at least $(\delta \cdot \gamma^\ell)/(2^{4\ell+5} \cdot \ell^{2\ell})n^{1/\ell} \geq k$, for sufficiently large n. Let w be any vertex in $V(H^*) \cap (W \setminus W_0)$. By our definition of W, there is a path R_w in $H' - (V(Q) \setminus \{y\})$ of length ℓ from y to w. Let $t = k - 1 - q - 2\ell$. Since $q \leq L + 1$ and $k \geq k_0 = 2\ell + L + 2$, we see $t \geq 0$. Since k is odd and q is even, we also see that t is even. Since H^* has minimum degree (much) larger than k, greedily we can build a path T of length t in H^* from w to some vertex w^* in $V(H^*) \cap (W \setminus W_0)$ such that T intersects $Q \cup R_w$ only in w. Now, let

$$C := uv \cup Q \cup R_w \cup T \cup P_{w^*}$$

By our definition of T, $Q \cup R_w \cup T$ is path. Also, since $w^* \in W \setminus W_0$, P_{w^*} is vertex disjoint from Q. Finally $P_{w^*} \setminus \{w^*\}$ does not contain any vertex of $S' \cup B$ and hence is vertex disjoint from $R_w \cup T$. Hence, C is a cycle in G of length $1 + q + 2\ell + t = k$, a contradiction. □

5 Concluding Remarks

1. Given integers $t, \ell \geq 2$, the *theta graph* $\theta_{t,\ell}$ is the graph consisting of t internally disjoint paths of length ℓ between two vertices. In particular, we have $\theta_{2,\ell} = C_{2\ell}$. Faudree and Simonovits [13] showed that for all $t, \ell \geq 2$, $\mathrm{ex}(n, \theta_{t,\ell}) = O(n^{1+1/\ell})$ (the case $t = 2$ was first proved by Bondy-Simonovits [5]). Conlon [9] showed that for each $\ell \geq 2$, there exists a t_0 such that for all $t \geq t_0$, $\mathrm{ex}(n, \theta_{t,\ell}) = \Omega(n^{1+1/\ell})$, the leading coefficients of which were further improved by Bukh-Tait [8]. Jiang, Ma and Yepremyan [19] showed that for all t, ℓ, there exists a constant $c = c(t, \ell)$ such that for all $m \leq n$

$$z(m, n, \theta_{t,\ell}) \leq \begin{cases} c \cdot [(mn)^{\frac{\ell+1}{2\ell}} + m + n] & \text{if } \ell \text{ is odd,} \\ c \cdot [m^{\frac{1}{2}+\frac{1}{\ell}} n^{\frac{1}{2}} + m + n] & \text{if } \ell \text{ is even.} \end{cases}$$

   For the case $t = 2$, the above bound was first proved by Naor-Verstraëte [25], and a different form of the upper bound on $z(m, n, \theta_{2,\ell})$ was obtained by Jiang-Ma [18]. On the other hand, using first moment deletion method it is not hard to show that

   Proposition 5.1. *Let $\varepsilon > 0$ be any real. Let $\ell \geq 2$. There exists a t_0 such that for all $t \geq t_0$, if ℓ is odd then*

$$z(m, n, \theta_{t,\ell}) \geq \Omega(m^{\frac{\ell+1}{2\ell}-\varepsilon} n^{\frac{\ell+1}{2\ell}-\varepsilon}),$$

   and if ℓ is even then

$$z(m, n, \theta_{t,\ell}) \geq \Omega(m^{\frac{1}{2}+\frac{1}{\ell}-\varepsilon} n^{\frac{1}{2}-\varepsilon}).$$

   Proof. Consider the bipartite random graph $G \in G(m, n, p)$ with p to be chosen later. Let $q = \lfloor \ell/2 \rfloor$. Let $X = e(G)$ and Y denote the number of copies of $\theta_{t,k}$ in G. We have $\mathbb{E}(X) = mnp$. If ℓ is odd, then $\ell = 2q+1$ and $\mathbb{E}[Y] \leq [m]_{tq+1}[n]_{tq+1}p^{t(2q+1)} < (1/2)m^{tq+1}n^{tq+1}p^{t(2q+1)}$. We now choose p so that $\mathbb{E}[X] \geq 2\mathbb{E}[Y]$. It suffices to set $p = m^{\frac{-tq}{2tq+t-1}} n^{\frac{-tq}{2tq+t-1}}$. Since

$\mathbb{E}(X-Y) \geq (1/2)\mathbb{E}[X]$, there exists a (m,n)-bipartite graph G for which $X-Y \geq (1/2)mnp = (1/2)(mn)^{\frac{tq+t-1}{2tq+t-1}}$. By deleting one edge from copy of $\theta_{t,\ell}$ in G, we obtained a (m,n)-bipartite graph G' that is $\theta_{t,\ell}$-free and satisfies

$$e(G') \geq (1/2)(mn)^{\frac{tq+t-1}{2tq+t-1}} = (1/2)(mn)^{\frac{\ell+1-(2/t)}{2\ell-(2/t)}}.$$

For sufficiently large t, we have $e(G') \geq (1/2)(mn)^{\frac{\ell+1}{2\ell}-\varepsilon}$, as desired.

For even integers $\ell = 2q$, the analysis is similar, except that we use the bound $\mathbb{E}[Y] \leq [m]_{tq+1}[n]_{t(q-1)+1}+[m]_{t(q-1)+1}[n]_{tq+1})p^{2tq} < (1/2)m^{t(q-1)+1}n^{tq+1}p^{2tq}$. We omit the details. □

It is quite likely using the random algebraic method used in [9], one could show that for each $\ell \geq 2$, there exist a t_0 such that for all $t \geq t_0$ if ℓ is odd then $z(m,n,\theta_{t,\ell}) \geq \Omega(m^{\frac{\ell+1}{2\ell}}n^{\frac{\ell+1}{2\ell}})$ and if ℓ is even then $z(m,n,\theta_{t,\ell}) \geq \Omega(m^{\frac{1}{2}+\frac{1}{\ell}}n^{\frac{1}{2}})$. In any case, Proposition 5.1 already shows that $\theta_{t,\ell}$ is not (α,β)-quasi-smooth and hence is also not (α,β)-smooth. (As far as we know, this is the first example of a family of bipartite graphs which are not (α,β)-smooth.) However, $\theta_{t,\ell}$-free graphs have similar expansion properties as $C_{2\ell}$-free graphs (see [19], Lemma 4.1). By using Lemma 4.1 in [19] instead of Lemma 3.1 in this paper, one can develop an analogous lemma as Lemma 3.2. Then using essentially the same proof as that of Theorem 1.7, one can show the following.

Theorem 5.2. *Let $t,\ell \geq 2$. Let $\delta > 0$ be any real. Let $k_0 = 3\ell(8\ell/\delta)^\ell + 2\ell + 2$. For all odd integers $k \geq k_0$ and n sufficiently large the following is true. If G is an n-vertex $\{\theta_{t,\ell}, C_k\}$-free graph with minimum degree at least $\delta n^{1/\ell}$, then G is bipartite.*

2. Our proof method works for a slightly broader family than (α,β)-quasi-smooth graphs and theta graphs. Suppose $\mathcal{F}$ is a family of bipartite graphs satisfying the following property (P1).

   (P1). For any $\delta > 0$, there are constants K and μ such that for every n-vertex $\mathcal{F}$-free graph G with minimum degree $\delta\,\mathrm{ex}(n,\mathcal{F})/n$ and for each vertex u in G, there are at least μn vertices within distance K from u.

   Then analogous theorems as Theorems 1.6 and 1.7 hold for $\mathcal{F}$-free graphs.

3. Let $\mathcal{F}$ be an (α,β)-smooth family with $2 > \alpha > \beta \geq 1$ and let $k \geq k_0$ be a (large) odd integer. Allen-Keevash-Sudakov-Verstraëte [1] proved that for any $\epsilon > 0$ and sufficiently large n, any n-vertex extremal $\mathcal{F} \cup \{C_k\}$-free graph can be made bipartite by deleting at most ϵn^α edges. This implies that $\mathrm{ex}(n, \mathcal{F} \cup \{C_k\}) = z(n,\mathcal{F}) + o(n^\alpha)$ and thus Theorem 1.3 holds.

   As a direct application of Theorem 1.6, one can strengthen Theorem 1.3 by deleting only $O(n^{1+\beta-\alpha})$ vertices (of low degree) to make the extremal graph bipartite. This gives further evidence to an affirmative answer to the question of Allen et al. [1] that whether the extremal n-vertex $\mathcal{F} \cup \{C_k\}$-free graph G in Theorem 1.3 is bipartite (for sufficiently large n).

   Theorem 5.3. *Let $\mathcal{F}$ be an (α,β)-smooth family with $2 > \alpha > \beta \geq 1$. Then there exists k_0 such that for any odd $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F}\cup\{C_k\}$-free extremal graph can be made bipartite by deleting a set of $O(n^{1+\beta-\alpha})$ vertices, which together are incident to $O(n^\beta)$ edges. Therefore, $\mathrm{ex}(n, \mathcal{F} \cup \{C_k\}) = z(n,\mathcal{F}) + O(n^\beta)$.*

Proof. Let $2 > \alpha > \beta \geq 1$ and $\mathcal{F}$ be an (α, β)-*smooth* family with relative density ρ. Then by the remark after Definition 1.2, there exist constants $C_1 < C_2$ such that for sufficiently large n,

$$\rho(n/2)^\alpha + C_1 n^\beta \leq z(n, \mathcal{F}) \leq \rho(n/2)^\alpha + C_2 n^\beta.$$

Fix $\delta := \rho/2^{\alpha+3}$. Let k_0 be from Theorem 1.6 such that for any odd $k \geq k_0$ and sufficiently large m, any m-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\delta m^{\alpha-1}$ is bipartite.

Now consider any odd $k \geq k_0$ and sufficiently large n. Let G be an n-vertex extremal $\mathcal{F} \cup \{C_k\}$-free graph. Then

$$e(G) = \mathrm{ex}(n, \mathcal{F} \cup \{C_k\}) \geq z(n, \mathcal{F}) \geq \frac{\rho}{2^\alpha} n^\alpha + C_1 n^\beta. \tag{9}$$

Let $G_0 = G$. If there exists some vertex x of degree less than $\delta n^{\alpha-1}$ in G_0, then we delete the vertex x and rename the remaining subgraph as G_0. We repeat the above process until there is no such vertex in G_0. Let H denote the remaining induced subgraph of G and let $t = n - |V(H)|$. We note that as $\alpha < 2$ and n is sufficiently large, using (9) and $\delta = \frac{\rho}{2^{\alpha+3}}$,

$$e(H) \geq e(G) - t \cdot \delta n^{\alpha-1} \geq \left(\frac{\rho}{2^\alpha} n^\alpha + C_1 n^\beta\right) - n \cdot \delta n^{\alpha-1} = \frac{7}{8}\frac{\rho}{2^\alpha} n^\alpha + C_1 n^\beta. \tag{10}$$

Let $m = |V(H)|$. By definition, H is an m-vertex $\mathcal{F} \cup \{C_k\}$-free graph with minimum degree at least $\delta n^{\alpha-1} \geq \delta m^{\alpha-1}$. By taking n sufficiently large, we can make m large enough to apply Theorem 1.6 to conclude that H is bipartite. Since H is also, $\mathcal{F}$-free, we have

$$e(H) \leq z(m, \mathcal{F}) = z(n-t, \mathcal{F}) \leq \frac{\rho}{2^\alpha}(n-t)^\alpha + C_2 n^\beta. \tag{11}$$

Comparing (10) and (11), we see that $n - t \geq \frac{3}{4}n$. By (9), (10) and (11), we have

$$\left(\frac{\rho}{2^\alpha} n^\alpha + C_1 n^\beta\right) - t \cdot \delta n^{\alpha-1} \leq \frac{\rho}{2^\alpha}(n-t)^\alpha + C_2 n^\beta$$

Recall that $\delta = \rho/2^{\alpha+3}$. Rearranging the above inequality, we get

$$\frac{\rho}{2^\alpha} n^\alpha - \frac{\rho}{2^\alpha}(n-t)^\alpha - t \cdot \delta n^{\alpha-1} \leq (C_2 - C_1) n^\beta.$$

By the Mean Value Theorem, for some $n - t \leq n' \leq n$, this is equivalent to

$$\frac{\rho\alpha}{2^\alpha} t (n')^{\alpha-1} - t \cdot \delta n^{\alpha-1} \leq (C_2 - C_1) n^\beta.$$

Since $n - t \geq \frac{3}{4}n$ and $\frac{\rho\alpha}{2^\alpha}(\frac{3}{4})^{\alpha-1} > 2 \cdot \frac{\rho}{2^{\alpha+3}}$, the above inequality yields

$$\frac{\rho}{2^{\alpha+3}} t n^{\alpha-1} \leq (C_2 - C_1) n^\beta.$$

So, $t = O(n^{1+\beta-\alpha})$ and in obtaining H from G at most $t \cdot \delta n^{\alpha-1} = O(n^\beta)$ edges are removed. $\square$

In fact, the above proof can be generalized a bit further as the following. Suppose $\mathcal{F}$ is a family of bipartite graphs satisfying the property (P1) and the following property (P2).

(P2). There exists some constants $\lambda > 0$ and $2 > \alpha > \beta \geq 1$ such that $z(n, \mathcal{F}) = \lambda n^\alpha + O(n^\beta)$.

Then Conjecture 1.1 holds for $\mathcal{F}$ in the following form: for any odd $k \geq k_0$ and sufficiently large n, any n-vertex $\mathcal{F} \cup \{C_k\}$-free extremal graph can be made bipartite by deleting a set of $O(n^{1+\beta-\alpha})$ vertices, which together are incident to $O(n^\beta)$ edges.

In particular, this also applies to $\mathcal{F} = \{C_4, C_6, ..., C_{2\ell}\}$ for $\ell \in \{2, 3, 5\}$.

4. One could also prove our main theorems using the sparse regularity lemma ([22], [26]). However, the proofs would be more technical and would involve longer buildups. We chose to present a proof that avoids the use of sparse regularity. It seems, however, in order to make more progress on the original conjecture of Erdős and Simonovits (Conjecture 1.1), for instance to verify the conjecture for (α, β)-quasi-smooth families, sparse regularity lemma may still be an effective tool. This is because for (α, β)-quasi-smooth families $\mathcal{F}$, like for (α, β)-smooth families (see [1]), there is a transference of density from an $\mathcal{F}$-free host graph to the corresponding cluster graph.

References

[1] P. Allen, P. Keevash, B. Sudakov and J. Verstraëte, Turán numbers of bipartite graphs plus an odd cycle, *J. Combin. Theory Ser. B* **106** (2014), 134-162.

[2] N. Alon, L. Rónyai and T. Szabó, Norm-graphs: variations and applications, *J. Combin. Theory Ser. B* **76** (1999), 280-290.

[3] N. Alon and J. H. Spencer, The Probabilistic Method, John Wiley & Sons, 3rd edition, 2016.

[4] B. Andrásfai, P. Erdős and V.T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.* **8** (1974), 205-218.

[5] J. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combin. Theory Ser. B* **16** (1974), 97-105.

[6] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281-285.

[7] B. Bukh, Extremal graphs without exponentially-small bicliques, arXiv:2107.04167

[8] B. Bukh and M. Tait, Turán numbers of theta graphs, *Combin. Probab. Comput.* **29** (2020), 495-507.

[9] D. Conlon, Graphs with few paths of prescribed length between any two vertices, *Bull. Lond. Math. Soc.* **51** (2019), 1015-1021.

[10] P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 51–57.

[11] P. Erdős and M. Simonovits, Compactness results in extremal graph theory, *Combinatorica* **2** (1982), 275-288.

[12] P. Erdős and H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.

[13] R. Faudree and M. Simonovits, On a class of degenerate extremal graph problems, *Combinatorica*, **3**(1983), 83–93.

[14] Z. Füredi, An upper bound on Zarankiewicz' problem, *Combin. Probab. Comput.* **1** (1996), 29-33.

[15] Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Combin. Theory Ser. A*, **75** (1996), 141-144.

[16] Z. Füredi and M. Simonovits, The history of the degenerate (bipartite) extremal graph problems, *Erdős centennial, Bolyai Soc. Math. Stud.* **25**, 169-264, János Bolyai Math. Soc., Budapest, 2013. See also arXiv:1306.5167.

[17] S. Janson, T. Luczak and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[18] T. Jiang and J. Ma, Cycles of given lengths in hypergraphs, *J. Combin. Theory Ser. B* **133** (2018), 54-77.

[19] T. Jiang, J. Ma and L. Yepremyan, On Turán exponents of bipartite graphs, *Combin. Probab. Comput.*, published online, https://doi.org/10.1017/S0963548321000341. See also arXiv:1806.02838.

[20] P. Keevash, B. Sudakov and J. Verstraëte, On a conjecture of Erdős and Simonovits: Even cycles, *Combinatorica* **33** (2013), 699-732.

[21] J. Kollár, L. Rónyai and T. Szabó, Norm-graphs and bipartite Turán numbers, *Combinatorica* **16** (1996), 399-406.

[22] Y. Kohayakawa, Szemerédi's regularity lemma for sparse graphs, in: F. Cucker, M. Shub (eds.), Foundations of Computational Mathematics, Springer-Verlag, Berlin, 1997, pp. 295-352.

[23] T. Kövári, V.T. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50-57.

[24] S. Letzter, Hypergraphs with no tight cycles, arXiv:2106:12082 (2021).

[25] A. Naor and J. Verstraëte, A note on bipartite graphs without $2k$-cycles, *Combin. Probab. Comput.* **14** (2005), 845-849.

[26] A. Scott, Szemerédi's Regularity Lemma for matrices and sparse graphs, *Combin. Probab. Comput.* **20** (2011), 455-466.

[27] J. Verstraëte, On arithmetic progressions of cycle lengths in graphs, *Combin. Probab. Comput.* **9** (2000), 369-373.