Proof of a conjecture of Voss on bridges of longest cycles

Jie Ma* Rongxing Xu†

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Abstract

Bridges are a classical concept in structural graph theory and play a fundamental role in the study of cycles. A conjecture of Voss from 1991 asserts that if disjoint bridges $B_1, B_2, \ldots, B_k$ of a longest cycle L in a 2-connected graph overlap in a tree-like manner (i.e., induce a tree in the *overlap graph* of L), then the total *length* of these bridges is at most half the length of L. Voss established this for $k \leq 3$ and used it as a key tool in his 1991 monograph on cycles and bridges. In this paper, we confirm the conjecture in full via a reduction to a cycle covering problem.

1 Introduction

Let G be a graph and H a subgraph of G. An H*-bridge* of G is either (i) an edge in $E(G)\setminus E(H)$ with both endpoints in $V(H)$, or (ii) a subgraph consisting of a component D of $G - V(H)$ together with all edges between $V(D)$ and $V(H)$. For an H-bridge B, the vertices in $V(H) \cap V(B)$ are called the *attachments* of B. In this paper, we often consider H to be a cycle.

The concept of bridges has naturally emerged in the development of graph theory, particularly in the study of cycles. As emphasized by Bondy in his influential survey [14] (p. 58), "bridges clearly play a very important role in the study of paths and circuits, and it can be argued that their role is central."

A cornerstone result in the study of cycles is Tutte's theorem [33], which strengthens Whitney's theorem [36] by asserting that every 4-connected planar graph is Hamiltonian. A key ingredient in Tutte's celebrated proof is the so-called Bridge Lemma, which characterizes the bridges of certain cycles in planar graphs by their attachments. Since then, bridges have facilitated numerous generalizations and refinements concerning Hamiltonicity, including results of Thomassen [30] (a small omission was corrected by Chiba and Nishizeki [11]), Thomas and Yu [27, 28], Kawarabayashi and Ozeki [21], as well as [19, 24, 25, 29]. Beyond Hamiltonicity,

*School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China. Research supported by National Key Research and Development Program of China 2023YFA1010201 and National Natural Science Foundation of China grant 12125106. Email: jiema@ustc.edu.cn.

†School of Mathematical Sciences, Zhejiang Normal University, Jinhua, Zhejiang, 321000, China. Research supported by National Natural Science Foundation for Young Scientists of China grant 12401472. Email: xurongxing@zjnu.edu.cn.

bridges have also played a central role in the study of longest cycles in general graphs, as explored in [5, 8, 9, 10, 18, 37], among others.

In light of these advances, an important direction of research has been to understand how cycles interact with their bridges and how these bridges are arranged along the cycle. This leads to the notion of the *overlap graph*. Let L be a cycle in a graph G. Two L-bridges B_1 and B_2 are said to *overlap* if L cannot be partitioned into two subpaths L_1 and L_2 such that the attachments of B_i lie entirely on L_i for each $i = 1, 2$. The *overlap graph* of G with respect to L, denoted $O_G(L)$, is the graph whose vertices correspond to the L-bridges, with an edge between two vertices if and only if the corresponding bridges overlap. Using this concept, Tutte [34] gave a characterization of planar graphs, proving that a graph G is planar if and only if, for every cycle L, the overlap graph $O_G(L)$ is bipartite. This characterization underlies most planarity-testing algorithms (see [2, 13, 17]). Voss [35] further showed that for any cycle L in a 3-connected graph G, the overlap graph $O_G(L)$ is connected.

In his monograph [35], Voss investigated various problems on cycles, with particular emphasis on the role of bridges. To provide a measure on the size of a bridge, he [35] introduced the following parameter (which is called the span of a bridge in [14]).

Definition 1 *For a subgraph H in a graph G, the* **length** $\lambda(B)$ *of an H-bridge B is the maximum number of edges in a tree within B whose leaves are exactly the attachments of B.*

For a cycle L, an L-bridge has length one if and only if it is a chord of L. Thomassen's Chord Conjecture (see, e.g., [1, 4, 31]) then asserts that every longest cycle L in a 3-connected graph contains such a bridge. The conjecture remains open, with significant progress in [6, 20, 32, 38].

Voss [35] proposed the following conjecture on longest cycles L in a graph G, aiming to provide quantitative control over the size of L-bridges relative to the length of L (i.e., the *circumference* of G). The conjecture is also discussed in Bondy's comprehensive survey on cycles [14] (see Conjecture 5.11).

Conjecture 2 (Voss [35], p. 54) *Let G be a 2-connected graph with a longest cycle L. Let $B_1, B_2, \ldots, B_k$ be L-bridges that are pairwise vertex-disjoint and induce a tree in $O_G(L)$. Then*

$$\sum_{i=1}^{k} \lambda(B_i) \leq \lfloor |E(L)|/2 \rfloor.$$

The cases $k \leq 3$ were established by Voss himself [35], who used them as key tools in his study of problems and properties of longest cycles (see Chapters 3, 7, and 11 of [35]).

We note that, if true, the inequality is best possible. For $k = 1$, consider $G = K_{2,3}$, where the longest cycle has length 4 and its unique bridge has length 2. For $k \geq 2$, let G be the graph obtained from a $2k$-cycle $L = v_1 v_2 \cdots v_{2k} v_1$ by adding the edge $v_1 v_{k+1}$ and, for each $i \in \{2, 3, \ldots, k\}$, the edge $v_i v_{2k+2-i}$. Then G has exactly k L-bridges whose total length is k, which equals half of $|E(L)| = 2k$. We also remark that the conjecture fails if the subgraph of $O_G(L)$ induced by $B_1, B_2, \ldots, B_k$ is disconnected or contains a cycle, as shown in Fig. 1. To be precise, we see that in the left graph, $O_G(L)$ is disconnected and $\lambda(B_1) + \lambda(B_2) = 4 > |E(L)|/2 = 3$, while in the right graph (the Petersen graph), $O_G(L)$ contains a triangle and $\lambda(B_1) + \lambda(B_2) + \lambda(B_3) = 5 > |E(L)|/2 = 9/2$.

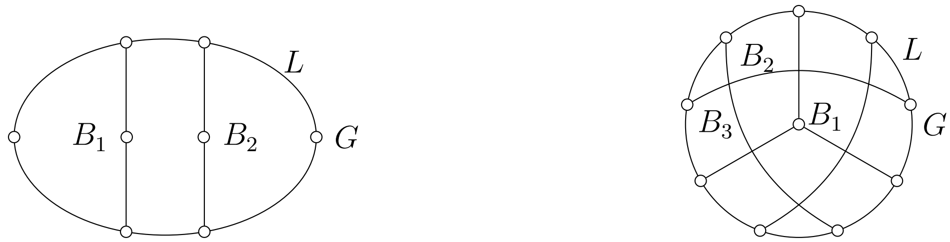


Fig. 1. Two examples for Conjecture 2

The main result of this paper resolves Conjecture 2 completely.

Theorem 3 *Conjecture 2 holds for all positive integers k.*

It is worth emphasizing that we prove this result by reducing the problem to one involving specified cycle coverings, which we then solve. To state this result formally, we need to introduce one more concept: the *symmetric difference* of two subgraphs H_1 and H_2 of a graph G, denoted $H_1 \triangle H_2$, which is the subgraph induced by the edge set $E(H_1) \triangle E(H_2)$. The following is the corresponding result on cycle coverings.

Theorem 4 *Let L be a cycle in a 2-connected graph G, and let $T_1, T_2, \ldots, T_k$ be L-bridges that are pairwise vertex-disjoint and induce a tree in $O_G(L)$, with each T_i being a tree whose leaves are exactly the attachments of T_i. Then there exists a collection $\mathcal{C}$ of cycles in G such that each edge of L lies in exactly two cycles of $\mathcal{C}$, each edge not in L lies in at least four cycles of $\mathcal{C}$, and for each cycle $C \in \mathcal{C}$, $L \triangle C$ is a cycle.*

Since the proof is short, we present this reduction immediately.

Proof of Theorem 3, assuming Theorem 4. Let $B_1, B_2, \ldots, B_k$ be L-bridges that satisfy the conditions of Theorem 3. For each L-bridge B_i, let T_i be the largest tree in B_i whose leaves are the attachments of B_i. Define H as the union of L and T_i for all $1 \le i \le k$, which is 2-connected. By Theorem 4, there exists a family $\mathcal{C}$ of cycles for H as described. Since L is a longest cycle in G and $L \triangle C$ is a cycle for every $C \in \mathcal{C}$, we have $|E(L \triangle C)| \le |E(L)|$. Note that $|E(L \triangle C)| = |E(L) \setminus E(C)| + |E(C) \setminus E(L)|$ and $|E(L)| = |E(L) \setminus E(C)| + |E(L) \cap E(C)|$. Thus $|E(C) \setminus E(L)| \le |E(L) \cap E(C)|$ for every $C \in \mathcal{C}$. Observe that $E(C) \setminus E(L) = \bigcup_{i=1}^{k} E(C) \cap E(T_i)$ and by the assumption, $B_1, B_2, \ldots, B_k$ are pairwise vertex-disjoint, so we have $\sum_{i=1}^{k} |E(C) \cap E(T_i)| = |E(C) \setminus E(L)| \le |E(L) \cap E(C)|$ for every $C \in \mathcal{C}$.

From the properties of the cycle family $\mathcal{C}$, each edge in L lies in exactly two cycles of $\mathcal{C}$, and each edge in $\bigcup_{i=1}^{k} T_i$ lies in at least four. Therefore, we can derive that

$$4 \sum_{i=1}^{k} \lambda(B_i) = 4 \sum_{i=1}^{k} |E(T_i)| \le \sum_{C \in \mathcal{C}} \sum_{i=1}^{k} |E(C) \cap E(T_i)| \le \sum_{C \in \mathcal{C}} |E(C) \cap E(L)| = 2|E(L)|,$$

which finishes the proof of Theorem 3. ■

The remainder of the paper is organized as follows. In Section 2, we prove Theorem 3 using Theorem 4 and then present a slightly stronger version of it that provides additional information on the cycle family for inductive arguments. Section 3 contains preliminary results needed for this stronger version (Theorem 5). In Section 4, we give the full proof of Theorem 5. Finally, we discuss some related open problems in the last section.

2 A strengthened version of Theorem 4

Let L be a cycle in a graph G. A natural idea for proving Theorem 4 is to use induction on the number of L-bridges. In the base case, where there are only one or two L-bridges, one can explicitly construct the corresponding family of cycles. On the other hand, for the case with at least three L-bridges, one may consider using induction hypothesis on (G_1, L) and (G_2, L), where G_1 and G_2 are subgraphs of G with less L-bridges than G. This gives two families of cycles, $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then one can construct a new family $\mathcal{C}$ of cycles, satisfying the theorem, based on those in $\mathcal{C}_1 \cup \mathcal{C}_2$. However, to ensure that the induction works, both $O_L(G_1)$ and $O_L(G_2)$ must be trees. So we have to add some edges for certain cases in this process. Then, in the construction of $\mathcal{C}$, we also need to address the cycles which contain the new edges. We will either delete these cycles or combine some pair of cycles by taking their symmetric difference. This necessitates a deeper understanding of the cycles involved, which motivates us to consider directed cycles instead.

We begin by defining four types of directed cycles (or, briefly, *dicycles*), which play a central role in this paper. Throughout, we denote by $\overrightarrow{H}$ a directed graph whose underlying graph is H. For two vertices x and y, we write (x, y) (or $x \to y$) to denote an arc from x to y, that is, an arc with tail x and head y.

Let L be a cycle in a graph G, whose vertices are arranged in cyclic order (often assumed *clockwise* in a planar drawing of L). We denote by v^+ (resp. v^-) the next (resp. previous) vertex of v on L in this order. Let C be a cycle of G that contains an edge ab, where a is a vertex on L and b is not. We say that a dicycle $\overrightarrow{C}$ is of *type* ij with respect to a if the following conditions are satisfied.

- $ij = 00$: (b, a) and (a, a^+) are in $\overrightarrow{C}$;
- $ij = 01$: (a, b) and (a^+, a) are in $\overrightarrow{C}$;
- $ij = 10$: (b, a) and (a, a^-) are in $\overrightarrow{C}$;
- $ij = 11$: (a, b) and (a^-, a) are in $\overrightarrow{C}$.

For convenience, we interpret $i = 0$ (resp., $i = 1$) as indicating that the dicycle contains a^+ (resp., a^-). Similarly, $j = 0$ (resp., $j = 1$) indicates that the dicycle contains the arc (b, a) (resp., (a, b)). Note that for each attachment v and $ij \in \mathbb{Z}_2^2$, it is possible that there are different dicycles of type ij with respect to v.

Given a graph G, let $\mathcal{C}(G)$ be the family of all possible dicycles of G. For a subfamily $\mathcal{C} \subseteq \mathcal{C}(G)$ and an edge $e \in E(G)$, let

$$\mathcal{C}_e = \{\overrightarrow{C} \in \mathcal{C} : e \in E(C)\}.$$

We are now ready to present the strengthened form of Theorem 4, the proof of which will occupy the remainder of the paper.

Theorem 5 *Let L be a cycle in a 2-connected graph G and let $T_1, T_2, \ldots, T_s$ be all the L-bridges that are pairwise vertex-disjoint and induce a tree in $O_G(L)$, with each T_i being a tree. Then there exists a subfamily $\mathcal{C} \subseteq \mathcal{C}(G)$ such that the following conditions hold.*

(C1) $|\mathcal{C}_e| = 2$ *for every* $e \in E(L)$ *and* $|\mathcal{C}_e| \geq 4$ *for every* $e \in E(G) \setminus E(L)$.

(C2) For each dicycle in $\mathcal{C}$ *with underlying graph* C, $L \triangle C$ *is a cycle.*

(C3) Every dicycle in $\mathcal{C}$ *contains either no or exactly two attachments of each* L*-bridge.*

(C4) For any $ij \in \mathbb{Z}_2^2$ *and any vertex* x *which is an attachment of some* L*-bridge, there is exactly one dicycle in* $\mathcal{C}$ *which is of type* ij *with respect to* x.

By considering the underlying graphs of the dicycles in $\mathcal{C}$, it is straightforward to see that Theorem 4 already follows from (C1) and (C2).

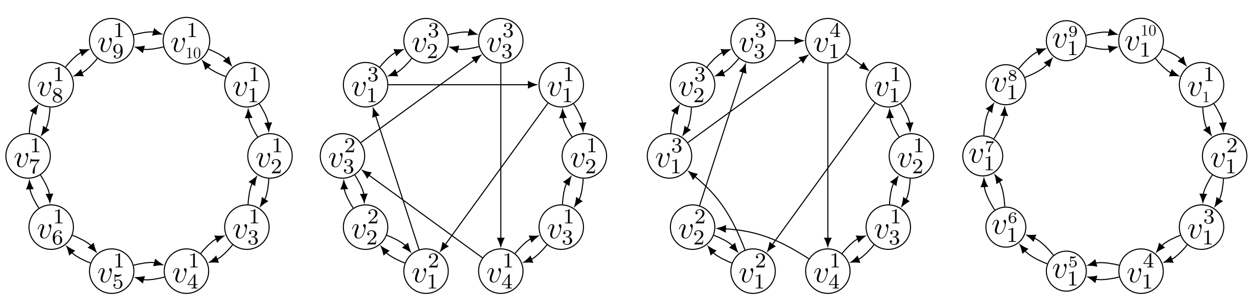
3 Preliminary for the proof of Theorem 5

In this section, we introduce several definitions and lemmas that will be useful later. In particular, we will present a method for constructing a special family of paths that covers a tree. For this purpose, we define an auxiliary digraph and introduce some of its properties in Section 3.1. Then, in Section 3.2, we use this auxiliary digraph to construct a family of paths in which each edge is covered at least four times.

3.1 An auxiliary digraph

Let n and k be two positive integers with $k \leq n$. We say a k-tuple $\eta = (p_1, p_2, \ldots, p_k)$ is a *k-partition* of n if $p_i \geq 1$ for each $1 \leq i \leq k$ and $\sum_{i=1}^{k} p_i = n$. Given a k-partition η of n, we define a multidigraph D_η with vertex-set $\bigcup_{i=1}^{k}\{v_1^i, v_2^i, \ldots, v_{p_i}^i\}$ and arcs as follows:

- $k = 1$. We add the arcs (v_i^1, v_{i+1}^1) and (v_{i+1}^1, v_i^1) for each $i = 1, 2, \ldots, n$, where we interpret v_{n+1}^1 as v_1^1.
- $k \geq 2$. We add (1) the arcs (v_j^i, v_{j+1}^i) and (v_{j+1}^i, v_j^i) for all $1 \leq i \leq k$ and $1 \leq j \leq p_i - 1$ (if $p_i = 1$, then this step is skipped); (2) the arcs (v_1^i, v_1^{i+1}) and $(v_{p_i}^i, v_{p_{i+1}}^{i+1})$ for all $1 \leq i \leq k$, where we interpret v_1^{k+1} as v_1^1 and $v_{p_{k+1}}^{k+1}$ as $v_{p_1}^1$.



(a) $\eta = (10)$ **(b)** $\eta = (4, 3, 3)$ **(c)** $\eta = (4, 2, 3, 1)$ **(d)** $\eta = (1, 1, \ldots, 1)$

Fig. 2. Examples of D_η, where each η is a partition of 10.

It is easy to check that D_η is a digraph in which every vertex has both in-degree 2 and out-degree 2. Note that if $p_i = p_{i+1} = 1$ for some i, then D_η contains multiple arcs. See Fig. 2

for examples. For convenience, we denote by G_η the underlying multigraph of D_η, which may contain parallel edges.

Lemma 6 *For each arc (u,v) in D_η, there exist two dicycles containing (u,v), and these two dicycles share only this arc.*

Proof. Suppose η is a k-partition. The statement obviously holds for $k=1$, one dicycle of length 2 and the other of length n suffice. Thus we assume that $k \geq 2$. It suffices to describe such two dicycles.

First assume that $u = v_j^i$ and $v = v_{j+1}^i$ for some $1 \leq i \leq k$ with $p_i \geq 2$ and $1 \leq j \leq p_i - 1$. It is clear that $v_j^i \to v_{j+1}^i \to v_j^i$ and

$$v_j^i \to v_{j+1}^i \to \cdots \to v_{p_i}^i \to v_{p_{i+1}}^{i+1} \to \cdots \to v_1^{i+1} \to v_1^{i+2} \to v_1^{i+3} \to \cdots \to v_1^i \to v_2^i \to \cdots \to v_j^i$$

are the desired dicycles (the upper indices are taken modulo k).

Thus without loss of generality, we may assume that $u = v_1^i$ and $v = v_1^{i+1}$. Then $v_1^i \to v_1^{i+1} \to v_1^{i+2} \to \cdots \to v_1^i$ and

$$v_1^i \to v_1^{i+1} \to v_2^{i+1} \to \cdots \to v_{p_{i+1}}^{i+1} \to v_{p_{i+2}}^{i+2} \to \cdots \to v_{p_i}^i \to v_{p_i-1}^i \to \cdots v_1^i$$

are the desired dicycles. We remark that the arcs (v_1^j, v_1^{j+1}) and $(v_{p_j}^j, v_{p_{j+1}}^{j+1})$ are essentially different if $p_j = p_{j+1} = 1$, in which case they are multi-arcs sharing the same tail and head by the definitions of D_η. ■

Given a digraph $D(G)$ with underlying graph G, for $S, T \subsetneq V(G)$, we denote by $[S,T]_{D(G)}$ the set of arcs with tails in S and heads in T, and $[S,T]_G$ the edges of G joining S and T. For a subset $\emptyset \neq X \subsetneq V(G)$, we denote by $\overline{X}$ the vertex-set $V(G) \setminus X$.

Lemma 7 *For any k-partition η of n, $1 \leq k \leq n$ and any $\emptyset \neq X \subsetneq V(D_\eta)$, we have $|[X, \overline{X}]_{G_\eta}| \geq 4$.*

Proof. Assume for some η and a subset $\emptyset \neq X \subsetneq V(D_\eta)$, we have $|[X, \overline{X}]_{G_\eta}| \leq 2$. Without loss of generality, assume that $[X, \overline{X}]_{D_\eta} \neq \emptyset$ and $(u,v) \in [X, \overline{X}]_{D_\eta}$ is an arc, then for any dicycle in D_η containing (u,v), it must contain an arc in $[\overline{X}, X]_{D_\eta}$. This contradicts Lemma 6, since we are unable to find two dicycles containing (u,v) but only sharing (u,v). Thus $|[X, \overline{X}]_{G_\eta}| \geq 3$. On the other hand, if $|[X, \overline{X}]_{G_\eta}|$ is odd, then the degree sum of the graph induced by X is odd as each vertex in G_η has even degree, but then it contradicts the Handshaking Lemma. Thus we have $|[X, \overline{X}]_{G_\eta}| \geq 4$. ■

3.2 A family of dipaths of tree with labeled leaves

Given a tree T, let $\mathcal{P}(T)$ be the family of all possible dipaths of T between leaves. For a dipath $\overrightarrow{P} \in \mathcal{P}(T)$ and an edge $e \in E(T)$, we say $\overrightarrow{P}$ covers e if $e \in E(P)$. For $\mathcal{P} \subseteq \mathcal{P}(T)$ and an edge $e \in E(T)$, let $\mathcal{P}_e = \{\overrightarrow{P} \in \mathcal{P} : \overrightarrow{P} \text{ covers } e\}$.

In this section, we will construct a subfamily $\mathcal{P} \subseteq \mathcal{P}(T)$, using the auxiliary digraph which we introduced in the beginning of Section 3.1, such that each edge of T is covered by at least four dipaths in $\mathcal{P}$.

For a tree T and two distinct vertices $u, v \in T$, we denote by $T[u, v]$ the unique path in T between u and v, by $\overrightarrow{T}[u, v]$ the corresponding dipath from u to v. We denote by $\partial(T)$ the set of leaves of T.

Recall that D_η is the auxiliary digraph with respect to a k-partition $\eta = (p_1, p_2, \ldots, p_k)$ of some integer n. For a tree T and a k-partition $\eta = (p_1, p_2, \ldots, p_k)$ of $|\partial(T)|$, an *η-labeling* ℓ of $\partial(T)$ is a bijection from $\partial(T)$ to $V(D_\eta)$.

Given a tree T, a k-partition $\eta = (p_1, p_2, \ldots, p_k)$ of $|\partial(T)|$ and an η-labeling ℓ of $\partial(T)$, we define $\mathcal{P}(T, \eta, \ell)$ to be the collection of all dipaths $\overrightarrow{T}[u, v]$, where u and v are leaves of T such that $(\ell(u), \ell(v))$ is an arc in the auxiliary digraph D_η.

The following lemma analyzes the number of dipaths that cover an edge in $\mathcal{P}(T, \eta, \ell)$.

Lemma 8 *Let T be a tree, $\eta = (p_1, p_2, \ldots, p_k)$ be a k-partition of $|\partial(T)|$ and ℓ be a η-labeling of $\partial(T)$. Then for any edge $e \in E(T)$, $|\mathcal{P}_e(T, \eta, \ell)| \geq 4$.*

Proof. Assume $e \in E(T)$, T_1 and T_2 are the two components of $T - e$. Let $X_1 = \{\ell(v) | v \in \partial(T_1)\}$ and $X_2 = \{\ell(v) | v \in \partial(T_2)\}$. Since ℓ is a one-to-one correspondence from $\partial(T)$ to $V(D_\eta)$, and $\partial(T_1) \cap \partial(T_2) = \emptyset$, $\partial(T) = \partial(T_1) \cup \partial(T_2)$, it follows that $V(D_\eta) = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$, that is $X_2 = \overline{X}_1$ in D_η. By Lemma 7, $|[X_1, X_2]_{G_\eta}| \geq 4$. This implies that there exist at least four distinct ordered vertex pairs $(a, b) \in \partial(T_1) \times \partial(T_2)$ such that $(\ell(a), \ell(b)) \in [X_1, X_2]_{D_\eta} \cup [X_2, X_1]_{D_\eta}$. For each such pair (a, b), either $\overrightarrow{T}[a, b] \in \mathcal{P}(T, \eta, \ell)$ or $\overrightarrow{T}[b, a] \in \mathcal{P}(T, \eta, \ell)$, and both $\overrightarrow{T}[a, b]$ and $\overrightarrow{T}[b, a]$ cover the edge e. Thus, $|\mathcal{P}_e(T, \eta, \ell)| \geq 4$. ∎

4 Proof of Theorem 5

For convenience, we say $\mathcal{C}$ is *feasible* for (G, L) if $\mathcal{C} \subseteq \mathcal{C}(G)$ and $\mathcal{C}$ satisfies (C1)-(C4). For two vertices u and v on L, let $L[u, v]$ be the segment of L from u to v in clockwise direction. We denote by $\overrightarrow{L}[u, v]$ the copy of $L[u, v]$ directed from u to v, and $\overleftarrow{L}[u, v]$ the copy of $L[u, v]$ directed from v to u. For a subgraph H of G, let $\overrightarrow{C}|_H$ be the subdigraph (not necessarily connected) of $\overrightarrow{C}$ whose underlying graph is induced by the edge set $E(C) \cap E(H)$, and let $\mathcal{C}|_H = \{\overrightarrow{C}|_H : \overrightarrow{C} \in \mathcal{C}\}$.

Basic step: $s = 1$

Assume that $v_1, v_2, \ldots, v_n$ are leaves of T_1, arranged in clockwise cyclic order on L. We construct $\mathcal{C} \subseteq \mathcal{C}(G)$ as follows: for each $1 \leq i \leq n$, we add two dicycles

$$\overrightarrow{T}[v_i, v_{i+1}] \cup \overleftarrow{L}[v_i, v_{i+1}] \text{ and } \overrightarrow{T}[v_{i+1}, v_i] \cup \overrightarrow{L}[v_i, v_{i+1}]$$

to $\mathcal{C}$, where the indices are taken modulo n. We shall check that $\mathcal{C}$ satisfies (C1)-(C4). Note that (C2)-(C4) are straightforward, we omit the verification and only focus on (C1).

It is clear that there is a one-to-one correspondence between $\mathcal{C}|_T$ and $\mathcal{P}(T, \eta, \ell)$, where $\eta = (n)$ is a 1-partition of n, and $\ell : \partial(T) \to \{v_1, v_2, \ldots, v_n\}$ is the η-labeling of $\partial(T)$. Therefore, by Lemma 8, $|\mathcal{C}_e| = |\mathcal{P}_e(T, \eta, \ell)| \geq 4$ for every $e \in E(G) \setminus E(L)$. On the other hand, observe that by our construction, $|\mathcal{C}_e| = 2$ for each $e \in E(L)$. Thus $\mathcal{C}$ satisfies (C1).

Basic step: $s = 2$

In this case, G contains exactly two L-bridges T_1 and T_2. Let

$$u_1^1, \dots, u_{p_1}^1, v_1^1, \dots, v_{q_1}^1, u_1^2, \dots, u_{p_2}^2, v_1^2, \dots, v_{q_2}^2, \dots, u_1^k, \dots, u_{p_k}^k, v_1^k, \dots, v_{q_k}^k$$

be all the attachments listed in clockwise cyclic order on L, where $p_i, q_i \geq 1$ for each $1 \leq i \leq k$, the vertices u_j^i's are the attachments (leaves) of T_1, and the vertices v_j^i's are the attachments (leaves) of T_2. Note that $k \geq 2$ because the overlap graph $O_G(L)$ is connected. Thus $\eta_1 = (p_1, p_2, \dots, p_k)$ is a k-partition of $|\partial(T_1)|$, and $\eta_2 = (q_1, q_2, \dots, q_k)$ is a k-partition of $|\partial(T_2)|$. Let ℓ_i be the corresponding η_i-labeling of $\partial(T_i)$ for $i = 1, 2$.

We now construct $\mathcal{C}$ as follows (See Fig. 3 for illustration). For $1 \leq i \leq k$, we add the following dicycles to $\mathcal{C}$, where the (superscript) index 0 is interpreted as k, and $k+1$ as 1,

(1) $\overrightarrow{T_1}[u_j^i, u_{j+1}^i] \cup \overleftarrow{L}[u_j^i, u_{j+1}^i]$ and $\overrightarrow{T_1}[u_{j+1}^i, u_j^i] \cup \overrightarrow{L}[u_j^i, u_{j+1}^i]$ for each $1 \leq j \leq p_i - 1$ if $p_i \geq 2$.

(2) $\overrightarrow{T_2}[v_j^i, v_{j+1}^i] \cup \overleftarrow{L}[v_j^i, v_{j+1}^i]$ and $\overrightarrow{T_2}[v_{j+1}^i, v_j^i] \cup \overrightarrow{L}[v_j^i, v_{j+1}^i]$ for each $1 \leq j \leq q_i - 1$ if $q_i \geq 2$.

(3) $\overrightarrow{T_2}[v_1^i, v_1^{i+1}] \cup \overleftarrow{L}[u_{p_{i+1}}^{i+1}, v_1^{i+1}] \cup \overrightarrow{T_1}[u_{p_{i+1}}^{i+1}, u_{p_i}^i] \cup \overrightarrow{L}[u_{p_i}^i, v_1^i]$.

(4) $\overrightarrow{T_2}[v_{q_{i-1}}^{i-1}, v_{q_i}^i] \cup \overrightarrow{L}[v_{q_i}^i, u_1^{i+1}] \cup \overrightarrow{T_1}[u_1^{i+1}, u_1^i] \cup \overleftarrow{L}[v_{q_{i-1}}^{i-1}, u_1^i]$.

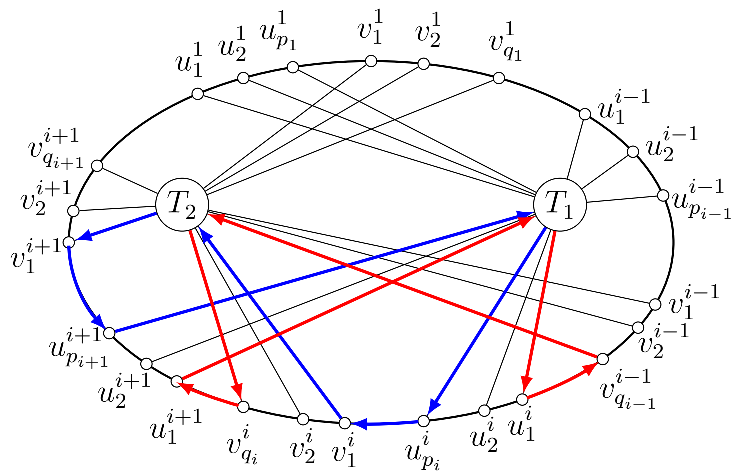


Fig. 3. Illustration for the construction of $\mathcal{C}$. The blue dicycle corresponds to (3) and the red one corresponds to (4).

By the construction of $\mathcal{C}$, it is easy to check that $\mathcal{C}$ satisfies (C3)-(C4), we omit the details and only focus on the verification of (C1) and (C2).

Observe that for $i = 1, 2$, the underlying paths in $\mathcal{C}|_{T_i}$ and $\mathcal{P}(T_i, \eta_i, \ell_i)$ are identical, so there is a natural one-to-one correspondence between $\mathcal{C}|_{T_i}$ and $\mathcal{P}(T_i, \eta_i, \ell_i)$. Thus by Lemma 8, for each edge $e \in E(T_i)$, $|\mathcal{C}_e| = |\mathcal{P}_e(T_i, \eta_i, \ell_i)| \geq 4$. On the other hand, $|\mathcal{C}_e| = 2$ for each edge $e \in E(L)$. Therefore $\mathcal{C}$ satisfies (C1).

The dicycles constructed in (1) and (2) satisfy (C2) obviously, we focus on the rest two classes. For $\overrightarrow{C} = \overrightarrow{T_2}[v_1^i, v_1^{i+1}] \cup \overleftarrow{L}[u_{p_{i+1}}^{i+1}, v_1^{i+1}] \cup \overrightarrow{T_1}[u_{p_{i+1}}^{i+1}, u_{p_i}^i] \cup \overrightarrow{L}[u_{p_i}^i, v_1^i]$, we have that

$$C \triangle L = L[v_1^{i+1}, u_{p_i}^i] \cup T_1[u_{p_i}^i, u_{p_{i+1}}^{i+1}] \cup L[v_1^i, u_{p_{i+1}}^{i+1}] \cup T_2[v_1^i, v_1^{i+1}]$$

is a cycle. For $\overrightarrow{C'} = \overrightarrow{T_2}[v_{q_{i-1}}^{i-1}, v_{q_i}^i] \cup \overrightarrow{L}[v_{q_i}^i, u_1^{i+1}] \cup \overrightarrow{T_1}[u_1^{i+1}, u_1^i] \cup \overleftarrow{L}[v_{q_{i-1}}^{i-1}, u_1^i]$,

$$C' \triangle L = L[u_1^{i+1}, v_{q_{i-1}}^{i-1}] \cup T_2[v_{q_{i-1}}^{i-1}, v_{q_i}^i] \cup L[u_1^i, v_{q_i}^i] \cup T_1[u_1^i, u_1^{i+1}]$$

is also a cycle. Thus $\mathcal{C}$ satisfies (C2).

Induction step: $s \geq 3$

Recall that $T_1, T_2, \ldots, T_s$ are all the L-bridges. We first claim the following.

Claim 1 *There exist two vertices u and v on L such that the segment $L[u, v]$ contains all the attachments of some T_i, which is a leaf in $O_G(L)$, and contains no attachments of any other L-bridge that does not overlap with T_i.*

Proof. We choose two vertices u, v such that $L[u, v]$ is as short as possible and contains all the attachments of some T_i which is a leaf in $O_G(L)$. By this choice, both u and v are attachments of T_i. Note that for any other T_j which does not overlap with T_i, if $L[u, v]$ contains one attachment of T_j, then it contains all the attachments of T_j, otherwise T_i and T_j would overlap. Suppose the claim is not true. Then $L[u, v]$ must contain all the attachments of T_j for some $j \neq i$, where T_j does not overlap with T_i. It is clear that $\partial(T_j)$ lies in $L[u^+, v^-]$ as the L-bridges are pairwise disjoint. We claim that $L[u^+, v^-]$ contains all the attachments of some L-bridge other than T_i that is a leaf in $O_G(L)$, which contradicts the choice of u and v. Indeed, if T_j is a leaf in $O_G(L)$, then we are done. Assume T_j is not a leaf. Since $O_G(L)$ is a tree, there exists a new leaf T_k such that the unique path P_{jk} in $O_G(L)$ between T_j and T_k does not contain the vertex T_i. Then by an easy inductive argument, it is easy to see that any $T_\ell \in V(P_{jk})$ (note that T_ℓ is not adjacent to T_i, i.e., T_ℓ and T_i are not overlap) has an attachment in $L[x, y]$. Thus all attachments of T_k are contained in $L[u^+, v^-]$. ■

Without loss of generality, by Claim 1, we may assume that u and v are two vertices on L such that $L[u, v]$ contains all the attachments of T_1 and $L[u, v]$ is as short as possible, and T_1 is a leaf in the overlap graph $O_G(L)$ that overlaps only with T_2. Assume

$$u_1^1, \ldots, u_{p_1}^1, v_1^1, \ldots, v_{q_1}^1, u_1^2, \ldots, u_{p_2}^2, v_1^2, \ldots, v_{q_2}^2, \ldots, v_1^{k-1}, \ldots, v_{q_{k-1}}^{k-1}, u_1^k, \ldots, u_{p_k}^k,$$

are the attachments listed in clockwise cyclic order on $L[u, v]$, where $k \geq 2$, $p_i, q_i \geq 1$, the vertices u_j^i's are attachments of T_1, and the vertices v_j^i's are (partial) attachments of T_2.

For two attachments x, y on L, we say that x and y *witness* each other if there are no other attachments on the segments $L[x, y]$ or $L[y, x]$. By our choice of u and v, we have $u = u_1^1$ and $v = u_{p_k}^k$; hence, only u_1^1 or $u_{p_k}^k$ can possibly witness attachments of T_i for $3 \leq i \leq s$.

The remainder of the proof is divided into three subsections, depending on how many vertices in $\{u_1^1, u_{p_k}^k\}$ can witness attachments of T_i for some $3 \leq i \leq s$.

4.1 Neither u_1^1 nor $u_{p_k}^k$ witness attachments of T_i for $3 \leq i \leq s$

Starting from u_1^1 and moving in the counterclockwise direction, let a be the first attachment of T_2 that can be witnessed by u_1^1. Starting from $u_{p_k}^k$ and moving in the clockwise direction, let b be the first attachment of T_2 that may witness an attachment of T_i for $3 \leq i \leq s$. See Fig. 4 for an illustration. Let T_{21} be the subtree of T_2 whose leaves are precisely the attachments on $L[b, a]$, and let T_{22} be the subtree of T_2 whose leaves are precisely the attachments on $L[a, b]$. We have the following.

Claim 1a $E(T_2) = E(T_{21}) \cup E(T_{22})$.

Proof. By the definition of T_{21} and T_{22}, $\partial(T_2) = \partial(T_{21}) \cup \partial(T_{22})$ and $\partial(T_{21}) \cap \partial(T_{22}) \neq \emptyset$. Since $E(T_{21}) \cup E(T_{22}) \subseteq E(T_2)$, it suffices to show that $E(T_2) \subseteq E(T_{21}) \cup E(T_{22})$. Suppose to the contrary that there exists an edge $e \in E(T_2)$, but $e \notin E(T_{21}) \cup E(T_{22})$. Then either T_{21} and T_{22} are contained in the two components of $T_2 - e$, respectively, or $T_{21} \cup T_{22}$ is contained in one component of $T_2 - e$. The former case contradicts the condition that $\partial(T_{21}) \cap \partial(T_{22}) \neq \emptyset$, and the latter case contradicts the condition that $\partial(T_2) = \partial(T_{21}) \cup \partial(T_{22})$. ■

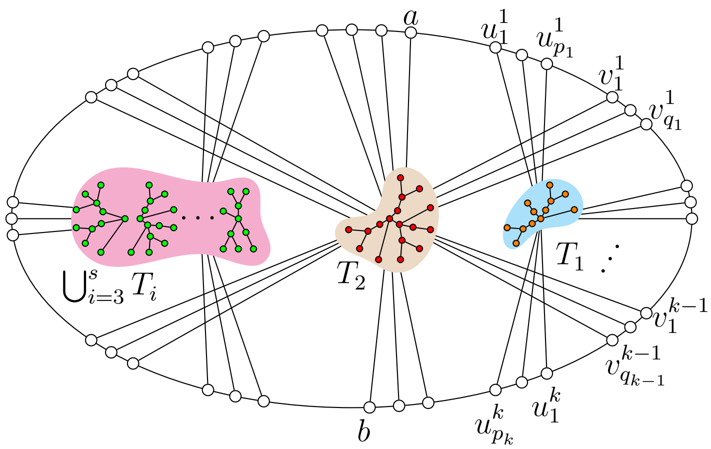


Fig. 4. Case 1

Let G_1 be the subgraph of G induced by the edges in $L, T_{21}, T_3, \ldots, T_s$ and G_2 be the subgraph of G induced by the edges in L, T_{22}, T_1. By induction hypothesis, there exists $\mathcal{C}^i \subseteq \mathcal{C}(G_i)$ which is feasible for (G_i, L), $i = 1, 2$. We have the following.

Claim 1b *The following statements hold.*

- $\overrightarrow{C}_1 = \overrightarrow{T_2}[b, a] \cup \overrightarrow{L}[a, b]$ *is a dicycle of type* 00 *with respect to* a, *of type* 11 *with respect to* b *in* $\mathcal{C}^1$;
- $\overrightarrow{C}_2 = \overrightarrow{T_2}[a, b] \cup \overleftarrow{L}[a, b]$ *is a dicycle of type* 01 *with respect to* a, *of type* 10 *with respect to* b *in* $\mathcal{C}^1$;
- $\overrightarrow{C}_3 = \overrightarrow{T_2}[a, b] \cup \overrightarrow{L}[b, a]$ *is a dicycle of type* 11 *with respect to* a, *of type* 00 *with respect to* b *in* $\mathcal{C}^2$;
- $\overrightarrow{C}_4 = \overrightarrow{T_2}[b, a] \cup \overleftarrow{L}[b, a]$ *is a dicycle of type* 10 *with respect to* a, *of type* 01 *with respect to* b *in* $\mathcal{C}^2$;

Proof. We only prove the first statement, the rest can be proven similarly. By (C4), there exists a dicycle $\overrightarrow{C} \in \mathcal{C}^1$ which is of type 00 with respect to a. Hence (a, a^+) is an arc in $\overrightarrow{C}$. As in G_1, there is not attachments except a and b in $L[a, b]$, so by (C3), $\overrightarrow{C} = \overrightarrow{T_2}[b, a] \cup \overrightarrow{L}[a, b]$. It is clear that this dicycle is of type 11 with respect to b. ■

Then we let $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2 \setminus \{\vec{C}_1, \vec{C}_2, \vec{C}_3, \vec{C}_4\}$, where $\vec{C}_i$ is defined as in Claim 1b. We now verify that $\mathcal{C}$ satisfies (C1)-(C4).

For each edge $e \in E(L)$, $|\mathcal{C}_e^1| = |\mathcal{C}_e^2| = 2$, and it is also covered by exactly two dicycles in $\{\vec{C}_1, \vec{C}_2, \vec{C}_3, \vec{C}_4\}$, so $|\mathcal{C}_e| = |\mathcal{C}_e^1| + |\mathcal{C}_e^2| - 2 = 2$. Consider an arbitrary edge $e \in E(G) \setminus E(L)$. Since $E(T_2) = E(T_{21}) \cup E(T_{22})$ by Claim 1a, we have $e \in E(G_1) \cup E(G_2)$. If $e \in E(G) \setminus [E(T_2[a, b]) \cup E(L)]$, then $e \in E(G_i) \setminus E(L)$ for some $i \in \{1, 2\}$ and $e \notin E(C_j)$ for any $j \in \{1, 2, 3, 4\}$. Therefore $|\mathcal{C}_e| \geq |\mathcal{C}_e^i| \geq 4$, where the last inequality follows from (C1) for (G^i, L). If $e \in E(T_2[a, b])$, since $E(T_2[a, b]) \subseteq E(T_{21}) \cap E(T_{22})$, and $e \in E(C_i)$ for each $i \in \{1, 2, 3, 4\}$, then we can obtain

$$|\mathcal{C}_e| = |\mathcal{C}_e^1| + |\mathcal{C}_e^2| - 4 \geq 4 + 4 - 4 = 4.$$

Therefore, (C1) holds.

Since each dicycle in $\mathcal{C}^1 \cup \mathcal{C}^2$ satisfies (C2) by induction, $\mathcal{C}$ is a subset of $\mathcal{C}^1 \cup \mathcal{C}^2$, $\mathcal{C}$ satisfies (C2) too.

By Claim 1b and that (C4) holds for $\mathcal{C}^1$, $\vec{C}_1$ and $\vec{C}_2$ are the only two dicycles in $\mathcal{C}^1$ containing vertices in $L[a^+, b^-]$. Thus every dicycle in $\mathcal{C}^1 \setminus \{\vec{C}_1, \vec{C}_2\}$ contains no attachments of T_1. On the other hand, as $\mathcal{C}^1$ satisfies (C3) by induction hypothesis, every dicycle in $\mathcal{C}^1 \setminus \{\vec{C}_1, \vec{C}_2\}$ contains either no or two attachments of T_i for $i = 2, 3, \ldots, s$. Similarly, we can verify that each dicycle in $\mathcal{C}^2 \setminus \{\vec{C}_3, \vec{C}_4\}$ contains no attachments of T_i for $i = 3, 4, \ldots, s$ and contains either no or exactly two attachments of each of T_1 and T_2. Thus $\mathcal{C}$ satisfies (C3).

For each $ij \in \mathbb{Z}_2^2$ and attachment w distinct from a or b, every dicycle of type ij with respect to w in $\mathcal{C}^1$ or $\mathcal{C}^2$ is kept in $\mathcal{C}$. For a or b, there are exactly two dicycles of type ij with respect to it in $\mathcal{C}^1 \cup \mathcal{C}^2$, but we removed one for each type by Claim 1b. Thus, $\mathcal{C}$ satisfies (C4).

4.2 Exactly one of u_1^1 and $u_{p_k}^k$ witnesses attachments of T_i for $3 \leq i \leq s$

Without loss of generality, we assume that only u_1^1 witnesses a vertex w, which is an attachment of T_i for some $3 \leq i \leq s$. Starting from $u_{p_k}^k$ and moving in the clockwise direction, let b be the first attachment of T_2 that witnesses an attachment of T_i for some $3 \leq i \leq s$. Let x be the unique vertex in T_2 adjacent to b. We construct a new graph G^* obtained from G by subdividing the edge ww^+ to waw^+ and adding a new edge between x and the new vertex a. See Fig. 5 for illustration. Let L' be the corresponding subdivision of L. Define $T_2^* = T_2 \cup xa$, and let T_{21}^* be the subtree of T_2^* whose leaves are precisely the attachments in $L'[b, a]$, and T_{22}^* be the subtree of T_2^* whose leaves are precisely the attachments in $L'[a, b]$. By arguments similar to those in Claim 1a, we obtain the following.

Claim 2a $E(T_2^*) = E(T_{21}^*) \cup E(T_{22}^*)$.

Let G_1^* be the subgraph of G^* induced by the edges in $L', T_{21}^*, T_3, \ldots, T_s$ and G_2^* be the subgraph of G^* induced by the edges in L', T_{22}^*, T_1. See again Fig. 5 for illustration. By induction hypothesis, there exists $\mathcal{C}^1 \in \mathcal{C}(G_1^*)$ that is feasible for (G_1^*, L'). Instead of applying induction on (G_2^*, L'), we construct $\mathcal{C}^2$ that is feasible for (G_2^*, L'), following the construction in the basic step with $s = 2$. More precisely, T_{22}^* plays the role of T_2, a plays the role of $v_{q_k}^k$

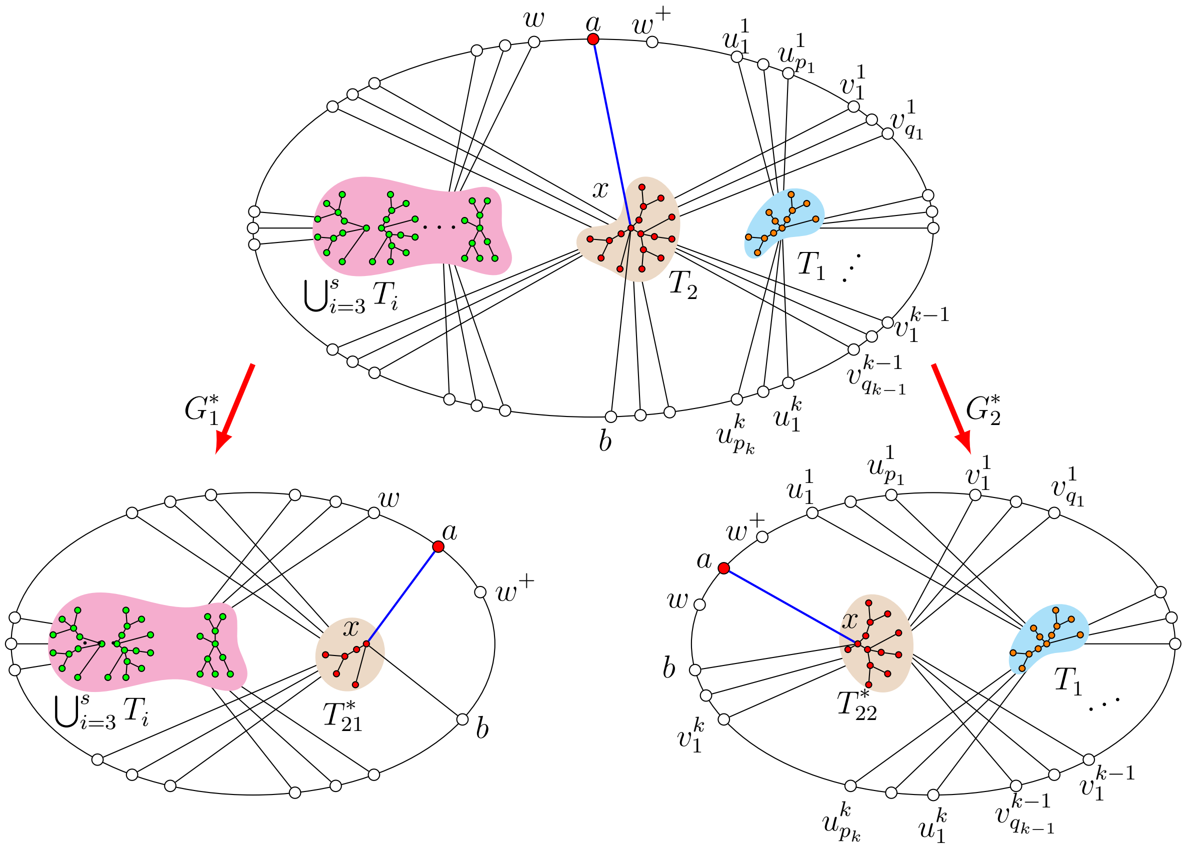


Fig. 5. Illustration for the construction of G^*, G_1^* and G_2^*.

and b plays the role of $v^k_{q_k-1}$, and the remaining attachments of T^*_{22} along $L[u^k_{p_k}, b^-]$ play the role of $v^k_1, v^k_2, \ldots, v^k_{q_k-2}$ in order, respectively.

Note that both a and b are two attachments if T^*_{2i} in G^*_i for $i = 1, 2$. By (C4) for (G^*_1, L') and (G^*_2, L'), for any $ij \in \mathbb{Z}^2_2$ and $t \in \{1, 2\}$, there is a unique dicycle in $\mathcal{C}^t$ which is of type ij with respect to a, we denote such dicycle as $\overrightarrow{C}^{ij}_t(a)$, and denote the other attachment of T_2 contained in $\overrightarrow{C}^{ij}_t(a)$ as a^{ij}_t. We denote $\overrightarrow{C}^{ij}_t(b)$ similarly. Also, we denote by $\mathcal{C}^i(a)$ all the dicycles in $\mathcal{C}^i$ containing a for $i = 1, 2$. The following claim lists six of the dicycles in $\mathcal{C}^1(a) \cup \mathcal{C}^2(a)$ and some properties of the rest two.

Claim 2b *The following statements hold.*

(1) $\overrightarrow{C}^{00}_1(a) = \overrightarrow{C}^{11}_1(b) = (b, x) \cup (x, a) \cup \overrightarrow{L}[a, b]$.

(2) $\overrightarrow{C}^{01}_1(a) = \overrightarrow{C}^{10}_1(b) = (a, x) \cup (x, b) \cup \overleftarrow{L}[a, b]$.

(3) $\overrightarrow{C}^{10}_2(a) = \overrightarrow{C}^{01}_2(b) = (b, x) \cup (x, a) \cup \overleftarrow{L}[b, a]$.

(4) $\overrightarrow{C}^{11}_2(a) = \overrightarrow{C}^{00}_2(b) = (a, x) \cup (x, b) \cup \overrightarrow{L}[b, a]$.

(5) $\overrightarrow{C}^{00}_2(a) = \overrightarrow{T^*_2}[v^{k-1}_{q_{k-1}}, a] \cup \overrightarrow{L'}[a, u^1_1] \cup \overrightarrow{T_1}[u^1_1, u^k_1] \cup \overleftarrow{L'}[v^{k-1}_{q_{k-1}}, u^k_1]$. *(Hence* $a^{00}_2 = v^{k-1}_{q_{k-1}}$*)*

(6) $\overrightarrow{C}^{01}_2(a) = \overrightarrow{T^*_2}[a, v^1_{q_1}] \cup \overrightarrow{L'}[v^1_{q_1}, u^2_1] \cup \overrightarrow{T_1}[u^2_1, u^1_1] \cup \overleftarrow{L'}[a, u^1_1]$. *(Hence* $a^{01}_2 = v^1_{q_1}$*)*

(7) Neither $\overrightarrow{C}_1^{11}(a)$ nor $\overrightarrow{C}_1^{10}(a)$ contain the vertex b.

Proof. The first two can be proven as in the proof of Claim 1b. The third through sixth statements follow from the construction of $\mathcal{C}^2$. We focus on the last one, and only prove that $\overrightarrow{C}_1^{11}(a)$ does not contain b, the other part can be shown similarly. Suppose to the contrary, b is contained in $\overrightarrow{C}_1^{11}(a)$. Since $\overrightarrow{C}_1^{11}(a)$ is of type 11 with respect to a, we know that $L'[a,b]$ is not contained in $\overrightarrow{C}_1^{11}(a)$, hence the cycle $L'[a,b] \cup T_{21}^*[a,b]$ is contained in $L' \triangle C_1^{11}(a)$. As $\mathcal{C}^1$ holds for (C2), it must be that $L' \triangle C_1^{11}(a) = L'[a,b] \cup T_{21}^*[a,b]$. This implies that $\overrightarrow{C}_1^{11}(a) = L'[b,a] \cup T_{21}^*[a,b]$. Hence $\overrightarrow{C}_1^{11}(a)$ contains all the attachments of T_{21}^*. Since T_2 overlaps with some T_i for $i \in \{3,4,\ldots,s\}$, T_{21}^* must contain another attachment other than a and b. So $\overrightarrow{C}_1^{11}(a)$ contains at least three attachments of T_{21}^*, which violates (C3). ■

Now we are going to construct $\mathcal{C}$ that is feasible for (G, L). To do this, we introduce several auxiliary subgraphs and notations that will be used in the construction.

Given two dicycles $\overrightarrow{C}_1$ and $\overrightarrow{C}_2$ such that their intersection $C_1 \cap C_2$ is a path containing at least one edge, and such that the orientations of $\overrightarrow{C}_1$ and $\overrightarrow{C}_2$ on this common path are opposite, we define $\overrightarrow{C}_1 \triangle \overrightarrow{C}_2$ to be the digraph whose underlying graph is $C_1 \triangle C_2$, with arc directions inherited from $\overrightarrow{C}_1 \cup \overrightarrow{C}_2$. It is clear that $\overrightarrow{C}_1 \triangle \overrightarrow{C}_2$ is also a dicycle.

Let $Q_1 = T_2[x, a_1^{11}] \cap T_2[x, a_2^{00}]$ and $Q_2 = T_2[x, a_1^{10}] \cap T_2[x, a_2^{01}]$. Note that both Q_1 and Q_2 are paths in T_2, and may consist of only the single vertex x. Since $C_1^{11}(a)$ and $C_2^{00}(a)$ share common edges and vertices only within T_2, it follows that

$$C_1^{11}(a) \cap C_2^{00}(a) = xa \cup Q_1.$$

Similarly, we have that

$$C_1^{10}(a) \cap C_2^{01}(a) = xa \cup Q_2.$$

On the other hand, by the definitions of $\overrightarrow{C}_1^{11}(a)$ and $\overrightarrow{C}_2^{00}(a)$, the orientations of these two dicycles are opposite on $xa \cup Q_1$. Thus $\overrightarrow{C}_1^{11}(a) \triangle \overrightarrow{C}_2^{00}(a)$ is also a dicycle. Let $\overrightarrow{C}_1^{\text{new}}$ be the dicycle obtained from $\overrightarrow{C}_1^{11}(a) \triangle \overrightarrow{C}_2^{00}(a)$ by contracting $w \to a \to w^+$ to $w \to w^+$. Similarly, let $\overrightarrow{C}_2^{\text{new}}$ be the dicycle obtained from $\overrightarrow{C}_1^{10}(a) \triangle \overrightarrow{C}_2^{01}(a)$ by contracting $w^+ \to a \to w$ to $w^+ \to w$. See Fig. 6 for illustration.

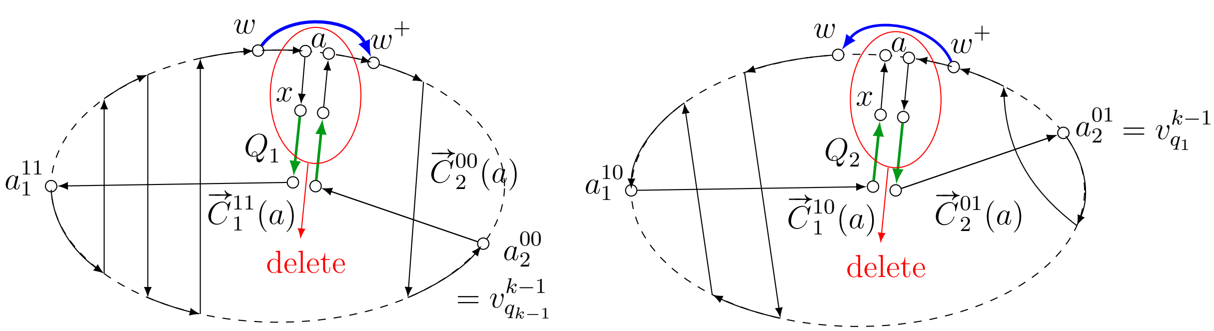


Fig. 6. Illustration for the construction of $\overrightarrow{C}_1^{\text{new}}$ (left) and $\overrightarrow{C}_2^{\text{new}}$ (right). The underlying graph of the green diptahs in the left figure are both Q_1, while those that in the right figure are both Q_2.

Now we construct $\mathcal{C}$ as follows, let

$$\mathcal{C} = \big(\mathcal{C}^1 \cup \mathcal{C}^2 \setminus [\mathcal{C}^1(a) \cup \mathcal{C}^2(a)]\big) \cup \{\vec{C}_1^{\text{new}}, \vec{C}_2^{\text{new}}\}.$$

In the rest of this subsection, we check that $\mathcal{C}$ satisfies (C1)-(C4).

We first show that $\mathcal{C}$ satisfies (C1). It is not difficult to verify that $|\mathcal{C}_e| = 2$ for each $e \in E(L)$. We then focus on edges in $E(G) \setminus E(L)$.

As each edge in Q_1 is also in $T_{21}^*[b, a_1^{11}] \cap T_{22}^*[b, a_2^{00}]$, and each edge in Q_2 is also in $T_{21}^*[b, a_1^{10}] \cap T_{22}^*[b, a_2^{01}]$, the following holds.

Claim 2c *For each edge $e \in E(Q_1) \cup E(Q_2) \cup \{xb\}$, we have $e \in E(T_{21}^* \cap T_{22}^*)$.*

Assume $e \in E(G) \setminus E(L)$, clearly $e \neq xa$. Then we consider the following cases.

- $e = xb$. Note that by the first four items of Claim 2b, e is contained in each of the cycles $C_1^{00}(a)$, $C_1^{01}(a)$, $C_2^{10}(a)$, $C_2^{11}(a)$. By the last three items of Claim 2b, e is not contained in each of $C_1^{10}(a)$, $C_1^{11}(a)$, $C_2^{00}(a)$, $C_2^{01}(a)$. On the other hand, by Claim 2c, $e \in E(G_1^*) \cap E(G_2^*)$. Therefore, $|\mathcal{C}_e| = |\mathcal{C}_e^1| + |\mathcal{C}_e^2| - 4 \geq 4$.

- $e \in E(Q_1) \cup E(Q_2)$. By Claim 2c, $e \in E(G_1^*) \cap E(G_2^*)$. On the other hand, e is not contained in any of the four dicycles $\vec{C}_1^{00}(a)$, $\vec{C}_1^{01}(a)$, $\vec{C}_2^{10}(a)$, $\vec{C}_2^{11}(a)$, but it is possible that e is in $C_1^{11}(a) \cap C_2^{00}(a)$ or $C_1^{10}(a) \cap C_2^{01}(a)$ as $e \in E(Q_1) \cup E(Q_2)$. Hence $|\mathcal{C}_e| \geq |\mathcal{C}_e^1| + |\mathcal{C}_e^2| - 4 \geq 4$.

- $e \in E(G) \setminus [E(L) \cup E(Q_1) \cup E(Q_2) \cup \{bx\}]$. Note that in this case, it is not possible that $e \in E(C_1^{11}(a)) \cup E(C_2^{00}(a))$ but $e \notin E(C_1^{\text{new}})$ as $e \notin E(Q_1)$. Similarly, it is not possible that $e \in E(C_1^{10}(a)) \cup E(C_2^{01}(a))$ but $e \notin E(C_2^{\text{new}})$ as $e \notin E(Q_2)$. Then by Claim 2a, e is contained in $E(G_i^*) \setminus (E(L') \cup \{xa\})$ for some $i \in \{1, 2\}$. On the other hand, e is not contained in any of the four dicycles $\vec{C}_1^{00}(a)$, $\vec{C}_1^{01}(a)$, $\vec{C}_2^{10}(a)$, $\vec{C}_2^{11}(a)$. Thus, as $\mathcal{C}^i$ is feasible for (G_i^*, L'), we have $|\mathcal{C}_e| \geq |\mathcal{C}_e^i| \geq 4$.

Therefore, we proved that $\mathcal{C}$ satisfies (C1).

By our construction of $\mathcal{C}$, to check that $\mathcal{C}$ satisfies (C2), it suffices to check that both $C_1^{\text{new}} \triangle L$ and $C_2^{\text{new}} \triangle L$ are cycles. Observe that $C_1^{11}(a) \cap C_2^{00}(a) = Q_1 \cup xa$ and $C_1^{10}(a) \cap C_2^{01}(a) = Q_2 \cup xa$. Since $C_t^{ij}(a) \triangle L'$ is a cycle for $ij \in \mathbb{Z}_2^2$, $t \in \{1, 2\}$, we decompose four of them to edge-disjoint paths as follows.

$$\begin{aligned}
C_1^{11}(a) \triangle L' &= xa \cup Q_1 \cup L'[a, b] \cup R_1, \\
C_2^{00}(a) \triangle L' &= xa \cup Q_1 \cup L'[b, a] \cup R_2, \\
C_1^{10}(a) \triangle L' &= xa \cup Q_2 \cup L'[a, b] \cup R_3, \\
C_2^{01}(a) \triangle L' &= xa \cup Q_2 \cup L'[b, a] \cup R_4,
\end{aligned}$$

where R_1 and R_3 are in G_1^*, R_2 and R_4 are in G_2^*. In addition, R_1 and R_3 share only the vertices b and one endpoint of Q_1 distinct from x, and similarly, R_2 and R_4 share only the vertices b and one endpoint of Q_2 distinct from x. It follows that $C_1^{\text{new}} \triangle L = R_1 \cup R_2$ and $C_2^{\text{new}} \triangle L = R_3 \cup R_4$ are both cycles. Thus $\mathcal{C}$ satisfies (C2).

For (C3), we only need to focus on C_1^{new} and C_2^{new}, the rest can be verified similarly as in Section 4.1. It is clear that C_1^{new} contains two attachments of T_2, which are a_1^{11} and a_2^{00}, and

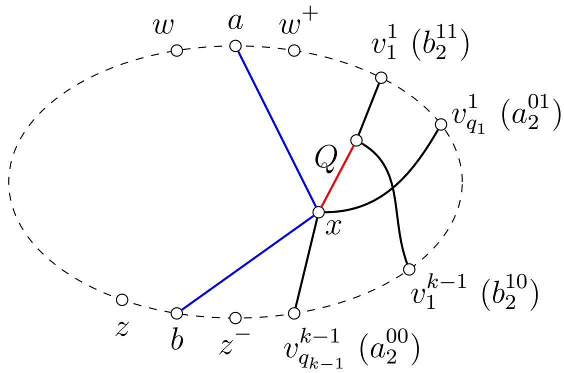


Fig. 7. Illustration for the choice of x and xa, xb. The red path is Q.

contains either no or two attachments of T_j for $j \neq 2$ as (C3) holds for $C_1^{11}(a)$ and $C_2^{00}(a)$. Similarly, we can verify that C_2^{new} satisfies (C3).

Now we show that $\mathcal{C}$ satisfies (C4). This statement holds for b by the similar arguments as in Section 4.1. Thus we assume w is an attachment distinct from b. If w is not contained in any dicycle in $\mathcal{C}^1(a) \cup \mathcal{C}^2(a)$, then each dicycle containing w is kept in $\mathcal{C}$. So we assume that w is contained in at least one dicycle in $\mathcal{C}^1(a) \cup \mathcal{C}^2(a)$. By Claim 2b, w is not contained in any of $\overrightarrow{C}_1^{00}(a)$, $\overrightarrow{C}_1^{01}(a)$, $\overrightarrow{C}_2^{11}(a)$ and $\overrightarrow{C}_2^{10}(a)$, each of which contains two attachments a and b of T_2^*. If w is contained in $\overrightarrow{C}_1^{11}(a)$ (resp., $\overrightarrow{C}_2^{00}(a)$) which is the unique dicycle of type ij with respect to w for some $ij \in \mathbb{Z}_2^2$, then $\overrightarrow{C}_1^{\text{new}}$ is also the unique dicycle of type ij with respect to w. If w is contained in $\overrightarrow{C}_1^{10}(a)$ (resp., $\overrightarrow{C}_2^{00}(a)$) which is the unique dicycle of type ij with respect to w for some $ij \in \mathbb{Z}_2^2$, then $\overrightarrow{C}_2^{\text{new}}$ is also the unique dicycle of type ij with respect to w.

4.3 Both u_1^1 and $u_{p_k}^k$ witness attachments of T_i for $3 \leq i \leq s$

In this case, let w be the first attachment that u_1^1 witnesses when moving in the counter-clockwise direction, and let z be the first attachment that $u_{p_k}^k$ witnesses when moving in the clockwise direction. Here, w and z are attachments of T_i and T_j, respectively, for some $3 \leq i, j \leq s$ (possibly with $i = j$).

We choose a non-leaf vertex x in T_2 as follows. If $v_1^1 = v_{q_1}^1 = v_1^{k-1} = v_{q_{k-1}}^{k-1}$ (when $k = 2$ and $q_1 = 1$), then let x be the unique neighbor of $v_{q_1}^1$ in T_2. Otherwise, let x be a non-leaf vertex in the subtree of T_2 whose leaves are exactly $\{v_1^1, v_{q_1}^1, v_1^{k-1}, v_{q_{k-1}}^{k-1}\}$. We choose x so that its degree in this subtree is as large as possible. In addition, if $v_{q_1}^1 \neq v_{q_{k-1}}^{k-1}$, then make sure that x is in $T_2[v_{q_1}^1, v_{q_{k-1}}^{k-1}]$. See Fig. 7 for illustration.

We construct a new graph G^* which obtained from G by subdividing ww^+ to waw^+ and adding a new edge between x and the new vertex a, and subdividing zz^- to zbz^- and adding an edge between x and the new vertex b. Let L' be the corresponding subdivision of L. Let $T_2^*, T_{21}^*, T_{22}^*, G_1^*, G_2^*, \mathcal{C}^1, \mathcal{C}^2, \overrightarrow{C}_t^{ij}(a), a_t^{ij}$ be defined or obtained as last section. Similarly, we define $\overrightarrow{C}_t^{ij}(b)$ and b_t^{ij} for b as $\overrightarrow{C}_t^{ij}(a)$ and a_t^{ij}, respectively. Obviously, Claim 2a also holds here.

Let $Q_1 = T_2[x, a_1^{11}] \cap T_2[x, a_2^{00}]$, $Q_2 = T_2[x, a_1^{10}] \cap T_2[x, a_2^{01}]$, $Q_3 = T_2[x, b_1^{00}] \cap T_2[x, b_2^{11}]$ and $T_2[x, b_1^{10}] \cap T_2[x, b_2^{01}]$. The following is obviously true.

Claim 3a *Every edge in* $E(Q_1) \cup E(Q_2) \cup E(Q_3) \cup E(Q_4) \cup \{xa, xb\}$ *is also in* $E(T^*_{21} \cap T^*_{22})$*.*

By (C3) and (C4) for $\mathcal{C}^1$ and the construction of $\mathcal{C}^2$, Claim 2b also holds here. In addition, we have the following.

Claim 3b *The following statements hold.*

(1) $\overrightarrow{C}^{10}_2(b) = \overrightarrow{T^*_2}[v^{k-1}_1, b] \cup \overleftarrow{L'}[u^k_{p_k}, b] \cup \overrightarrow{T_1}[u^k_{p_k}, u^{k-1}_{p_{k-1}}] \cup \overrightarrow{L'}[u^{k-1}_{p_{k-1}}, v^{k-1}_1]$*. (Hence* $b^{10}_2 = v^{k-1}_1$*)*

(2) $\overrightarrow{C}^{11}_2(b) = \overrightarrow{T^*_2}[b, v^1_1] \cup \overleftarrow{L'}[u^1_{p_1}, v^1_1] \cup \overrightarrow{T_1}[u^1_{p_1}, u^k_{p_k}] \cup \overrightarrow{L'}[u^k_{p_k}, b]$*. (Hence* $b^{11}_2 = v^1_1$*)*

Thus by Claim 2b and Claim 3b, $a^{00}_2 = v^{k-1}_{q_{k-1}}$, $a^{01}_2 = v^1_{q_1}$, $b^{10}_2 = v^{k-1}_1$ and $b^{11}_2 = v^1_1$. Let T_a be the subtree of T_2 whose leaves are precisely $\{a^{00}_2, a^{01}_2, a^{10}_1, a^{11}_1\}$, and let T_b be the subtree of T_2 whose leaves are precisely $\{b^{00}_1, b^{01}_1, b^{10}_2, b^{11}_2\}$. We claim the following.

Claim 3c $E(Q_1) \cap E(Q_2) \cap E(T_a) = \emptyset$ *and* $E(Q_3) \cap E(Q_4) \cap E(T_b) = \emptyset$*. In particular, if* x *is in* T_a*, then* $E(Q_1) \cap E(Q_2) = \emptyset$*. Similarly, if* x *is in* T_b*, then* $E(Q_3) \cap E(Q_4) = \emptyset$*.*

Proof. Assume to the contrary, $e \in E(Q_1) \cap E(Q_2)$ and also $e \in E(T_a)$. Then $T_a - e$ contains two components, say T_{a1} and T_{a2}. Let y be the vertex in $Q_1 \cap Q_2 \cap T_a$ that is closest to x along Q_1. Clearly, y is also in Q_2 for otherwise, there is a cycle in T_2. Without loss of generality, assume $y \in V(T_{a1})$. By the definition of Q_1 and Q_2, it must be that all of $a^{00}_2, a^{01}_2, a^{10}_1, a^{11}_1$ are contained in T_{a2} as $e \in\in E(Q_1) \cap E(Q_2)$. However, by definition, T_a is the minimal subtree of T_2 containing all of these four vertices. This implies that $T_a \subseteq T_{a2}$ and hence does not contain y, which contradicts the assumption that $y \in V(T_{a1})$.

Similarly, we can prove $E(Q_3) \cap E(Q_4) \cap E(T_b) = \emptyset$.

Note that if x is in T_a, and $E(Q_1) \cap E(Q_2) \neq \emptyset$, then $E(Q_1) \cap E(Q_2) \cap E(T_a) \neq \emptyset$, a contradiction. Similarly, we can show that if x is in T_b, then $E(Q_3) \cap E(Q_4) = \emptyset$. ■

Claim 3d *Each edge in* $E(G)$ *is contained in at most two of* Q_1, Q_2, Q_3, Q_4*.*

Proof. First assume that $v^{k-1}_{q_{k-1}} = v^1_{q_1}$, it follows that $a^{00}_2 = a^{01}_2$ (by the fifth and sixth items of Claim 2b), hence T_a has at most three leaves. Thus at least one of Q_1 and Q_2 is empty. Without loss of generality, assume $Q_1 = \emptyset$. If $v^1_1 = v^1_{q_1} = v^{k-1}_1 = v^{k-1}_{q_{k-1}}$, then it is easy to see that $Q_1 = Q_2 = Q_3 = Q_4 = \emptyset$, the claim is obviously true. Otherwise, by the choice of x, we know that x must lie in $T_2[v^1_1, v^{k-1}_1]$ and hence in T_b. By Claim 3c, $E(Q_3) \cap E(Q_4) = \emptyset$. Together with the fact that $Q_1 = \emptyset$, the claim holds.

Assume that $v^{k-1}_{q_{k-1}} \neq v^1_{q_1}$. By the choice of x, we have that x is in $T_2[v^1_{q_1}, v^{k-1}_{q_{k-1}}] = T_2[a^{01}_2, a^{01}_2]$, hence in T_a. By Claim 3c, $E(Q_1) \cap E(Q_2) = \emptyset$.

If x is also in T_b, then again by Claim 3c, $E(Q_3) \cap E(Q_4) = \emptyset$. Thus there is no edge could be in three of $E(Q_1), E(Q_2), E(Q_3), E(Q_4)$, the claim follows.

Assume that x is not contained in T_b. Then, by the choice of x, the subpaths $T_2[a^{00}_2, a^{01}_2]$ and $T_2[b^{10}_2, b^{11}_2]$ are vertex-disjoint, see Fig. 7 for illustration. Since T_2 is a tree, there exists a unique path Q in T_2 joining these two subpaths. By the choice of x, it must be one endpoint of Q, and we denote the other endpoint by y. Clearly, y is in $T_2[b^{10}_2, b^{11}_2]$, and hence it is also in T_b. Most importantly, we know that $Q_3 \cap Q_4 = Q$ by Claim 3c and the definition of Q_3, Q_4, Q.

By the choice of x, both $Q_1 \subset T_2[a_2^{00}, a_2^{01}]$ and $Q_2 \subset T_2[a_2^{00}, a_2^{01}]$, but Q is edge-disjoint to $T_2[a_2^{00}, a_2^{01}]$, so we have

$$E(Q_3) \cap E(Q_4) \cap (E(Q_1) \cup E(Q_2)) = E(Q) \cap (E(Q_1) \cup E(Q_2)) = \emptyset.$$

Together with the fact that $E(Q_1) \cap E(Q_2) = \emptyset$, the claim follows. ■

Now we shall construct $\mathcal{C}$ that is feasible for (G, L). Let

$$\mathcal{C} = \left(\mathcal{C}^1 \cup \mathcal{C}^2 \setminus [\mathcal{C}^1(a) \cup \mathcal{C}^2(a) \cup \mathcal{C}^1(b) \cup \mathcal{C}^2(b)]\right) \cup \{\overrightarrow{C}_1^{\text{new}}, \overrightarrow{C}_2^{\text{new}}, \overrightarrow{C}_3^{\text{new}}, \overrightarrow{C}_4^{\text{new}}\},$$

where $\overrightarrow{C}_1^{\text{new}}$ and $\overrightarrow{C}_2^{\text{new}}$ are defined as last subsection, $\overrightarrow{C}_3^{\text{new}}$ and $\overrightarrow{C}_4^{\text{new}}$ are defined as follows.

- $\overrightarrow{C}_3^{\text{new}}$ is obtained from $\overrightarrow{C}_1^{00}(b) \triangle \overrightarrow{C}_2^{11}(b)$ by contracting $z^- \to b \to z$ to $z^- \to z$.
- $\overrightarrow{C}_4^{\text{new}}$ is obtained from $\overrightarrow{C}_1^{01}(b) \triangle \overrightarrow{C}_2^{10}(b)$ by contracting $z \to b \to z^-$ to $z \to z^-$.

The verification of $\mathcal{C}$ satisfying (C2)-(C4) is analogous to the last subsection, except that there are additional dicycles $\overrightarrow{C}_3^{\text{new}}$ and $\overrightarrow{C}_4^{\text{new}}$, which need to be verified for (C2)-(C3), and one additional vertex b to be considered for (C4). We therefore omit the repeated arguments. We can also deduce that $|\mathcal{C}_e| = 2$ for $e \in E(L)$ and $|\mathcal{C}_e| \geq 4$ for each $e \in E(G) \setminus [E(L) \cup E(T_{21}^* \cap T_{22}^*)]$ as in that part. Thus it remains to verify (C1) for the edges in $[E(T_{21}^* \cap T_{22}^*)] \setminus \{xa, xb\}$.

By Claims 2b, 3b, and the definition of Q_1, Q_2, Q_3, Q_4, we have the following.

$$C_1^{11}(a) \cap C_2^{00}(a) = xa \cup Q_1, \quad C_1^{10}(a) \cap C_2^{01}(a) = xa \cup Q_2,$$
$$C_1^{00}(b) \cap C_2^{11}(b) = xb \cup Q_3, \quad C_1^{01}(b) \cap C_2^{10}(b) = xb \cup Q_4.$$

Therefore, by Claims 3a, 3d, $|\mathcal{C}_e| \geq |\mathcal{C}_e^1| + |\mathcal{C}_e^2| - 4 \geq 4$ for each edge $e \in [E(T_{21}^* \cap T_{22}^*)] \setminus \{xa, xb\}$.

This completes the proof of Theorem 5. ■

5 Concluding remarks

As shown in Voss [35], the case $k \leq 3$ of Conjecture 2 provides a useful estimate of the number of edges in certain subgraphs of the bridges of a longest cycle, which has proved valuable in many problems on cycles. Hence, we would expect that resolving this conjecture (i.e., Theorem 3) may lead to further applications in the study of cycles.

One potential application of Theorem 3 concerns the size of the intersection of two longest cycles in highly connected graphs. A well-known conjecture (see [14, 16]), often attributed to Scott Smith, asserts the following.

Conjecture 9 *In k-connected graphs, any two longest cycles intersect in at least k vertices.*

There has been extensive research on this conjecture; see [7, 16, 26] and recent results [15, 22]. Let C and D be two longest cycles in a 2-connected graph G (so that they intersect in at least two vertices), and let $H = C \cup D$ denote the 2-connected subgraph formed by their union. The C-bridges of H are all subpaths of D, and existing approaches to Conjecture 9 often analyze how these bridges are arranged along C. In this context, Conjecture 9 aligns closely with the essence of Conjecture 2, and our main result may provide a useful tool for studying this structure in more detail. This conjecture is also closely related to the famous conjecture of Lovász on the circumference of vertex-transitive graphs (see [3, 12, 15, 22, 23] for details).

Another potential application of Theorem 3 is related to a problem of Babai [3], which asks about the intersection size of two longest cycles in a 3-connected cubic graph.

Problem 10 (Babai [3], Problem 2) *Let $f(c)$ denote the largest integer with the following property: if a 3-connected graph has circumference c, then any two longest cycles of the graph intersect in at least $f(c)$ vertices. Does $f(c) \to \infty$ as $c \to \infty$?*

Suppose C and D are two longest cycles of a 3-connected cubic graph G, and let $H = C \cup D$, which is a 2-connected subgraph of G. Since G is cubic, all C-bridges of H, say $B_1, \ldots, B_k$, are vertex-disjoint subpaths of D, whose union is exactly $E(D) \setminus E(C)$, so that $\sum_{i=1}^{k} \lambda(B_i) = |E(D) \setminus E(C)|$. Now suppose that $O_H(C)$ forms a tree. By Theorem 3, we then have $\sum_{i=1}^{k} \lambda(B_i) \leq |E(C)|/2$. Combining these, we obtain

$$|E(C \cap D)| = |E(D)| - |E(D) \setminus E(C)| = |E(C)| - \sum_{i=1}^{k} \lambda(B_i) \geq |C|/2.$$

This bound is certainly too strong to hold in general. A more practical approach is to analyze the overlap graph of $O_H(C)$ and attempt to decompose it into small trees, so that Theorem 3 can be applied to each tree in a meaningful way.

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