

A clique version of the Erdős-Gallai stability theorems

Jie Ma*

Long-Tu Yuan†

Abstract

Combining Pósa’s rotation lemma with a technique of Kopylov in a novel approach, we prove a generalization of the Erdős-Gallai theorems on cycles and paths. This implies a clique version of the Erdős-Gallai stability theorems and also provides alternative proofs for some recent results.

1 Introduction

The *circumference* $c(G)$ of a graph G is the length of a longest cycle in G . For $s \geq 2$, let $N_s(G)$ denote the number of unlabeled copies of the clique K_s in G . For integers $n \geq k \geq 2a$, let $H(n, k, a)$ be the n -vertex graph whose vertex set is partitioned into three sets A, B, C such that $|A| = a, |B| = n - k + a$ and $|C| = k - 2a$ and the edge set consists of all edges between A and B together with all edges in $A \cup C$. Let $h_s(n, k, a) = N_s(H(n, k, a))$.

The celebrated Erdős-Gallai theorem [2] states that any n -vertex graph G with $c(G) < k$ has at most $\frac{k-1}{2}(n-1)$ edges. This was improved by Kopylov [6] by showing that any n -vertex 2-connected graph G with $c(G) < k$ has at most $\max\{h_2(n, k, 2), h_2(n, k, \lfloor \frac{k-1}{2} \rfloor)\}$ edges. Combined with the results in [4], Füredi, Kostochka, Luo and Verstraëte [5] proved a stability version of Kopylov’s theorem, which says that for any 2-connected graph G with $c(G) < k$, if $e(G)$ is close to the above maximum number from Kopylov’s theorem, then G must be a subgraph of some well-specified graphs. This was further extended in [8] to stability results of 2-connected graphs of any given minimum degree. On the other hand, Luo [7] generalized Kopylov’s theorem by showing that the number of s -cliques in any n -vertex 2-connected graph G with $c(G) < k$ is at most $\max\{h_s(n, k, 2), h_s(n, k, \lfloor \frac{k-1}{2} \rfloor)\}$.

The aim of this paper is to study a new approach and provide some potential tools in this line of research. Following this approach, our main result, Theorem 4.1, considers a general stability setting. To get into the statement, it requires an entangled family of some specified graphs which we will define in Section 2. However, we would like to point out that using Theorem 4.1, one can not only derive alternate proofs of many recent results in [4, 5, 8], but also infer some new results. One such new result is the following stability result of Luo’s theorem [7] on the number of s -cliques, which also can be viewed as a clique version of one of the main results in [4] (see Theorem 1.4 therein).

Theorem 1.1. *Let $k \geq 5$, $\ell = \lfloor (k-1)/2 \rfloor$, $2 \leq s \leq \max\{2, \ell-1\}$ and $n \geq n_0(\ell)$ be integers.¹ Let G be an n -vertex 2-connected graph with $c(G) < k$. Then $N_s(G) \leq h_s(n, k, \ell-1)$ unless*

(a) $s = 3$ and $k \in \{9, 10\}$,

(b) $k = 2\ell + 1$, $k \neq 7$, and $G \subseteq H(n, k, \ell)$, or

*School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China. Email: jiema@ustc.edu.cn. Supported in part by National Natural Science Foundation of China grant 11622110, the project “Analysis and Geometry on Bundles” of Ministry of Science and Technology of the People’s Republic of China, and Anhui Initiative in Quantum Information Technologies grant AHY150200.

†School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, Shanghai 200240, China. Email: ltyuan@math.ecnu.edu.cn. Supported in part by National Natural Science Foundation of China grant 11901554.

¹We remark that we do not pursue a precise expression on the constant $n_0(\ell)$, which needs to be quite large in the proof. Instead, we pose a conjecture in the end of this paper which would help extend Theorem 1.1 to all values of n .

(c) $k = 2\ell + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most ℓ .²

We would like to mention that Theorem 4.1 can in fact imply a refined version of Theorem 1.1.

The organization of this paper is as follows. In Section 2, we give a formal definition of a family of graphs for the use of our characterization. In Section 3, we prove the key lemma for our proofs which builds on an integration of Pósa's rotation lemma and Kopylov's proof in [6]. In Section 4, we prove our main result Theorem 4.1. In Section 5, we show how to use Theorem 4.1 to deduce Theorem 1.1 as well as some main results in [4, 5, 8]. We also pose a related conjecture for further research to conclude this paper. For the proof ideas, we would like to suggest readers to first look through the odd k case, in which the family defined in Section 2 as well as the proofs will be much more concise. Throughout the rest of the paper, let $k \geq 5$ be an integer and $\ell = \lfloor (k - 1)/2 \rfloor$.

2 Notation and a family of graphs

2.1 Notation

The general notation used in this paper is standard (see, e.g., [1]). For a graph G , let $\omega(G)$ be the order of a maximum clique in G . For disjoint subsets $A, B \subseteq V(G)$, we denote $G(A, B)$ to be the induced bipartite subgraph of G with parts A, B . Let $E(A, B) = E(G(A, B))$ for short. When defining a graph, we will only specify these adjacent pairs of vertices, that says, if a pair $\{a, b\}$ is not discussed as a possible edge, then it is assumed to be a non-edge.

Denote by $N_G(x)$ the set of neighbors of x in G and let $d_G(x)$ be the size of $N_G(x)$. For $U \subseteq V(G)$, let $N_U(x) = N_G(x) \cap U$ and $d_U(x) = |N_U(x)|$. Let $P = x_1x_2 \cdots x_m$ be a path in G . For $x \in V(G)$, let $N_P(x) = N_G(x) \cap V(P)$ and $N_P[x] = N_P(x) \cup \{x\}$, with $d_P(x) := |N_P(x)|$. For $x_i, x_j \in V(P)$, we use x_iPx_j to denote the subpath of P between x_i and x_j . For $x \in V(P)$, denote x^- and x^+ to be the immediate predecessor and immediate successor of x on P , respectively. For $S \subseteq V(P)$, let $S^+ = \{x^+ : x \in S\}$ and $S^- = \{x^- : x \in S\}$. We call $(x_i, x_j)_P$ a *crossing pair* of P if $i < j$, $x_i \in N_P(x_m)$ and $x_j \in N_P(x_1)$. If there is no ambiguity, we write this pair as (i, j) for short. We call a path a *crossing path* if it has a crossing pair. Let $j - i - 1$ be the *length* of the crossing pair (i, j) . A crossing pair (i, j) is *minimal* in P if $x_h \notin N_P(x_1) \cup N_P(x_m)$ for each $i < h < j$. For $S \subseteq V(G)$, we call P an *S -path* if $x_1, x_m \in S$.

For an integer α and a graph G , the α -*disintegration* of G , denoted by $H(G, \alpha)$, is the graph obtained from G by recursively deleting vertices of degree at most α until that the resulting graph has no such vertex.³

2.2 A family of graphs

Let $m \geq k \geq 5$ and $1 \leq r \leq \ell$ be integers. We now devote the rest of this subsection to the definition of a family $\mathcal{F}(m, k, r)$ of some delicate graphs F , each of which has a Hamilton path and satisfies $|V(F)| = m$ and $c(F) < k$.⁴ We divide $\mathcal{F}(m, k, r)$ into the following four classes, namely Types I, II, III and IV. Along the way, we also define some very special graphs (see Figure 1).

Type I: Let $k = 2\ell + 1$ be odd and $r \leq \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type I satisfies:

- $V(F) = A \cup B \cup C \cup D$,
- $F[A]$ and $F[B]$ are cliques on r vertices,
- $F[C]$ is empty with $|C| = \ell - r + 1$,

²A *star forest* is a graph in which every component is a star.

³One can see that $H(G, \alpha)$ is unique in G and has minimum degree at least $\alpha + 1$ (if non-empty).

⁴For the parameter r , roughly speaking we may view it as something close to $\omega(F)$, though its own meaning will be clear in the proof of Lemma 3.2. Readers may treat the coming lengthy definition as a handout and skip to next sections.

- $F[D]$ is empty when $|C| \geq 3$, and $F[D]$ is a path when $|C| = 2$,⁵
- each vertex in $A \cup B$ is adjacent to each vertex in C , and
- $F[C \cup D]$ is a C -path.

Type II: Let $k = 2\ell + 2$ be even and $r \leq \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type II satisfies:

- $V(F) = A \cup B \cup C \cup D$,
- $|A| \in \{r, r + 1\}$ and $F[B]$ is a clique on r vertices,
- $F[C]$ is empty with $|C| = \ell - r + 1$,
- $F[D]$ is a path when $|C| = 2$, and $F[D]$ consists of at most two independent edges and some isolated vertices when $|C| \geq 3$ such that one of the following holds:
 - $F[D]$ is empty when $|A| = r + 1$,
 - $F[D]$ contains a unique edge when $|A| = r$, or
 - $F[D]$ consists of two independent edges when $|A| = r = \ell - 2$.
- each vertex in A has degree exactly ℓ in $F[A \cup C]$ ⁶ and each vertex in B has degree exactly ℓ in $F[B \cup C]$, and
- $F[C \cup D]$ is a C -path satisfying that if $|A| = r + 1$ then each end-vertex of $F[C \cup D]$ is adjacent to some vertex of A . In particular, we denote the graph with $|A| = r = \ell - 2$ and $|D| = 3$ by $F_0(m, k, r)$, the graph with $|A| = r = \ell - 2$ and $|D| = 4$ by $F_4(m, k, r)$, and the graph with $F[A]$ being a star on three vertices by $F_5(m, k, 2)$.

Type III: Let $k = 2\ell + 2$ be even and $r \leq \ell - 1$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type III satisfies:

- $V(F) = A \cup B \cup C \cup D$,
- $F[A]$ and $F[B]$ are cliques on r vertices,
- $F[C]$ is empty with $|C| = \ell - r + 1$,
- $F[D]$ is empty when $|C| \geq 3$, and $F[D]$ consists of a path and an isolated vertex when $|C| = 2$,
- each vertex in $A \cup B$ is adjacent to each vertex in C , and
- $F[C \cup D]$ consists of at most two vertex-disjoint paths such that one of the following holds:
 - $F[C \cup D]$ consists of a C -path and an isolated vertex $x \in D$ such that x is adjacent to exactly two vertices x_1, x_2 of A (denote this graph by $F_1(m, k, r)$),
 - $F[C \cup D]$ is a path with the end-vertex $y \in D$ such that y is an isolated vertex in $F[D]$ and is adjacent to exactly one vertex y_1 in A (denote this graph by $F_2(m, k, r)$), or
 - $F[C \cup D]$ consists of a path with distinct end-vertices $z, z' \in D$ and a path with end-vertices in C satisfying that $|D| = \ell - r + 1$ and z, z' is adjacent to exactly one vertex z_1, z'_1 in A , respectively, with $z_1 \neq z'_1$ (denote this family of graphs by $\mathcal{F}_3(m, k, r)$).

Type IV: Let $k = 2\ell + 2$ be even and $r = \ell$. Each graph $F \in \mathcal{F}(m, k, r)$ of Type IV satisfies:

- $V(F) = A \cup B \cup C$,

⁵An isolated vertex will also be viewed as a (trivial) path in this paper.

⁶Note that if $r = 1$ and $|A| = 2$, then $F[A] = K_2$ (by the fact that F contains a Hamilton path).

- $F[A]$ and $F[B]$ are cliques on $\ell - 1$ vertices, and
- $F[C]$ induces a cycle with three distinct vertices w_1, w_2, w such that $w_1 w_2 \in E(F[C])$, $w w_i \notin E(F[C])$ for $i \in \{1, 2\}$, w_1 is adjacent to each vertex of A , w_2 is adjacent to each vertex of B , and w is adjacent to each vertex of $A \cup B$.

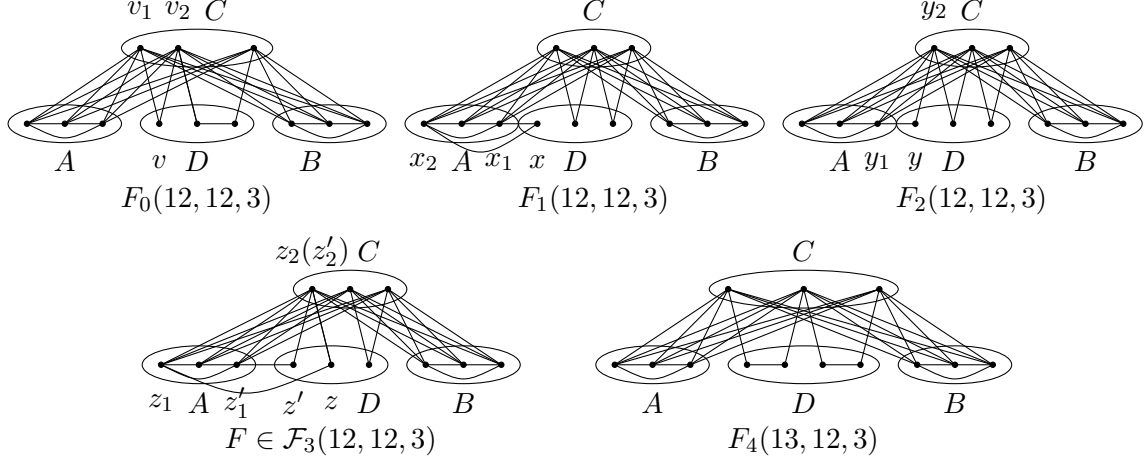


Figure 1. Some special graphs in the family $\mathcal{F}(m, k, r)$

We point out that by definition, there is a Hamilton path in each $F \in \mathcal{F}(m, k, r)$ starting from A and ending at B . Also, if k is odd, then all graphs in $\mathcal{F}(m, k, r)$ have Type I. Furthermore, $F_4(m, k, \ell - 2) = F_4(k + 1, k, \ell - 2)$ is the only graph in $\mathcal{F}(m, k, r)$ with $m > k$ and $r \leq \ell - 2$.

3 A generalization of Pósa's lemma

The following well-known lemma is due to Pósa [9] and is extensively used in extremal graph theory.

Lemma 3.1 (Pósa [9]). *Let G be a 2-connected graph and $P = x_1 x_2 \cdots x_m$ be a path in G . Then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m)\}$. Moreover, let i be the minimum integer such that x_i is adjacent to x_m and j be the maximum integer such that x_j is adjacent to x_1 . If $j = i$, then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m) + 1\}$. In addition, if $j < i$, then G contains a cycle of length at least $\min\{m, d_P(x_1) + d_P(x_m) + 2\}$.*

The following lemma, which combines the ideas of Pósa's lemma [9] and Kopylov's work [6], is the key technical part in the proof of our main theorem. As we shall see in later sections, under certain circumstances, to find some particular useful subgraphs will be enough to determine the whole structure of graphs. Recall that $k \geq 5$ and $\ell = \lfloor (k - 1)/2 \rfloor$.

Lemma 3.2. *Let G be a 2-connected graph with $c(G) < k$. Suppose that the $(\ell - 1)$ -disintegration H of G is non-empty and the longest H -path in G has $m \geq k$ vertices. Then G contains a subgraph $F \in \mathcal{F}(m, k, r)$ for some $r \leq \ell$.*

Proof. We devote the rest of this section to the proof of this lemma. Suppose to the contrary that G does not contain any subgraph in $\mathcal{F}(m, k, r)$ with $r \leq \ell$. Let \mathcal{P} be the family of all longest H -paths in G . We proceed by showing a sequence of claims in what follows.

Claim 1. *Every $P = x_1 x_2 \cdots x_m \in \mathcal{P}$ satisfies the following properties.*

- (i) $N_H(x_1) \subseteq N_P(x_1)$ and $N_H(x_m) \subseteq N_P(x_m)$,
- (ii) $d_P(x_1) \geq d_H(x_1) \geq \ell$ and $d_P(x_m) \geq d_H(x_m) \geq \ell$, and

(iii) $N_P^-(x_1) \cap N_P[x_m] = \emptyset$ and $N_P^+(x_m) \cap N_P[x_1] = \emptyset$.

Proof. Suppose to the contrary that there exists $y \in N_H(x_1) \setminus N_P(x_1)$. Clearly, yx_1Px_m is an H -path longer than P , a contradiction. Therefore, we have $N_H(x_1) \subseteq N_P(x_1)$. Similarly, we have $N_H(x_m) \subseteq N_P(x_m)$. Note that H is the $(\ell - 1)$ -disintegration of G and H is non-empty. It follows that $d_P(x_1) \geq d_H(x_1) \geq \ell$ and $d_P(x_m) \geq d_H(x_m) \geq \ell$.

Suppose to the contrary that $N_P^-(x_1) \cap N_P[x_m] \neq \emptyset$. Let x_i be a vertex in $N_P^-(x_1) \cap N_P[x_m]$. Then $x_1Px_ix_mPx_{i+1}x_1$ is a cycle of length m in G , a contradiction. Therefore, we have $N_P^-(x_1) \cap N_P[x_m] = \emptyset$. Similarly, we have $N_P^+(x_m) \cap N_P[x_1] = \emptyset$. \square

Let $N_H^-(x_1) = N_P^-(x_1) \cap V(H)$ and $N_H^+(x_m) = N_P^+(x_m) \cap V(H)$.

Claim 2. Let $P = x_1x_2 \cdots x_m$ be a crossing path in \mathcal{P} and (i, j) be a minimal crossing pair of P . Let

$$U_i = N_H[x_1] \cup (N_H^+(x_m) \setminus \{x_{i+1}\}) \text{ and } V_j = N_H[x_m] \cup (N_H^-(x_1) \setminus \{x_{j-1}\}).$$

Then the following properties hold.

- (i) $U_i \subseteq V(x_1Px_i) \cup V(x_jPx_m)$ and $V_j \subseteq V(x_1Px_i) \cup V(x_jPx_m)$,
- (ii) $m - k < j - i - 1 \leq m - 2\ell$, i.e., $2\ell \leq |V(x_1Px_i) \cup V(x_jPx_m)| \leq 2\ell + 1$, and
- (iii) $|V(x_1Px_i) \cup V(x_jPx_m) \setminus U_i| = |V(x_1Px_i) \cup V(x_jPx_m) \setminus V_j| \leq 1$. In particular, if k is odd or $d_H(x_1) + d_H(x_m) = 2\ell + 1$ or $j - i - 1 = m - 2\ell$, then $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$.

Proof. By definition of a minimal crossing pair, we can easily obtain (i). Since $c(G) < k$ and $x_1Px_ix_mPx_jx_1$ is a cycle of length $m - (j - i - 1)$, we have that $m - k < j - i - 1$. Suppose that $j - i - 1 > m - 2\ell$, i.e., $|V(x_1Px_i) \cup V(x_jPx_m)| < 2\ell$. Since $|N_P^-(x_1) \setminus \{x_{j-1}\}| \geq \ell - 1$ and $|N_P[x_m]| \geq \ell + 1$, it follows that $N_P^-(x_1) \cap N_P(x_m) \neq \emptyset$, a contradiction. Therefore, we have $j - i - 1 \leq m - 2\ell$, proving (ii). Lastly, (iii) should follow from the fact that $|U_i| = |V_j| \geq 2\ell$ easily. \square

Next we consider the neighbors of end-vertices of a path with a crossing pair. The following claim strengthens Claim 1(iii) and Claim 2 and will be used many times throughout the proof.

Let $N_P^{+1}(x_m) = N_P^+(x_m)$ and $N_P^{+i}(x_m) = (N_P^{+(i-1)}(x_m))^+$ for $i \geq 2$.

Claim 3. Let $P = x_1x_2 \cdots x_m$ be a crossing path with $d_P(x_1) \geq \ell$ and $d_P(x_m) \geq \ell$. If k is even, then x_1 is adjacent to all vertices but at most one in $V(P) \setminus (\bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m) \cup \{x_1\})$. Moreover, if $d_P(x_1) = |V(P) \setminus (\bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m) \cup \{x_1\})|$, then x_1 is adjacent to each vertex of $V(P) \setminus (\bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m) \cup \{x_1\})$. In particular, if k is odd, then x_1 is adjacent to each vertex of $V(P) \setminus (\bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m) \cup \{x_1\})$.

Proof. Clearly, since $c(G) < k$, we have $N_P[x_1] \subseteq V(P) \setminus \bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m)$. Since P has a crossing pair, we have $|V(P) \setminus \bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m)| \leq m - (d_P(x_m) - 1) - (m - k + 1) = k - d_P(x_m)$. If $d_P(x_1) \geq \ell$, $d_P(x_m) \geq \ell$ and k is even, then $|V(P) \setminus \bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m)| \leq \ell + 2$. Hence, using $d_P(x_1) \geq \ell$, x_1 must be adjacent to all vertices but at most one in $V(P) \setminus (\bigcup_{i=1}^{m-k+1} N_P^{+i}(x_m) \cup \{x_1\})$. The proof for the rest of Claim 3 is similar and omitted. \square

For $P = x_1x_2 \cdots x_m \in \mathcal{P}$, let $s_P = \min\{h : x_{h+1} \in N_P(x_m)\}$ and $t_P = \max\{h : x_{h-1} \in N_P(x_1)\}$.⁷

Claim 4. Let $P = x_1x_2 \cdots x_m$ be a crossing path in \mathcal{P} with a minimal crossing pair (i, j) . If $x_s \in V(H)$ and $x_{s+1} \in N_P(x_1)$, then x_1 cannot be adjacent to two consecutive vertices of x_jPx_{t-1} . Similarly, if $x_t \in V(H)$ and $x_{t-1} \in N_P(x_m)$, then x_m cannot be adjacent to two consecutive vertices of $x_{s+1}Px_i$.

⁷When there is no ambiguity, we often omit the subscript index in s_P and t_P (such as in the coming claim).

Proof. By symmetry between x_1 and x_m , we will prove the first statement. Suppose to the contrary that x_1 is adjacent to x_q and x_{q+1} for some $j \leq q \leq t-2$. We consider the path $R = x_s P x_1 x_{s+1} P x_m$. It follows from $x_s, x_m \in H$ that $R \in \mathcal{P}$. By the maximality of m , we have $N_H[x_s] \subseteq V(R)$ and $N_H[x_m] \subseteq V(x_{s+1} R x_m)$. R has a crossing pair, as otherwise we have $|V(x_1 P x_s)| \geq |N_R[x_s]| \geq \ell + 1$ and hence $x_1 P x_i x_m P x_j x_1$ is a cycle of length at least $|V(x_1 P x_{s+1})| + |N_R^+(x_m) \setminus \{x_{i+1}\}| + |\{x_q, x_{q+1}\}| = \ell + 1 + \ell - 1 + 2 \geq k$, a contradiction. Note that $x_q, x_{q+1} \in N_P[x_1] \subseteq V(P) \setminus \bigcup_{i=1}^{\theta} N_P^i(x_m)$, where $\theta = m - k + 1$. Thus by Claim 3, x_s must be adjacent to one of x_q, x_{q+1} . Suppose that x_s is adjacent to x_q . Then $x_s x_q P x_{s+1} x_m P x_{q+1} x_1 P x_s$ is a cycle of length m , a contradiction. Therefore, x_s is adjacent to x_{q+1} . Then $x_s x_{q+1} P x_m x_{s+1} P x_q x_1 P x_s$ is a cycle of length m , a contradiction. \square

Now according to the parity of k , we divide the remaining proof into two subsections. First, we consider the odd case, whose proof is comparably easier, yet revealing essential ideas of our arguments.

3.1 k is odd.

In this subsection, we have $k = 2\ell + 1$. By Claims 1 and 2, $N_H(x_1) = N_P(x_1)$ and $N_H(x_m) = N_P(x_m)$.

Claim 5. *There exists a crossing path in \mathcal{P} .*

Proof. Suppose to the contrary that all paths in \mathcal{P} are non-crossing. Let $P = x_1 x_2 \cdots x_m \in \mathcal{P}$. Let α be the maximum integer such that x_α is adjacent to x_1 and β be the minimum integer such that x_β is adjacent to x_m . Note that $\alpha \leq \beta$. By Lemma 3.1, G contains a cycle of length at least $\min\{m, 2\ell + 1\} \geq k$, a contradiction. \square

By Claim 5, there is a crossing path $P \in \mathcal{P}$. Let (i_1, j_1) and (i_2, j_2) be two minimal crossing pairs of P such that i_1 is as small as possible and j_2 is as large as possible.⁸

Claim 6. *P has a unique minimal crossing pair (i, j) with $j - i - 1 = m - k + 1$ when $m \geq k + 1$. Moreover, if $m = k$, then each minimal crossing pair (i', j') in P satisfies that $j' - i' - 1 = 1$.*

Proof. Assume that $m \geq k + 1$. Suppose to the contrary that there exist two minimal crossing pairs in P , say $i_1 < j_1 \leq i_2 < j_2$. By Claim 2(ii), we have that $j_1 - i_1 - 1 \geq 2$ and $j_2 - i_2 - 1 \geq 2$. Since $V(x_{i_2+1} P x_{j_2-2}) \cap ((N_P^-(x_1) \setminus \{x_{j_1-1}\}) \cup N_P[x_m]) = \emptyset$, it follows that $x_1 P x_{i_1} x_m P x_{j_1} x_1$ is a cycle of length at least k , a contradiction. Let $m = k$. Then by Claim 2(ii) again, each minimal crossing pair (i', j') satisfies that $j' - i' - 1 = 1$. \square

Claim 7. *$i_1 = s + 1$ and $j_2 = t - 1$.*

Proof. Let (i, j) be a minimal crossing pair of P . We may assume that $j < t - 1$, since otherwise $j_2 = t - 1$. By the choices of s, t , we have $x_{s+1} \in N_P(x_m)$ and $x_{t-1} \in N_P(x_1)$. Since k is odd, Claim 2(iii) gives us $x_s, x_{s+1} \in N_H(x_1)$. Thus it follows from Claim 4 that x_1 is not adjacent to x_{t-2} . By Claim 2(iii) again, x_m is adjacent to x_{t-3} . Therefore, $(t-3, t-1)$ is a minimal crossing pair in P . So $j_2 = t - 1$. Similarly, we have $i_1 = s + 1$. \square

Now we are ready to finish the proof of Lemma 3.2 when k is odd. By Claim 2(iii), we have $V(x_1 P x_s) \subseteq N_H[x_1]$ and $V(x_t P x_m) \subseteq N_H[x_m]$. In particular, this shows $x_s, x_t \in V(H)$. Note that $x_{s+1} \in N_H(x_m)$ and $x_{t-1} \in N_H(x_1)$. By Claim 4, x_1 is not adjacent to consecutive vertices of $V(x_{j_1} P x_{t-1})$ and x_m is not adjacent to consecutive vertices of $V(x_{s+1} P x_{i_2})$. By Claim 7, we derive that $i_1 = s + 1$ and $j_2 = t - 1$. Let $A = V(x_1 P x_s)$, $B = V(x_t P x_m)$, $C = \{x_{s+1}, x_{s+3}, \dots, x_{t-3}, x_{t-1}\}$ (if $(i_1, j_1) = (i_2, j_2)$, then $C = \{x_{s+1}, x_{t-1}\}$) and $D = V(P) \setminus (A \cup B \cup C)$. Combining the above arguments with Claims 2(iii) and 6, we have $N_H[x_1] = A \cup C$ and $N_H[x_m] = B \cup C$. Consider the crossing paths $R_\gamma = x_\gamma P x_1 x_{\gamma+1} P x_m$ for $2 \leq \gamma \leq s$, it is clear that $R_\gamma \in \mathcal{P}$ is a crossing path. By Claim 3, the neighbors of x_γ in H are determined by the neighbors of x_m in R , that is $N_H[x_\gamma] = N_H[x_1]$. Similarly, $N_H[x_\lambda] = N_H[x_m]$ for $t \leq \lambda \leq m$. Now it is straightforward to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m, k, s)$ of Type I, a contradiction. This completes the proof of Lemma 3.2 for odd k .

⁸Note that it is possible that $(i_1, j_1) = (i_2, j_2)$.

3.2 k is even.

In this subsection, we have $k = 2\ell + 2$.

Claim 8. *There exists a crossing path in \mathcal{P} .*

Proof. Suppose to the contrary that all paths in \mathcal{P} are non-crossing. Let $P = x_1x_2 \cdots x_m \in \mathcal{P}$. Let α be the maximum integer such that x_α is adjacent to x_1 and β be the minimum integer such that x_β is adjacent to x_m . Note that $\alpha \leq \beta$.

If $\alpha < \beta$, then by Lemma 3.1, G contains a cycle of length at least $\min\{m, 2\ell + 2\} \geq k$, a contradiction. Therefore, $\alpha = \beta$. Since G is 2-connected, there exists a path Q in G with $V(Q) \cap V(P) = \{x_u, x_v\}$ for $1 \leq u < \alpha < v \leq m$. Let $p = \min\{h : h > u, x_h \in N_P(x_1)\}$ and $q = \max\{h : h < v, x_h \in N_P(x_m)\}$. Then $C_0 = x_1Px_uQx_vPx_mx_qPx_px_1$ is a cycle containing $N_P[x_1] \cup N_P[x_m]$. By Claim 1, C_0 has length at least $k - 1$. Note that $c(G) < k$. This forces that C_0 has length $k - 1$. It follows that $d_H(x_1) = d_H(x_m) = \ell$, $N_H(x_1) = V(x_2Px_u) \cup V(x_pPx_\alpha)$, $N_H(x_m) = V(x_\alphaPx_q) \cup V(x_vPx_{m-1})$, $V(C) = N_H[x_1] \cup N_H[x_m]$, and $Q = x_u x_v$.

For any $2 \leq \gamma \leq u - 1$, we consider the path $R_\gamma = x_\gamma Px_1 x_{\gamma+1} Px_m$. Clearly, $R_\gamma \in \mathcal{P}$. Also, by our assumption, R_γ is non-crossing. It follows that $N_H[x_\gamma] \subseteq V(x_1Px_\alpha)$. Suppose that x_γ has a neighbor y in $V(x_{u+1}Px_{p-1})$. Then $x_\gamma Px_1 x_{\gamma+1} Px_u Q x_v Px_m x_q Py x_\gamma$ is a cycle of length at least $k + 1$, a contradiction.

Therefore, we have that $N_H[x_\gamma] = N_H[x_1]$ for any $2 \leq \gamma \leq u - 1$. Suppose that $p < \alpha$ or $q > \alpha$. By symmetry, we may assume that $p < \alpha$. Then we have that $x_{\alpha-1} \in N_P(x_1)$. Now, we consider the path $L = x_u Px_{\alpha-1} x_1 Px_{u-1} x_\alpha Px_m$. Clearly, $L \in \mathcal{P}$. Note that $x_v \in N_L(x_u)$, $x_\alpha \in N_L(x_m)$ and x_α precedes x_v in L . It follows that L is a crossing path in \mathcal{P} , a contradiction.

The last paragraph implies that $p = \alpha$ and $q = \alpha$. Suppose that $u = \alpha - 1$ or $v = \alpha + 1$. By symmetry, we may assume that $x_\alpha x_u \in E(P)$. Now we consider the path $M = x_u Px_1 x_\alpha Px_m$. Clearly, $M \in \mathcal{P}$. Note that $x_v \in N_M(x_u)$, $x_\alpha \in N_M(x_m)$ and x_α precedes x_v in M . It follows that M is a crossing path in \mathcal{P} , a contradiction.

Thus, we may suppose that $u < \alpha - 1$ and $v > \alpha + 1$. Let $A = V(x_1Px_u)$, $B = V(x_vPx_m)$ and $C = V(P) \setminus (A \cup B)$. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m, k, \ell)$ of Type IV (with $k = 2\ell + 2$, $w = x_\alpha$ and $\{w_1, w_2\} = \{x_u, x_v\}$), a contradiction. \square

Let $P \in \mathcal{P}$ be a longest H -path with as many minimal crossing pairs as possible and subject to this, let (i, j) be a minimal crossing pair of P with largest length. Let (i_1, j_1) and (i_2, j_2) be two minimal crossing pairs of P such that i_1 is as small as possible and j_2 is as large as possible.

Claim 9. *There is a unique minimal crossing pair in P when $m \geq k + 2$. In particular, there are at most two minimal crossing pairs in P when $m = k + 1$. Moreover, for $m = k$, each minimal crossing pair $(i', j') \neq (i, j)$ in P satisfies $j' - i' = 2$.*

Proof. Assume that $m \geq k + 2$. Suppose to the contrary that there exist two minimal crossing pairs in P , that is $i_1 < j_1 \leq i_2 < j_2$. By Claim 2(ii), we have that $j_1 - i_1 - 1 \geq 3$ and $j_2 - i_2 - 1 \geq 3$. Note that $V(x_{i_2+1}Px_{j_2-2}) \cap ((N_P^-(x_1) \setminus \{x_{j_1-1}\}) \cup N_P[x_m]) = \emptyset$. It follows that $x_1Px_{i_1}x_mx_{j_1}x_1$ is a cycle of length at least k , a contradiction.

Assume that $m = k + 1$. Suppose to the contrary that there exist three minimal crossing pairs (α_1, β_1) , (α_2, β_2) and (α_3, β_3) in P . Without loss of generality, we may assume that $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \alpha_3 < \beta_3$. Note that $(V(x_{\alpha_2+1}Px_{\beta_2-2}) \cup V(x_{\alpha_3+1}Px_{\beta_3-2})) \cap ((N_P^-(x_1) \setminus \{x_{\alpha_1-1}\}) \cup N_P[x_m]) = \emptyset$. Then $x_1Px_{\alpha_1}x_mx_{\beta_1}x_1$ is a cycle of length at least k , a contradiction.

Therefore, $m = k$. By Claim 2(ii), we have that $j - i - 1 = 1$ or 2 . We may assume that $j - i - 1 = 2$, since otherwise the result follows by the choice of (i, j) . Hence Claim 2(iii) implies $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. Suppose to the contrary that there exists a minimal crossing pair (i', j') other than (i, j) in P with $j' - i' - 1 = 2$. It is clearly that $V(x_{i'+1}Px_{j'-2}) \cap ((N_P^-(x_1) \setminus \{x_{j-1}\}) \cup N_P[x_m]) = \emptyset$, contradicting $V(x_1Px_i) \cup V(x_jPx_m) = U_i = V_j$. \square

There are two possibilities for the size of m : $m \geq k + 1$ or $m = k$. We now split the rest of the proof into two cases based on these two possibilities.

3.2.1 $m \geq k + 1$.

Claim 10. $i_1 = s + 1$ and $j_2 = t - 1$.

Proof. By Claim 9, there are at most two minimal crossing pairs in P . Suppose that there are two minimal crossing pairs $(i_1, j_1), (i_2, j_2)$ in P . By Claim 9 again, we have that $m = k + 1$. By Claims 1, 2 and the choices of s, t , we have that $j_1 - i_1 - 1 = j_2 - i_2 - 1 = 2$, $x_s, x_{s+1} \in N_H(x_1)$ and $x_{t-1}, x_t \in N_H(x_m)$. Assume that $j_2 < t - 1$. Note that $x_s \in H$. It follows from Claim 4 that x_1 is not adjacent to x_{t-2} . By Claim 2, we have that x_m is adjacent to x_{t-3} . Therefore, $(t - 3, t - 1)$ is a minimal crossing pair in P , contradicting that there are two minimal crossing pairs. Hence, we have that $j_2 = t - 1$. Similarly, we have that $i_1 = s + 1$.

Thus, we may assume that there is a unique minimal crossing pair (i, j) in P . We will show that $i = s + 1$ and $j = t - 1$. Suppose that $d_H(x_1) + d_H(x_m) \geq 2\ell + 1$ or $m - (j - i - 1) = 2\ell$. Then, similarly as the case when k is odd, Claims 1(iii) and 4 imply that $i = s + 1$ and $j = t - 1$.

Now, suppose that $m - (j - i - 1) = 2\ell + 1$ and $d_H(x_1) + d_H(x_m) = 2\ell$. Then there exists a vertex $x_p \in V(x_1Px_i) \cup V(x_jPx_m)$ such that $x_p \notin N_H(x_1) \cup N_H^+(x_m)$. By symmetry between x_1 and x_m , we may assume that $1 \leq p \leq i$. Suppose to the contrary that $i > s + 1$. Since $x_t, x_{t-1} \in N_H(x_m)$, by Claim 4, we have that $x_{i-1} \notin N_P(x_m)$. Note that there is only one minimal crossing pair in P . It follows that $V(x_{s+2}Px_i) \cap N_P(x_1) = \emptyset$. This forces that $x_{i-2} \in N_P(x_m)$, $s = i - 3$ and $p = i$. Hence, $x_s, x_{s+1} \in N_H(x_1)$. By Claim 4 and $p = i$, we can easily deduce that $V(x_jPx_{m-1}) \subseteq N_H(x_m)$. We consider the path $R_1 = x_tPx_mx_{t-1}Px_1$. Clearly, $R_1 \in \mathcal{P}$ is a crossing path, and hence by Claim 3, x_t must be adjacent to one of x_{i-2}, x_{i-1} (as in the proof of Claim 4). Then $x_\gamma Px_1x_{t-1}Px_ix_mPx_tx_\gamma$ is a cycle of length at least $m - 1 \geq k$, where $\gamma \in \{i - 2, i - 1\}$, a contradiction. This contradiction shows that $i = s + 1$.

Next, we will show that $j = t - 1$. First, we suppose that $x_{j+1} \in N_P(x_1)$. By Claim 4, we have that $p = i - 1$, or x_1 is not adjacent to x_i , i.e., $p = i$. (a) x_1 is not adjacent to x_i . It follows that $V(x_1Px_{i-1}) \subseteq N_H[x_1]$. If $i \leq 3$, then $x_1x_jPx_ix_mPx_{j+1}x_1$ is a cycle of length $m - 1 \geq k$ vertices, a contradiction. Therefore, $i \geq 4$. Then we consider the path $R_3 = x_{i-2}Px_1x_{i-1}Px_m$. Clearly, $R_3 \in \mathcal{P}$ is a crossing path, and hence by Claim 3, x_{i-2} must be adjacent to one of $\{x_j, x_{j+1}\}$ (as in the proof of Claim 4). If x_{i-2} is adjacent to x_j , then $x_1Px_{i-2}x_jPx_ix_mPx_{j+1}x_1$ is a cycle of length $m - 1 \geq k$, a contradiction. Similarly, if x_{i-2} is adjacent to x_{j+1} , then $x_1Px_{i-2}x_{j+1}Px_mx_iPx_jx_1$ is a cycle of length $m - 1 \geq k$, a contradiction. (b) $p = i - 1$. Clearly, we have $x_{i-2} \in N_H(x_1)$. Suppose that x_{i-2} has a neighbor $y \in V(H)$ not in P . Then we consider the path $R_4 = yx_{i-2}Px_1x_iPx_m$. Clearly, $R_4 \in \mathcal{P}$ is a crossing path. Therefore, by Claim 3, y must be adjacent to one of $\{x_j, x_{j+1}\}$ (as in the proof of Claim 4). If y is adjacent to x_j (or x_{j+1}), then $x_1Px_{i-2}yx_jPx_ix_mPx_{j+1}x_1$ (or $x_1Px_{i-2}yx_{j+1}Px_mx_iPx_jx_1$) is a cycle of length at least k , a contradiction. Therefore, we have $N_H(x_{i-2}) \subseteq V(P)$. Then we consider the path $R_5 = x_{i-2}Px_1x_iPx_m$.⁹ Obviously, R_5 has a crossing pair. Moreover, it is easy to check that $d_{R_5}(x_{i-2}) \geq \ell$ and $d_{R_5}(x_m) \geq \ell$. Therefore, by Claim 3, x_{i-2} must be adjacent to one of $\{x_j, x_{j+1}\}$. If x_{i-2} is adjacent to x_j (or x_{j+1}), then $x_1Px_{i-2}x_jPx_ix_mPx_{j+1}x_1$ (or $x_1Px_{i-2}x_{j+1}Px_mx_iPx_jx_1$) is a cycle of length $m - 1 \geq k$, a contradiction. Thus, by (a) and (b), x_1 is not adjacent to x_{j+1} . By Claim 2(iii), x_m is adjacent to x_j . Since there is only one minimal crossing pair, x_1 is not adjacent to any vertex of $V(x_{j+1}Px_m)$, that is, $j = t - 1$. This completes the proof of the claim. \square

By Claim 9, there are at most two minimal crossing pairs in P . Suppose that there are two minimal crossing pairs (i_1, j_1) and (i_2, j_2) in P with $i_1 < j_1 \leq i_2 < j_2$. From Claim 2, we deduce $j_1 - i_1 - 1 = j_2 - i_2 - 1 = 2$. Applying Claim 9 again, we have $m = k + 1$. By the choices of s and t , $x_s, x_{s+1} \in N_H(x_1)$ and $x_t, x_{t-1} \in N_H(x_m)$. Combining Claim 2(iii) and 4, it is not hard to show that $j_1 = i_2$. Considering the paths $x_\gamma Px_1x_{\gamma+1}Px_m$ and $x_\lambda Px_{k+1}x_{\lambda-1}Px_1$, by Claim 3, we can determine the neighbors of x_γ and x_λ in H , that is $N_H[x_1] = N_H[x_\gamma]$ for $2 \leq \gamma \leq \ell - 2$ and $N_H[x_{k+1}] = N_H[x_\lambda]$ for $\ell + 6 \leq \lambda \leq k$. Let $A = V(x_1Px_{\ell-2})$, $B = V(x_{\ell+6}Px_{k+1})$, $C = \{x_{\ell-1}, x_{\ell+2}, x_{\ell+5}\}$ and $D = V(P) \setminus (A \cup B \cup C)$. It is easy to check that $G[V(P)]$ gives a copy in $F_4(k + 1, k, \ell - 2)$, a contradiction.

⁹Note that R_5 has $m - 1$ vertices. Thus $R_5 \notin \mathcal{P}$.

Therefore, there is only one minimal crossing pair (i, j) in P , that is $i = s+1$ and $j = t-1$ by Claim 10. It follows from Claim 2(ii) that $m - (j - i - 1) = 2\ell$ or $2\ell + 1$. Suppose that $m - (j - i - 1) = 2\ell$. Then applying Claim 3 as the last paragraph, it is not hard to show that $N_H[x_\gamma] = N_H[x_1]$ and $N_H[x_m] = N_H[x_\lambda]$ for $2 \leq \gamma \leq \ell - 1$, $m - \ell + 2 \leq \lambda \leq m - 1$. In fact, consider the crossing path $x_2x_1x_3Px_m \in \mathcal{P}$, by Claim 3, x_2 is not adjacent to at most one of $\{x_{j-1}\} \cup N_H(x_1)$. If x_2 is adjacent to x_{j-1} , then $x_1x_{i-1}Px_2x_{j-1}Px_ix_mPx_jx_1$ is a cycle of length $m \geq k + 1$, a contradiction. Thus, we have $N_H[x_2] = N_H[x_1]$. Progressively and similarly, we can show that $N_H[x_\gamma] = N_H[x_1]$ and $N_H[x_m] = N_H[x_\lambda]$. Hence, it is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(m, k, \ell - 1)$ with Type II, a contradiction.

Therefore, $m - (j - i - 1) = 2\ell + 1$. Suppose that $d_H(x_1) + d_H(x_m) = 2\ell + 1$. Without loss of generality, let $d_H(x_1) = \ell + 1$ and $d_H(x_m) = \ell$. Then applying Claim 3 as the before, it is not hard to show that $N_H[x_\gamma] \subseteq N_H[x_1]$ for $2 \leq \gamma \leq \ell$ and $N_H[x_m] = N_H[x_\lambda]$ for $m - \ell + 2 \leq \lambda \leq m - 1$. $G[V(P)]$ gives a copy in $\mathcal{F}(m, k, \ell - 1)$ with Type II (note that there is a Hamilton path starting from x_1 and ending at x_m , also x_i is adjacent to x_{i-1} and x_1 is adjacent to x_j), a contradiction.

Now we may assume that $d_H(x_1) = d_H(x_m) = \ell$. Without loss of generality, there exists a vertex $x_p \notin N_H(x_1) \cup N_H^+(x_m)$ with $1 \leq p \leq i$. Claim 10 implies that $i = \ell + 1$ and $j = m - \ell + 1$. Also, note that $N_H[x_m] = \{x_{\ell+1}\} \cup V(x_{m-\ell+1}Px_m)$ and $N_H[x_1] = \{x_{m-\ell+1}\} \cup (V(x_1Px_{\ell+1}) \setminus \{x_p\})$. Then we consider the path $Q_\lambda = x_1Px_{\lambda-1}x_mPx_\lambda$, where $m - \ell + 2 \leq \lambda \leq m - 1$. Clearly, $Q_\lambda \in \mathcal{P}$ is a crossing path. As the previous proofs, we have $N_P[x_\lambda] \subseteq N_H[x_m] \cup \{x_{p-1}\}$ by Claim 3.

Claim 11. *For each $m - \ell + 2 \leq \lambda \leq m - 1$, we have $N_H[x_\lambda] = N_H[x_m]$.*

Proof. Suppose to the contrary that x_λ is adjacent to x_{p-1} . First we assume that $p < i$. Then $x_{p-1}Px_1x_{p+1}Px_{\lambda-1}x_mPx_\lambda x_{p-1}$ is a cycle of length $m - 1 \geq k$, a contradiction. Therefore, we have $p = i$. Suppose that x_λ is not adjacent to x_i . By Claim 3, x_λ is adjacent to x_{i-1} . Note that x_1 is adjacent to x_j . It follows that there is a minimal crossing pair of longer length in Q_λ , a contradiction. Therefore, x_λ is adjacent to x_i . Then we consider the path $L_\lambda = x_1Px_\lambda x_mPx_{\lambda+1}$. Clearly, $L_\lambda \in \mathcal{P}$ is a crossing path. By Claim 3, $x_{\lambda+1}$ must be adjacent to one of $\{x_{i-1}, x_i\}$. By the maximality of $j - i - 1$, $x_{\lambda+1}$ is adjacent to x_i . Then $x_{i-1}Px_1x_jPx_ix_{\lambda+1}Px_mx_{j+1}Px_\lambda x_{i-1}$ is a cycle of length m , a contradiction. Therefore, x_λ is not adjacent to $x_{i-1} = x_{p-1}$. This completes the proof of the claim. \square

Suppose that $x_p \notin H$. Let $2 \leq \gamma \leq \ell$ and $\gamma \neq p - 1$. Then we consider the path $M_\gamma = x_\gamma Px_1x_{\gamma+1}Px_m$. Clearly, $M_\gamma \in \mathcal{P}$ is a crossing path. Note that $x_p \notin N_H(x_\gamma)$. By Claim 3, we have that $N_H[x_1] = N_H[x_\gamma]$. Let $A = \{x_1, \dots, x_\ell\} \setminus \{x_p\}$, $B = \{x_{m-\ell+2}, \dots, x_m\}$, $C = \{x_i, x_j\}$ and $D = V(P) \setminus (A \cup B \cup C)$. It is easy to check that $G[V(P)]$ gives a copy of $F_1(m, k, \ell - 1)$ or $F_2(m, k, \ell - 1)$ of Type III, a contradiction.

Therefore, we have $x_p \in H$. Then $p \geq 3$. Let $A = \{x_1, \dots, x_\ell\}$ and $C = \{x_i, x_j\}$. Then we consider the path $T_\gamma = x_\gamma Px_1x_{\gamma+1}Px_m$ for $2 \leq \gamma \leq \ell$ with $\gamma \neq p - 1$. Clearly, $T_\gamma \in \mathcal{P}$ is a crossing path. Firstly, let $p = i$. By Claim 11, we may consider the path $Q_1 = x_{p-1}Px_1x_{m-\ell+1}Px_ix_{m-\ell+2}Px_m$. Clearly, $Q_1 \in \mathcal{P}$ is a crossing path. Let $\mathcal{P}_1 = \{Q_1\} \cup \{T_\gamma : 2 \leq \gamma \leq \ell\}$. Considering each path in \mathcal{P}_1 , by Claim 3, we have that each vertex of A in $G[A \cup C]$ has degree at least ℓ . Note that both vertices of C are adjacent to A . It is easy to check that $G[V(P)]$ gives a copy of $F \in \mathcal{F}(m, k, \ell - 1)$ of Type II, a contradiction. Secondly, let $p \leq \ell = i - 1$ and $\ell \geq 4$. Then we consider the crossing path $Q_2 = x_p Px_1x_{p+1}Px_m$. By the maximality of Q_2 , $N_H(x_p) \subseteq V(Q_2) = V(P)$. Since $x_p \notin N_H(x_1)$, by Claim 3, we have that x_p is adjacent to each vertex of $\{x_2, \dots, x_{\ell+1}\}$. Considering each path of T_γ with $2 \leq \gamma \leq \ell$ and $\gamma \neq p - 1$, and the crossing paths $x_{p-1}x_px_{p-2}Px_1x_{p+1}Px_m$ when $p \geq 4$ and $x_{p-1}x_px_{p+1}x_1x_{p+2}Px_m$ when $p = 3$, by Claim 3, each vertex of A in $G[A \cup C]$ has degree at least ℓ . It is easy to check that $G[V(P)]$ gives a copy of $F \in \mathcal{F}(m, k, \ell - 1)$ of Type II, a contradiction. Finally, $p \leq \ell$ and $\ell \leq 3$. Suppose that $m \geq k + 2$. Then $j = m - \ell + 1 \geq \ell + 5 \geq 2\ell + 2$. Thus $x_1Px_jx_1$ is a cycle of length at least $2\ell + 2 = k$, a contradiction. Now let $m = k + 1$, that is $j = m - \ell + 1$. If $\ell \leq 2$, then $x_1Px_jx_1$ is a cycle of length at least k , a contradiction. Let $\ell = 3$. This forces that $k = 8$ and $p = 3$. Suppose that $N_P(x_2) \geq 3$. Then G contains a copy of $F \in \mathcal{F}(9, 8, 2)$ ($A = \{x_1, x_2, x_3\}$) with Type II, a contradiction. Therefore, $N_P(x_2) = 2$. It follows that there is a vertex $z \in N_H(x_2) \setminus N_P(x_2)$.

Thus $N_H(z) = \{x_2, x_4, x_7\}$. Let $A = \{x_1\}$, $B = \{z\}$ and $C = \{x_2, x_4, x_7\}$. Then we consider the path $zx_2Px_7x_1$. It is easy to check that $G[V(P)]$ gives a copy in $F \in \mathcal{F}(8, 8, 1)$ with Type II, a contradiction. This completes the proof when k is even and $m \geq k + 1$.

3.2.2 $m = k$.

By Claim 2(ii), we have that $j - i - 1 = 1$ or 2 . Suppose that either $j - i - 1 = 2$ or $j - i - 1 = 1$ and $d_H(x_1) + d_H(x_k) = 2\ell + 1$. Then the same proof as in the Subsection 3.2.1 shows that $G[V(P)]$ gives a copy of $\mathcal{F}(k, k, s)$ with Type II, a contradiction.

Therefore, $j - i - 1 = 1$ and $d_H(x_1) = d_H(x_k) = \ell$. Without loss of generality, there exists a vertex $x_p \notin N_H(x_1) \cup N_H^+(x_k)$ with $1 \leq p \leq i$. By the definition of t , we have that $i \leq t - 2$. Now, subject to previous choices, we choose $P \in \mathcal{P}$ such that $|V(x_{s+2}Px_{t-2}) \cap \{x_p\}|$ is as large as possible.

Claim 12. $p \leq s + 1$.

Proof. Suppose to the contrary that $p > s + 1$, that is $s + 2 \leq p \leq t - 2$. Then x_s and x_{s+1} belong to $N_H(x_1)$, and x_{t-1} and x_t belong to $N_H(x_k)$. Clearly, we have that $x_p \notin N_H(x_1)$ and $x_{p-1} \notin N_H(x_k)$.

Suppose that $x_{p-1} \in N_H(x_1)$ and $x_p \in N_H(x_k)$. Let

$$C = \{x_{s+1}, x_{s+3}, \dots, x_{p-5}, x_{p-3}, x_{p+2}, x_{p+4}, \dots, x_{t-3}, x_{t-1}\}.$$

We shall show that x_1 and x_k are adjacent to each vertex of C . By Claim 2(iii), $p > s + 1$ and definition of s , x_1 and x_k are adjacent to x_{s+1} . By Claim 1(iii) and Claim 4, x_1 and x_k are not adjacent to x_{s+2} . Next, by Claim 2(iii) and x_1 is adjacent to x_{s+3} with $p > s + 3$ and hence by Claim 4, x_1 is not adjacent to x_{s+4} . Then it follows from Claim 2(iii) and 4 that x_k is adjacent to x_{s+3} and not adjacent to x_{s+4} with $p > s + 4$. Progressively, we can show that x_1 and x_k are adjacent to each vertex of $\{x_{s+1}, x_{s+3}, \dots, x_{p-5}, x_{p-3}\}$. Moreover, we also show that $i_1 = s + 1$. Similarly, we have that x_1 and x_k are adjacent to each vertex of $\{x_{p+2}, x_{p+4}, \dots, x_{t-3}, x_{t-1}\}$ and $j_2 = t - 1$. Thus, x_1 and x_k are adjacent to each vertex of C . Moreover, we have $s \equiv p$ modulo 2 and $t - 1 \equiv p$ modulo 2. Then we consider the paths $T_\gamma = x_\gamma Px_1 x_{\gamma+1} Px_k$ and $S_\lambda = x_\lambda Px_k x_{\lambda-1} Px_1$ for $2 \leq \gamma \leq s$ and $t \leq \lambda \leq k - 1$. Clearly, $T_\gamma, S_\lambda \in \mathcal{P}$. By Claims 1(iii), 2(iii), 4 and maximality of $j - i - 1$, we have that $N_H[x_\gamma] = N_H[x_1]$ and $N_H[x_\lambda] = N_H[x_k]$. Hence $x_{p-1}x_1Px_sx_{t-1}Px_px_kPx_tx_{s+1}Px_{p-1}$ is a cycle of length k , a contradiction.

Suppose that $x_{p-1} \in N_H(x_1)$ and $x_p \notin N_H(x_k)$. Then $p < i$ and $x_{p+1} \in N_H(x_1)$. Let

$$C = \{x_{s+1}, x_{s+3}, \dots, x_{p-1}, x_{p+1}, \dots, x_{t-3}, x_{t-1}\}.$$

Then the same proof as in the last paragraph shows that x_1 is adjacent to each vertex of C and x_k is adjacent to each vertex of $C \setminus \{x_{p-1}\}$. Let $A = V(x_tPx_k)$ and $B = V(x_1Px_s)$. Note that $d_H(x_1) = d_H(x_k) = \ell$. Then we have $|A| = s + 1$ and $|B| = s$. Considering the paths $x_\lambda Px_k x_{\lambda-1} Px_1 \in \mathcal{P}$ for $t \leq \lambda \leq k - 1$, by Claim 1(iii), we have that $N_H(x_\lambda) \subseteq A \cup C$. Moreover, considering the paths $x_\gamma Px_1 x_{\gamma+1} Px_k \in \mathcal{P}$ for $2 \leq \gamma \leq s$, similarly as the proof as in the last paragraph, we have that $N_H(x_\gamma) = B \cup C$. Note that x_k is adjacent to both end-vertices of $x_{s+1}Px_{t-1}$. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s)$ with Type II.

Therefore $x_{p-1} \notin N_H(x_1)$. If $x_p \notin N_P(x_k)$, then there is a minimal crossing pair (i', j') of length at least two ($x_{p-1}, x_p \notin N_P(x_k)$ and $x_{p-1}, x_p \notin N_P(x_k)$), a contradiction. Therefore we have that $x_p \in N_H(x_k)$. Let

$$C = \{x_{s+1}, x_{s+3}, \dots, x_{p-2}, x_p, x_{p+2}, \dots, x_{t-3}, x_{t-1}\}.$$

As the proofs before, x_k is adjacent to each vertex of C and x_1 is adjacent to each vertex of $C \setminus \{x_p\}$. Let $A = V(x_1Px_s)$ and $B = V(x_tPx_k)$. From $d_H(x_1) = d_H(x_k) = \ell$, we have $|A| = s$ and $|B| = s - 1$. Considering the paths $x_\gamma Px_1 x_{\gamma+1} Px_k \in \mathcal{P}$ and $x_\lambda Px_k x_{\lambda-1} Px_1 \in \mathcal{P}$ for $2 \leq \gamma \leq s$ and $t \leq \lambda \leq k - 1$, as the previous proofs, we have $N_H[x_\gamma] \subseteq A \cup C$ and $N_H[x_\lambda] = B \cup C$. Note that x_1 is adjacent to both end-vertices of $x_{s+1}Px_{t-1}$. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s - 1)$ with Type II, a contradiction. Thus we finish the proof of Claim 12. \square

The rest proof is similar as the case $m = k+1$. Let $2 \leq p \leq s+1$, that is, $|V(x_{s+2}Px_{t-2}) \cap \{x_p\}| = 0$. Then x_t and x_{t-1} belong to $N_H(x_k)$ and $p \leq i_1$. By Claim 4, x_k is not adjacent consecutive vertices of $V(x_{s+1}Px_{i_2})$. First, we will show that $i_1 = s+1$. Suppose that $i_1 > s+1$. Then by Claim 4, x_k is not adjacent to x_{s+2} , and hence $i_1 \geq s+3$. Also, since $2 \leq p \leq s+1$, by Claim 2(iii), x_1 is adjacent to x_{s+3} , that is $(s+1, s+3)$ is a crossing pair, contradicting $i_1 > s+1$.

Claim 13. $N_H[x_\lambda] = N_H[x_k]$ for $t \leq \lambda \leq k-1$.

Proof. By Claim 3, we have $N_H[x_\lambda] \subseteq N_H[x_k] \cup \{x_{p-1}\}$. If $d_H(x_\lambda) \geq \ell+1$, then we are done as the beginning of the case $m = k$ is even. Let $d_H(x_\lambda) = \ell$. Suppose that x_λ is adjacent to x_{p-1} . Let $p < s+1$. Since $x_p \notin N_H^+(x_m)$, x_1 is adjacent to x_{p+1} . Considering the path $P^\lambda = x_\lambda Px_k x_{\lambda-1} Px_1 \in \mathcal{P}$, by the maximality of the number of minimal crossing pairs of P , x_λ is adjacent to each vertex of $V(x_{t-1}Px_k)$. Hence, x_λ is not adjacent to a vertex $x \in N_H[x_k] \setminus V(x_{t-1}Px_k)$. Note that $N_{P^\lambda}(x_\lambda) \cup \{x\} = N_P(x_k) \cup \{x_{p-1}\}$. Thus the path $P^\lambda = x_\lambda Px_k x_{\lambda-1} Px_1 = y_k Py_1$ ($y_k = x_\lambda$ and $y_1 = x_1$) with a minimal crossing pair (i', j') satisfies $|V(y_{s'+2}Py_{t'-2}) \cap \{y_{p'}\}| = 1$, where $s' = \min\{h : y_{h+1} \in N_{P'}(y_k)\}$, $t' = \max\{h : y_{h-1} \in N_{P'}(y_1)\}$ and $\{y_{p'}\} = (V(y_1 Py_{t'}) \cup V(y_{j'} Py_k)) \setminus (N_{P^\lambda}[y_1] \cup N_{P^\lambda}^+(y_k))$, a contradiction to the choice of P . Now, let $p = s+1$. Note that $i_1 = s+1 = p$. Then the same proof as in Claim 11 shows that $N_P[x_k] = N_P[x_\lambda]$. This completes the proof of the claim. \square

Claim 14. $p = s+1$.

Proof. Suppose to the contrary that $p \leq s$. First we assume that $2 \leq p \leq s-1$. Since $\{x_s, x_{s+1}\} \subseteq N_H(x_1)$, by Claim 4, x_1 is not adjacent to any two consecutive vertices of $x_{s+1}Px_{t-1}$. Suppose that $x_p \notin H$. By Claims 3, we have $N_H[x_\gamma] = N_H[x_1]$ for $2 \leq \gamma \leq s$. Combining the fact that $N_H[x_\lambda] = N_H[x_k]$ for $t \leq \lambda \leq k-1$, it is easy to check that G contains either a copy of $F_1(k, k, s)$ or $F_2(k, k, s)$ with Type III, a contradiction. Therefore $x_p \in H$. Then $p \geq 3$ and $s \geq 4$. Let $A = \{x_1, \dots, x_s\}$ and $C = \{x_{s+1}, x_{s+3}, \dots, x_{t-3}, x_{t-1}\}$. For $x \in A \setminus \{x_{p-1}\}$, we can deduce $N_H[x] \subseteq A \cup C$ from Claim 3 as the previous proofs. By the maximality of P , $N_H(x_p)$ belongs to $V(P)$. It follows that x_1 is adjacent to each vertex of $(A \cup C) \setminus \{x_p\}$ and x_p is adjacent to each vertex of $(A \cup C) \setminus \{x_1\}$. Now, considering the path $x_{p-1}x_p x_{p-2} Px_1 x_{p+1} Px_m \in \mathcal{P}$, Claim 3 implies $N_H(x_{p-1}) \subseteq A \cup C$. Note that x_1 is adjacent to each vertex of C . Hence, it is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s)$ with Type II, a contradiction.

Therefore we can assume that $p = s$. Suppose that x_1 is not adjacent to any two consecutive vertices of $x_{j_1}Px_{t-1}$. Then the same proof as in the last paragraph shows that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s)$ with Type II, a contradiction. Therefore x_1 is adjacent to two consecutive vertices of $x_{j_1}Px_{t-1}$. Let λ be the minimum integer such that x_1 is adjacent to both x_λ and $x_{\lambda+1}$. By Claim 4, we have that $x_s \notin H$. Let $r = \min\{h : h \geq \lambda, x_h \in N_H(x_k)\}$. By Claim 1 and $p = s < i_1$, we have that $V(x_\lambda Px_r) \subseteq N_H(x_1)$. Hence, we have that $x_r, x_{r-1} \in N_H(x_1)$. It follows from Claim 4 that x_1 is not adjacent to any two consecutive vertices of $x_{r+2}Px_{t-1}$. Considering the path $x_\gamma Px_1 x_{\gamma+1} Px_k \in \mathcal{P}$ for $\gamma \in [2, s-2] \cup [\lambda, r-1]$, by $x_s \notin H$ and Claim 3, we have $N_P[x_1] = N_P[x_\gamma]$. Let $A = V(x_1 Px_{s-1}) \cup V(x_\lambda Px_{r-1})$, $B = V(x_{t+1}Px_k)$ and $C = \{x_{s+1}, x_{s+3}, \dots, x_{\lambda-2}, x_{r-1}, x_r, x_{r+2}, \dots, x_{t-2}, x_t\}$. Note that $|A| = |B| = k-t$. Moreover, $G[A]$ is a complete graph. It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}_3(k, k, k-t)$ with Type III, a contradiction. This proves Claim 14. \square

Therefore, we have $p = s+1$. Note that $x_p \in N_H(x_k)$. Then $x_p \in H$ and $p \geq 3$. First, we show that x_1 is not adjacent to any two consecutive vertices of $V(x_{j_1}Px_t)$. Suppose to the contrary that x_1 is adjacent to both of $\{x_\lambda, x_{\lambda+1}\}$ for some $\lambda \geq j_1$. Considering the path $x_{s-1}Px_1 x_s Px_k$, it follows from Claim 3 that x_{s-1} is adjacent to at least one of $\{x_\lambda, x_{\lambda+1}\}$. Hence, $P'_1 = x_s Px_\lambda x_{s-1} Px_1 x_{\lambda+1} Px_k \in \mathcal{P}$ (or $P'_1 = x_s Px_\lambda x_1 Px_{s-1} x_{\lambda+1} Px_k \in \mathcal{P}$, we omit the proof of this case) is a crossing path on k vertices ending at x_k . Note that x_k is not adjacent to both of $\{x_1, x_{\lambda-1}\}$. By Claim 3, x_s is adjacent at least one of $\{x_\lambda, x_{\lambda+1}\}$. Then $x_1 Px_s x_\lambda Px_{s+1} x_k Px_{\lambda+1} x_1$ (or $x_1 x_\lambda Px_{s+1} x_k Px_{\lambda+1} x_s Px_1$) is a cycle of length k , a contradiction. Therefore x_1 is not adjacent to any two consecutive vertices of $x_{j_1}Px_{t-1}$. Let $A = \{x_1, x_2, \dots, x_s\}$, $B = \{x_t, x_{t+1}, \dots, x_k\}$ and $C = \{x_{s+1}, x_{s+3}, \dots, x_{t-3}, x_{t-1}\}$. Then we have $N_H[x_1] = (A \cup C) \setminus \{x_{s+1}\}$ and $N_H[x_k] = B \cup C$. Since $N_H[x_k] = N_H[x_\lambda]$ for $t \leq \lambda \leq k-1$, by Claim

3, we have that $N_H[x_\gamma] \subseteq A \cup C$ for $2 \leq \gamma \leq s$. Note that x_s is adjacent to x_{s+1} . It is easy to check that $G[V(P)]$ gives a copy in $\mathcal{F}(k, k, s-1)$ with Type II, a contradiction. This completes the proof of Lemma 3.2. \square

4 Proof of the main result

For a family of graphs \mathcal{F} , we say a graph G is \mathcal{F} -free if it does not contain any $F \in \mathcal{F}$ as a subgraph. Let $F(\ell)$ be the graph obtained by taking a path $P_{2\ell-1}$ on $2\ell-1$ vertices and a disjoint copy of \overline{K}_3 , and joining each vertex of \overline{K}_3 to each vertex of the larger partite set in the unique bipartition of $P_{2\ell-1}$.

Let $k \geq 5$ and $\mathcal{K}_{k,0} = \emptyset$. For $1 \leq \alpha \leq \ell-2$, let $\mathcal{K}_{k,\alpha}$ be the family of the following graphs:¹⁰

- (a) $F \in \mathcal{F}(m, k, r)$ with $r \in \{1, \dots, \alpha\} \cup \{\ell-1, \ell\}$,
- (b) $F_0(k, k, \ell-2)$ and $F_4(k+1, k, \ell-2)$ when $k \geq 10$ is even and $\ell-\alpha \leq 3$,
- (c) $F_2(m, k, \alpha+1)$ with $\alpha+1 \leq \ell-2$ when k is even,
- (d) $F_5(m, k, 2)$ when $\alpha=1$ and k is even, and
- (e) $F(\ell)$ when k is even.

For a given family of graphs \mathcal{F} , we say a graph G is a *maximal* \mathcal{F} -free graph with $c(G) < k$ if, for any non-edge ab of G , $G+ab$ contains either a copy of $F \in \mathcal{F}$ or a cycle of length at least k .

The following theorem is the main result of this paper, from which one can derive Theorem 1.1 and some other results (such as the results of [4, 5, 8]), to be discussed in Section 5. Mainly, it says that by forbidding some family $\mathcal{K}_{k,\alpha}$, one can have a good understanding on structural properties of graphs with given circumference and relatively many s -cliques.

Theorem 4.1. *Let $k \geq 5$, $\alpha \geq 0$ and $\beta \geq 2$ be integers. Let G be an n -vertex 2-connected maximal $\mathcal{K}_{k,\alpha}$ -free graph with $c(G) < k$. If $\ell - \alpha \geq \beta$ and*

$$N_s(G) > \max\{h_s(n, k, \ell - \alpha), h_s(n, k, \beta)\}, \quad (1)$$

then we have either $\omega(G) > k - \beta$ or $|V(H(G, \ell - 1))| < k - \ell + \alpha$.

We note that if α or β is larger, then $\max\{h_s(n, k, \ell - \alpha), h_s(n, k, \beta)\}$ is smaller and presumably the structure of G becomes more complicated. Also we have $\omega(G) \leq k - 2$, so (b) does not occur when $\beta = 2$. Equivalently, Theorem 4.1 states that an n -vertex 2-connected graph G satisfying (1) with $\beta = 2$ and $|V(H(G, \ell - 1))| \geq k - \ell + \alpha$ contains either a copy of $K \in \mathcal{K}_{k,\alpha}$ or a cycle of length at least k .

4.1 Some facts on $\mathcal{F}(m, k, r)$ with $r \leq \ell - 2$

We need the following technical propositions.

Proposition 4.2. *Let G be an n -vertex connected graph with a non-edge c_1c_2 and $n \geq 6$. Assume that each vertex except c_1 and c_2 of G has degree $n-2$. Then the following hold:*

- (i) *For each $ab \in E(G)$, there is a Hamilton path starting from c_1 through ab and ending at c_2 .*
- (ii) *For each $v \in V(G) \setminus \{c_1, c_2\}$, there is a path on $n-1$ vertices starting from v ending at $\{c_1, c_2\}$.*
- (iii) *For each non-edge $ab \neq c_1c_2$ of G , there is a path starting from c_1 through ab and ending at c_2 on at least n vertices in $G+ab$ except when $\{d_G(c_1), d_G(c_2)\} = \{1, n-3\}$.*

Proof. Let $A = V(G) \setminus \{c_1, c_2\}$. We divide the vertex of A into A_0, A_1 and A_2 such that each vertex in A_0 is adjacent to both of $\{c_1, c_2\}$ and each vertex of A_i is not adjacent to c_i for $i = 1, 2$. Since each vertex of A has degree $n-2$, $G[A_1]$ and $G[A_2]$ are complete graphs and $G[A_0]$ is the complement of the graph of $|A_0|/2$ independent edges. Moreover, since G is connected, if $|A_0| = 0$, then $|A_1| \geq 1$ and $|A_2| \geq 1$. Note that $|A| \geq 4$. For each $ab \in E(G)$, we can easily find a Hamilton path starting

¹⁰If k is odd, then $\mathcal{K}_{k,\alpha}$ only contains graphs in $\mathcal{F}(m, k, r)$ with $r \in \{1, \dots, \alpha, \ell-1\}$.

from c_1 through ab and ending at c_2 (consider $|A_0| = 0$, $|A_0| = 2$ and $|A_0| \geq 4$ separately). Thus we finish the proof of (i). The proof of (ii) is similar. Now let ab be a non-edge. Adding the edge ab and deleting at most two independent edges between $\{a, b\}$ and $\{c_1, c_2\}$ such that each vertex of A in the obtained graph has degree $n - 2$. If the obtained graph is connected, then one can easily prove (iii) by applying (i). Assume that the obtained graph is not connected. Then $\{a, b\} \cap \{c_1, c_2\} \neq \emptyset$, and hence it is easy to see that $\{d_G(c_1), d_G(c_2)\} = \{1, n - 3\}$. Moreover, we have $G[A] = K_{n-2}$. The proof is complete. \square

Recall vertices $x, x_i, y, y_1, z_i, z'_i$ in those special graphs in $\mathcal{F}(m, k, r)$. For $F = F_0(m, k, r)$, we denote by v the isolated vertex in $F[D]$ and v_1, v_2 the neighbour of v in $F[C \cup D]$, respectively. For $F = F_2(m, k, r)$, we denote by y_2 the neighbour of y in $F[C \cup D]$. For $F \in \mathcal{F}_3(m, k, r)$, we denote by z_2, z'_2 the neighbour of z_1, z'_1 in $F[C \cup D]$, respectively. For $F = F_5(m, k, 2)$ with $F[A] = S_3$, where S_3 is a star in three vertices, we denote by u_1 the center of S_3 .

Proposition 4.3. *For $1 \leq r \leq \ell - 2$, each $F \in \mathcal{F}(m, k, r)$ satisfies the following:*

- (i) *Let $ab \in E(F)$. If $ab \in \{x_1x_2, z_1z_2, z'_1z'_2, vv_1, vv_2, y_1y_2\}$, $ab \in E(\{u_1\}, C)$ or $ab \in E(\{y_1\}, C)$ and $r \geq 2$, then there is a cycle of length $k - 2$ containing ab ; otherwise, there is a cycle of length $k - 1$ containing ab .*
- (ii) *For each non-edge ab in $A \cup B \cup D$, if $\{a, b\} \subseteq A$, $\{a, b\} \subseteq A \cup \{x\}$, $\{a, b\} \subseteq A \cup \{y\}$, $u_1 \in \{a, b\}$ or $y_1 \in \{a, b\}$ and $r \geq 2$, then $F + ab$ contains a cycle of length $k - 1$ containing ab ; otherwise, $F + ab$ contains a cycle of length at least k containing ab .*
- (iii) *For each non-edge ab between $A \cup B \cup D$ and C , $F + ab$ contains a cycle of length at least $k - 2$ containing ab . Moreover, if ab is between A and C with $u_1 \notin \{a, b\}$, then $F + ab$ contains a cycle of length at least $k - 1$ containing ab .*
- (iv) *Suppose that G is 2-connected with $c(G) < k$ and containing a copy of F . Then $G - A \cup B \cup C$ is a star forest.*

Proof. Note that $|C| = \ell - r + 1 \geq 3$. We only verify some special cases and leave other cases to readers.

(i). For $ab = x_1x_2$, since the longest path starting from x_1 ending at x_2 contains at most $|D| - 2$ vertices of D , the result follows. For $|A| = r + 1 \geq 4$ and k is even, the result follows from Proposition 4.2(i). Be careful! For $|A| = r + 1 = 2$ and even k , by the definition of $\mathcal{F}(m, k, r)$, we have $F[A] = K_2$.

(ii). Let $\{a, b\} \subseteq A \cup \{y\}$ be a non-adjacent pair. Then we have $y_1 \in \{a, b\}$ and $r \geq 2$. Since the longest path starting from y_1 in F is on at most $k - 1$ vertices when $r \geq 2$, the result follows. For $|A| = r + 1 \geq 4$ and k is even, the result follows from Proposition 4.2(ii) easily.

(iii). Let $F = F_4(k + 1, k, 3)$ and $\{a, b\}$ be non-adjacent pair between $A \cup B \cup C$ and D . Then the longest path starting from a ending at b contains all vertices of $A \cup B \cup C$ and at least one vertex of D . Thus, we have $c(F + ab) \geq k + 1 - 3 = k - 2$. The result follows. Let $|A| = r + 1 \geq 4$, k be even and c_1, c_2 be the end-vertices of $F[C \cup D]$. Without loss of generality, let $d_{F[A \cup C]}(c_1) \leq d_{F[A \cup C]}(c_2)$. If $F[A] = K_\ell$ and $d_{F[A \cup C]}(c_1) = 1$, then it is easy to see that (iii) holds (note that $|C| \geq 3$). Otherwise, the result follows from Proposition 4.2(iii).

(iv). Let X be a non-trivial component of $G - A \cup B \cup C$.¹¹ Since G is 2-connected with $c(G) < k$, by (i), (ii), and (iii), X is only connected to C . Note that, for any two vertices $s_1, s_2 \in C$, there is a path on at least $k - 3$ vertices starting from s_1 and ending at s_2 . The longest path starting from C through X ending at C is on at most four vertices. Then there is an edge uv in X which is connected to C by two independent edges. Moreover, $V(X) - \{u, v\}$ is an independent set of $G[X]$ and each vertex of $X - \{u, v\}$ is adjacent to the same vertex of $\{u, v\}$. Otherwise, it is not hard to show that G contains a cycle of length at least k , a contradiction. Thus $G[X]$ is a star. This finishes the proof of the proposition. \square

¹¹We say a component is *trivial* if it consists of a unique vertex.

Let E_{n-k+1} be the $(n-k+1)$ -vertex graph consisting of $\lfloor \frac{n-k+1}{2} \rfloor$ independent edges. Let $G(n, k, 3)$ be the graph obtained from a disjoint union of $F_4(k+1, k, \ell-2)$ and E_{n-k+1} by joining each vertex of the set C in $F_4(k+1, k, \ell-2)$ to each vertex in the set D and $V(E_{n-k+1})$. Denote by $g_s(n, k, 3)$ the number of unlabeled s -cliques of $G(n, k, 3)$. Recall that $h_s(n, k, r)$ is the number of unlabeled s -cliques of $H(n, k, r)$. Also recall that $F_4(m, k, \ell-2)$ is the only graph with $m > k$ and $r \leq \ell-2$. We need the following lemma to prove our main theorem.

Lemma 4.4. *Let G be a 2-connected graph on n vertices with $c(G) < k$. Let $m \geq k \geq 9$ and $1 \leq r \leq \ell-2$. Suppose that G contains a copy of $F \in \mathcal{F}(m, k, r)$. Then*

(i) *Let $\gamma = \min\{\ell-r+2, \ell\}$. If $F = F_2(k, k, r)$, then*

$$N_s(G) \leq \min\{h_s(n, k, \gamma), \dots, h_s(n, k, \ell)\},$$

(ii) *If $F = F_5(k, k, 2)$, then*

$$N_s(G) \leq h_s(n, k, \ell).$$

(iii) *If $F = F_0(k, k, \ell-2)$ or $F = F_4(k, k, \ell-2)$, then*

$$N_s(G) \leq \min\{g_s(n, k, 3), h_s(n, k, 4), \dots, h_s(n, k, \ell)\}.$$

(iv) *Otherwise,*

$$N_s(G) \leq \min\{h_s(n, k, \ell-r+1), \dots, h_s(n, k, \ell)\}.$$

Proof. We begin with a claim. Let

$$f_s(n, k, r) = \binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\ell-r+1}{s} + (n-k+\ell-r) \binom{\ell-r+1}{s-1}.$$

Claim. $f_s(n, k, r) \leq h_s(n, k, t)$ for any $t \geq \ell-r+1$.

Proof. Let $t \geq \ell-r+1$. We have

$$\begin{aligned} f_s(n, k, r) &= \binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\ell-r+1}{s} + (n-k+\ell-r) \binom{\ell-r+1}{s-1} \\ &\leq \binom{k-t}{s} + \binom{t+1}{s} - \binom{t}{s} + (n-k+t-1) \binom{t}{s-1} = h_s(n, k, t), \end{aligned}$$

where the second inequality follows by $\binom{k-\ell}{s} + \binom{\ell+1}{s} \leq \binom{k-t}{s} + \binom{t+1}{s}$ and the fact that $(n-k+t-1) \binom{t}{s-1} - \binom{t}{s}$ increases with t when $s \geq 2$. The proof is complete. \square

Let $F \in \mathcal{F}(m, k, r) \setminus \{F_2(k, k, r)\}$. Let G be an n -vertex 2-connected graph with $c(G) < k$ containing a copy of F and $X = G - F$. By Proposition 4.3(iv), X is a star forest. First, we consider the case: $|A| = r+1$ or k is odd, i.e., $F[C \cup D]$ is a C -path and C, D are empty sets. Since G is 2-connected with $c(G) < k$, by Proposition 4.3(i), (ii) and (iii), it is easy to check that X is an independent set. Moreover, if $F \neq F_5(k, k, 2)$, then it is easy to see that each $x \in X$ is only adjacent to C . Let $t \geq \ell-r+1$. Since the numbers of unlabeled s -cliques inside $A \cup B \cup C$ and unlabeled s -cliques incident with D are at most $\binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\ell-r+1}{s}$ and at most $(n-k+\ell-r) \binom{\ell-r+1}{s-1}$ respectively. By the claim, we have that for any $t \geq \ell-r+1$,

$$N_s(G) \leq \binom{k-\ell}{s} + \binom{\ell+1}{s} - \binom{\ell-r+1}{s} + (n-k+\ell-r) \binom{\ell-r+1}{s-1} \leq h_s(n, k, t).$$

Now let $F = F_5(k, k, 2)$. Then each vertex of X can only be adjacent to $\{u_1\} \cup C$. Thus it is easy to check that $N_s(G) \leq h_s(n, k, \ell)$.

Now we may suppose that k is even and $|A| = r \leq \ell-3$. Then $|C| \geq 4$. (a). $F[C \cup D]$ is a C -path, i.e., F is of Type II. Then there is a unique edge in $F[D]$. Clearly, by Proposition 4.3(i), (ii), and

(iii), each isolated vertex of $G[X \cup D]$ is only adjacent to C . Let uw be the unique edge in $F[D]$. Denoted by u_1 and w_1 the neighbours of u and w in $F[C \cup D]$ respectively. Since G is 2-connected with $c(G) < k$, each independent edge in $G[X \cup D]$ can only be adjacent to $\{u_1, w_1\}$. Moreover, the center of each star S_α with $\alpha \geq 3$ in $G[X \cup D]$ is adjacent to both of $\{u_1, w_1\}$ and the leaves of S_α is only adjacent to, without loss of generality, u_1 . Recall that $|C| \geq 4$. Thus, by the claim, it is not hard to show that $N_s(G) \leq f_s(n, k, \ell - r + 1) \leq h_s(n, k, t)$ for any $s \geq 2$ and $t \geq \ell - r + 1$. (b). $F = F_1(k, k, r)$. Since $c(G) < k$, it is easy to check that X is an independent set and each $x \in X$ is only adjacent to C or to $\{x_1, x_2\}$, the result follows similarly as before. (c). $F \in \mathcal{F}_3(k, k, r)$. Let $C_1 = C \cap P'$ and $C_2 = C \cap P^*$, where $P' \cup P^* = F[C \cup D]$. Then it is not hard to see that X is an independent set. By Proposition 4.3(i), (ii), and some observations (for each non-adjacent pair (a, b) between C and D , there is a path on at least $k - 1$ vertices), each $x \in X$ is only adjacent to $\{z_1, z_2\}$, $\{z'_1, z'_2\}$, C_1 or C_2 . Hence, by the claim we have $N_s(G) \leq f_s(n, k, \ell - r + 1) \leq h_s(n, k, t)$ for any $s \geq 2$ and $t \geq \ell - r + 1$. The result follows.

Let k be even and $|A| = r = \ell - 2$. Then $|C| = 3$. For $F \in \mathcal{F}(m, k, \ell - 2) \setminus \{F_0(k, k, \ell - 2), F_2(k, k, \ell - 2), F_4(k + 1, k, \ell - 2)\}$, similarly as previous arguments, we have $N_s(G) \leq h_s(n, k, \ell - 2)$. The result follows from the claim. Assume that $F = F_0(k, k, \ell - 2)$ or $F = F_4(k + 1, k, \ell - 2)$. Since G is 2-connected with $c(G) < k$, by Proposition 4.3(i), (ii), and (iii), each vertex of $G[D \cup X]$ is not adjacent to $A \cup B$. Moreover, for each star S_α with $\alpha \geq 3$, the center of the star is adjacent to at least two vertices of C and the leaves of S_α are adjacent to the same vertex $x \in C$. Furthermore, for other vertices, each of them is adjacent to all vertices of C . Let $t \geq 4$. Since $n \geq k$, basic calculations show that $\lfloor \frac{n-k+3}{2} \rfloor \left(\binom{5}{s} - \binom{3}{s} \right) + i \binom{4}{s} \leq (n - k + 4) \binom{4}{s-1} - \binom{4}{s}$, where $i = 1$ when $n - k + 3$ is odd, and $i = 0$ when $n - k + 3$ is even. Then, combining the above arguments, we have

$$\begin{aligned} N_s(G) \leq g_s(n, k, 3) &= 2 \binom{\ell + 1}{s} - \binom{3}{s} + \left\lfloor \frac{n - k + 3}{2} \right\rfloor \left(\binom{5}{s} - \binom{3}{s} \right) + i \binom{4}{s} \\ &\leq \binom{k - t}{s} + \binom{t}{s} + (n - k + 4) \binom{4}{s-1} - \binom{4}{s} \\ &\leq \binom{k - t}{s} + \binom{t}{s} + (n - k + t) \binom{t}{s-1} - \binom{t}{s} = h_s(n, k, t), \end{aligned}$$

where the third inequality holds from the fact that $(n - k + t) \binom{t}{s-1} - \binom{t}{s}$ increases with t . Thus we finish the proof for $r = \ell - 2$.

Finally, let $F = F_2(k, k, r)$. Since G is 2-connected with $c(G) < k$, by Proposition 4.3(i), (ii) and (iii), X is an independent set. If y is adjacent to exactly one vertex of A , then each vertex in $\{y\} \cup X$ can only be adjacent to vertices of $C \cup \{y_1\}$ and each vertex of B can only be adjacent to vertices of $C \cup \{y_1\}$. Similarly as the previous proof, we have $N_s(G) \leq f_s(n, k, \ell - r + 2)$. If y is adjacent to two vertices of A , then y can be adjacent to all vertices of A and each vertex of X can only be adjacent to C . Hence, we have $N_s(G) \leq f_s(n, k, \ell - r + 1)$ as before. Thus, it follows from the claim that $N_s(G) \leq h_s(n, k, t)$ for any $t \geq \ell - r + 2$. The proof is complete. \square

4.2 Proof of Theorem 4.1

Now we are ready for the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $k \geq 5$, $\alpha \geq 0$, $\beta \geq 2$, $\ell = \lfloor (k - 1)/2 \rfloor$ and $\ell - \alpha \geq \beta$. Let G be an n -vertex 2-connected maximal $\mathcal{K}_{k, \alpha}$ -free graph with $c(G) < k$ satisfying (1). Thus, if $xy \notin E(G)$, then either $G + xy$ contains a copy of $K \in \mathcal{K}_{k, \alpha}$, or a cycle of length at least k . Now suppose that $\omega(G) \leq k - \beta$ and $|V(H(G, \ell - 1))| \geq k - \ell + \alpha$. We will finish our proof by contradictions. Let $H = H(G, \ell - 1)$.

Claim. H is a complete graph.

Proof. Suppose not, there is a non-edge ab in H . We prove the claim in the following four cases.

Case 1. $G + ab$ contains a cycle of length at least k .

Then, by $a, b \in V(H)$, there is an H -path on at least k vertices. Thus, there exists a longest H -path P on $m \geq k$ vertices. If $\alpha = \ell - 2$ or $k \leq 8$, i.e., $\ell \leq 3$, then by Lemma 3.2, G contains a copy of $F \in \mathcal{F}(m, k, r)$, contradicting that G is $\mathcal{K}_{k, \alpha}$ -free. Hence, we may suppose $\alpha < \ell - 2$ and $k \geq 9$, i.e., $\ell \geq 4$. Since G is $\mathcal{K}_{k, \alpha}$ -free, by Lemma 3.2, G contains a copy of $F \in \mathcal{F}(m, k, r) \setminus \mathcal{K}_{k, \alpha}$.

Let k be odd, or $r \leq \ell - 3$ and $\alpha \geq 2$. Since G is 2-connected and $c(G) < k$, it follows from Lemma 4.4(iv) that

$$N_s(G) \leq \min\{h_s(n, k, \ell - r + 1), \dots, h_s(n, k, \ell)\} \leq h_s(n, k, \ell - \alpha),$$

a contradiction to (1). For $r \leq \ell - 3$ and $\alpha = 1$, combining Lemma 4.4(ii) and Lemma 4.4(iv) we can also easily get a contradiction.

Assume that $r = \ell - 2$ and $k \geq 10$ is even. Note that $\ell - \alpha \geq 3$. If $\ell - \alpha = 3$, i.e., $\alpha = \ell - 3$, then we have $F \in \mathcal{F}(m, k, \ell - 2) \setminus \{F_0(k, k, \ell - 2), F_2(k, k, \ell - 2), F_4(k + 1, k, \ell - 2), F_5(k, k, 2)\}$ (G is $\mathcal{K}_{k, \alpha}$ -free). By Lemma 4.4(iv), we have

$$N_s(G) \leq \min\{h_s(n, k, 3), h_s(n, k, 4), \dots, h_s(n, k, \ell)\} \leq h_s(n, k, 3),$$

a contradiction. Let $\ell - \alpha \geq 4$. Then $F \in \mathcal{F}(m, k, \ell - 2) \setminus \{F_2(k, k, \ell - 2)\}$. It follows from Lemma 4.4(iii) and (iv) that

$$N_s(G) \leq \min\{\max\{g_s(n, k, 3), h_s(n, k, 3)\}, h_s(n, k, 4), \dots, h_s(n, k, \ell)\} \leq h_s(n, k, \ell - \alpha),$$

which is also a contradiction to (1). This completes the proof of Case 1.

If $c(G + ab) \geq k$ or there is an H -path on at least k vertices, then by Case 1, we get a contradiction. Thus, in the following cases, it suffices to show that either $c(G + ab) \geq k$ or there is an H -path on at least k vertices.

Now, suppose that $G + ab$ contains a copy of $F \in \mathcal{K}_{k, \alpha}$. We divide the following proof into two cases basing on the value of r in $\mathcal{F}(m, k, r)$.

Case 2. $G + ab$ contains a copy of $F \in \mathcal{F}(k, k, r)$ for some $r \in \{1, 2, \dots, \alpha\}$ with $\alpha \leq \ell - 2$; $F_0(k, k, \ell - 2)$ or $F_4(k + 1, k, \ell - 2)$ when $k \geq 10$ is even and $\ell - \alpha \leq 3$; or $F_2(k, k, \alpha + 1)$ with $\alpha + 1 \leq \ell - 2$; or $F_5(k, k, 2)$ with $\alpha = 1$ and $\ell \geq 4$, i.e. $r = 2 \leq \ell - 2$.

Let $A \cup B \cup C \cup D$ be a partition of $V(F)$ in Section 2. By Proposition 4.3(i), for each edge ab of F , there is a path on $k - 1$ vertices starting from a and ending at b in F , except that $ab \in \{x_1x_2, z_1z_2, z'_1z'_2, vv_1, vv_2, y_1y_2\}$, $ab \in E(\{u_1\}, C)$ or $ab \in E(\{y_1\}, C)$ with $r \geq 2$. We may assume

$$N_H(a) \subseteq V(F) \text{ and } N_H(b) \subseteq V(F). \quad (2)$$

Otherwise, since $a, b \in H$, by Proposition 4.3(i), there is an H -path on at least k vertices, and we are done. Note that there is no edge in $F[C]$. We can choose $v \in \{a, b\} \cap (A \cup B \cup D)$. Then we have

$$v \in A \text{ and } N_H(v) \subseteq A \cup C. \quad (3)$$

Otherwise, since $|C| \leq \ell$, by Proposition 4.3(ii) we have $c(G + ab) \geq k$, and hence G contains an H -path on k vertices. Since a is not adjacent to b , $N_H(a) \geq \ell$, $N_H(b) \geq \ell$ and $|A \cup C| \leq \ell + 2$, it follows from (3) that each vertex in A has degree at least ℓ in $H[A \cup C]$. Thus G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $|A| = r + 1$ or a copy of $F(\ell)$ (when $|A| = 2$ and $e(H[A]) = 0$). Both are contradictions.

Let $ab \in \{x_1x_2, z_1z_2, z'_1z'_2, vv_1, vv_2, y_1y_2\}$, $ab \in E(\{u_1\}, C)$ or $ab \in E(\{y_1\}, C)$ with $r \geq 2$. Then by Proposition 4.3(i), there is a path on $k - 2$ vertices starting from a and ending at b in F . Thus for each $w_a \in N_H(a) \setminus V(F)$ and each $w_b \in N_H(b) \setminus V(F)$, we have

$$w_a = w_b = w. \quad (4)$$

Otherwise, there is an H -path starting from w_a ending at w_b on k vertices and we are done. Now, we consider the following six cases:

(2.1) Let $ab = x_1x_2$. First, a and b are not adjacent to any vertex of $(B \cup D) \setminus \{x\}$. Otherwise, by Proposition 4.3(ii), we can deduce that $c(G + ab) \geq k$, and hence we are done. Since $|A \cup C| = \ell + 1$ and a is not adjacent to b , $|N_H(a) \setminus (A \cup C)| \geq 1$ and $|N_H(b) \setminus (A \cup C)| \geq 1$. Thus by (4), we have $N_H(a) \setminus V(F) = N_H(b) \setminus V(F) = \{w\}$. Then $w \in V(H)$ and $A \cup C \subseteq V(H)$. Note that each vertex of B has degree ℓ in $G[A \cup B \cup C]$. Thus we have $B \subseteq V(H)$. So we can easily find an H -path on k vertices.

(2.2) Let $ab \in \{z_1z_2, z'_1z'_2\}$. Without loss of generality, let $ab = z_1z_2$. If $N_H(z_1) \subseteq V(F)$, then Proposition 4.3(ii) implies that $A \cup C \cup \{z\} \subseteq V(H)$. Note that each vertex of B has degree ℓ in $G[A \cup B \cup C]$. This implies that $B \subseteq V(H)$. Hence, there is a path on k vertices starting from B ending at z and we are done. Now we may suppose that there is a vertex $w \in N_H(z_1) \setminus V(F)$. If there is a vertex $w' \in N_H(w) \setminus V(F)$, then we can find a path on k vertices starting from w' ending at z_2 and hence we are done. Now let $N_H(w) \subseteq V(F)$. Then by Proposition 4.3(i) and (ii), if there is a vertex $z \in N_H(w) \setminus \{z_1, z_2\}$, then $c(G + ab) \geq k$. Thus $N_H(w) \subseteq \{z_1, z_2\}$ and $\ell = 2$, and hence $G = H$. Therefore, it is easy to find an H -path on at least k vertices.

(2.3) Let $ab \in \{vv_1, vv_2\}$. Then $|C| = 3$. Without loss of generality, let $ab = vv_1$. Then there is a vertex $w \in N_H(v) \setminus V(F)$. Otherwise, since $|C| \leq \ell$ and v is not adjacent to v_1 , we have $N_H(v) \cap (A \cup B \cup D) \neq \emptyset$. It follows from Proposition 4.3(ii) that $c(G + ab) \geq k$ and we are done. Again, it follows from Proposition 4.3(ii) that $C \subseteq N_H(w)$ or there is a vertex $w' \in N_H(w) \setminus V(F)$. Thus, in the former case, we have $A \cup B \cup C \subseteq V(H)$, and hence, there is a path on k vertices starting from w ending at B . In the later case, there is a path on k vertices starting from w' ending at v_1 . We are done in both cases.

(2.4) Let $ab = y_1y_2$. Then $r = \alpha + 1$. (a) $r \geq 2$. If y_1 is adjacent to B , then G contains a copy of $F \in \mathcal{F}(k, k, \alpha)$. If there is an $y' \in N_H(y_1) \cap D$, then $y' \in V(H)$; as $|C| = \ell - r + 1 \leq \ell - 1$, there is a vertex $w^* \in N_H(y')$. We can find an H -path on at least k vertices starting from w^* and ending at y_1 . Thus, there is a vertex $w \in N_H(y_1) \setminus V(F)$. If there is a vertex $w' \in N_H(w) \setminus V(F)$, then there is a path on k vertices starting from w' ending at y_2 , and we are done. Assume that $N_H(w) \subseteq V(F)$. Thus by Proposition 4.3(i), we have $N_H(w) \subseteq C$. Therefore, since $|C| \leq \ell - 1$, we have $C = N_H(w) \subseteq V(H)$. Hence, as before, we have $B \subseteq V(H)$. We can easily find a path on k vertices starting from w and ending at B . (b) $r = 1$. This case is similar as (a).

For $ab \in E(\{y_1\}, C)$ with $r \geq 2$ or $ab \in E(\{u_1\}, C)$, the proofs are essentially the same as the proof of (2.4) and thus we omit here.

Case 3. $G + ab$ contains a copy of $F \in \mathcal{F}(m, k, r)$ for $r \in \{\ell - 1, \ell\}$.

Let $G + ab$ contains a copy of $F \in \mathcal{F}(m, k, \ell - 1)$. Note that $\delta(F[A \cup B \cup C]) \geq \ell$ and $a, b \in V(H)$. It is clearly that $A \cup B \cup C \subseteq V(H)$. For k is odd, or $|A| = r = \ell - 1$ and k is even, since $F[A] = F[B] = K_{\ell-1}$, it is not hard to find an H -path on at least k vertices. Let k be even and $|A| = r + 1 = \ell$. (a). ab is incident with D . Let $a \in D$. Since $d_{F[A \cup C]}(w) \geq \ell$ for each $w \in A$, by Proposition 4.2(i) when $\ell \geq 4$ and by definition of $\mathcal{F}(m, k, \ell - 1)$ when $\ell = 2$, we can find a Hamilton C -path in $F[A \cup C]$. For $\ell = 3$, a simple observation shows that there is also a Hamilton C -path in $F[A \cup C]$. Thus there is path on at least k vertices starting from a , through the Hamilton path in $F[A \cup C]$, ending at B . We are done. (b). $ab \in F[A \cup C]$ or $ab \in F[B \cup C]$. Note that $A \cup B \cup C \subseteq V(H)$. The proofs (to be divided into cases: $ab \in \{x_1x_2, y_1y_2\}$, $ab \in E(\{u_1\}, C)$ or $ab \in E(\{y_1\}, C)$ with $r \geq 2$.) can be handled similarly as the proofs in Case 2.

Now, let k be even and $G + ab$ contains a copy of $F \in \mathcal{F}(m, k, \ell)$ for $m \geq k$. Let $X = A \cup B \cup \{w, w_1, w_2\}$. Since the degree of each vertex of $X \setminus \{a, b\}$ in $G[X]$ is at least ℓ , together with $a, b \in H$, we have $X \subseteq H$. Hence, if $ab \notin E(C)$ or $ab = w_1w_2$, then we can easily find an H -path on at least k vertices. If $ab \in E(C)$ and $ab \neq w_1w_2$, then there is a path on at least k vertices starting from a (or b), through w , A , w_1w_2 and ending at B (note that $ww_1, ww_2 \notin E(C)$).

Case 4. $G + ab$ contains a copy of $F(\ell)$ when k is even.

Let A and B be the partite sets of $P_{2\ell-1}$ in $F(\ell)$ with $|A| = \ell$ and $|B| = \ell - 1$. Let $C = V(F(\ell)) \setminus (A \cup B)$. If there is an edge in $G[C]$, then $G + ab$ contains a copy of $F \in \mathcal{F}(k, k, 1)$, and we are done by Case 2. Thus, we may assume that $G[C]$ is empty. Let $a \in A$ and $b \in C$. If b is adjacent to A , then $G + ab$ contains a copy of $F \in \mathcal{F}(k, k, 1)$, and hence we are done. Since $|C| \leq \ell$ and a is not adjacent to b , there is a vertex $w \in N_H(b) \setminus (A \cup B \cup C)$. If there is a $w' \in N_H(w) \setminus (A \cup B \cup C)$, then there is a path on k vertices starting from w' ending at a . Thus, $N_H(w) \subseteq (A \cup B \cup C)$. Then, it is not hard to see that $c(G + ab) \geq k$, and we are done. We omit the proof when $a \in A$ and $b \in B$. We complete our proof of the claim. \square

Let $|V(H)| = m$. Since $V(H(G, \ell - 1)) \geq k - \ell + \alpha$ and $\omega(G) \leq k - \beta$, we have $k - \ell + \alpha \leq m \leq k - \beta$. Apply to the graph G the process of $(k - m)$ -disintegration. Let $H' = H(G, k - m)$. If $H' = H$, then

$$N_s(G) \leq \binom{m}{s} + (n - m) \binom{k - m - 1}{s - 1} \leq \max\{h_s(n, k, \ell - \alpha), h_s(n, k, \beta)\},$$

a contradiction to (1). If $H' \neq H$, then there exists a vertex $b \in V(H')$ which is not adjacent to a vertex $a \in V(H)$. We divide the proof into the following two cases: (a). Adding ab , the obtained graph contains a cycle of length at least k . Then there is a path in G on at least k vertices starting in H and ending in H' . Let $P = xPy$ be a longest such path with $x \in V(H)$ and $y \in V(H')$. Then we have $d_P(a) \geq m - 1$ and $d_P(b) \geq k - m + 1$. It follows from Lemma 3.1 that $c(G) \geq k$, a contradiction. (b). Adding ab , the obtained graph contains a copy of $K \in \mathcal{K}_{k, \alpha}$. Note that H is a complete graph on $m \geq k - \ell + \alpha$ vertices, $d_{H'}(b) \geq k - m + 1$ and $c(G) \leq k - 1$. Similarly as the Cases 2, 3 and 4, we can find a path on at least k vertices starting from H and ending at H' (actually the situation here will be easier than previous cases). Thus, by Lemma 3.1 again, we have $c(G) \geq k$. This final contradiction completes the proof of Theorem 4.1. \square

5 Implications

In this section, we shall use Theorem 4.1 to deduce Theorem 1.1 and equivalent statements of some main results in [4, 5, 8]. We need the following result proved by Fan [3].

Theorem 5.1 (Fan [3]). *Let G be an n -vertex 2-connected graph and ab be an edge in G . If the longest path starting from a and ending at b in G has at most r vertices, then $e(G) \leq \frac{(r-3)(n-2)}{2} + 2n - 3$. Moreover, the equality holds if and only if $G - \{a, b\}$ is a vertex-disjoint union of copies of K_r .*

First, we can derive a more general result concerning the number of cliques from Theorem 4.1.

Corollary 5.2. *Let G be an n -vertex 2-connected graph with minimum degree $\delta \geq 2$. Let $k \geq 9$ and $\ell - 1 \geq \delta + 1$.¹² If $c(G) < k$ and*

$$N_s(G) > \max\{h_s(n, k, \ell - 1), h_s(n, k, \delta + 1)\}, \quad (5)$$

then one of the following holds:

- (a) G contains a copy of
 - (a.1) $F \in \mathcal{F}(m, k, r)$ with $r \in \{1, \ell - 1, \ell\}$, or
 - (a.2) $F_0(10, 10, 2)$ or $F_4(11, 10, 2)$ when $k = 10$, or
 - (a.3) $F_2(m, k, 2)$ or $F_5(m, k, 2)$ when k is even, or
 - (a.4) $F(\ell)$ when k is even;
- (b) G is a subgraph of the graph $Z(n, k, \delta)$,¹³
- (c) G is a subgraph of $H(n, k, \delta)$.

¹²If $5 \leq k \leq 8$, then $\ell - 1 < \delta + 1$. By (5), it follows from Luo's theorem that G contains a cycle of length at least k .

¹³The graph $Z(n, k, \delta)$ denotes the vertex-disjoint union of a clique $K_{k-\delta}$ and some cliques $K_{\delta+1}$'s, where any two cliques share the same two vertices.

Proof. If (a) holds, then we are done. Thus we may suppose that G is $\mathcal{K}_{k,1}$ -free. Let $J \supseteq G$ be a maximal $\mathcal{K}_{k,1}$ -free with $c(J) < k$. Suppose that $|V(H(J, \ell - 1))| \leq k - \ell$. Then

$$N_s(J) \leq (n - k + \ell) \binom{\ell - 1}{s - 1} + \binom{k - \ell}{s} = h_s(n, k, \ell - 1)$$

contradicting (5). Thus we have $|V(H(J, \ell - 1))| \geq k - \ell + 1$. Clearly, $N_s(J) > \max\{h_s(n, k, \ell - 1), h_s(n, k, \delta + 1)\}$ and $\delta(J) \geq \delta$. Applying Theorem 4.1 with $\alpha = 1$ and $\beta = \delta + 1$, we have that $\omega(J) \geq k - \delta$.

It suffices to show that either (b) or (c) holds. Let K be a maximum clique in J . Then there is a non-edge x_1x_m with $x_1 \in K$ and $x_m \in J - K$. Then by the maximality of J , we may first suppose that there is a longest path $P = x_1x_2 \dots x_m$ on $m \geq k$ vertices starting from x_1 and ending at x_m . Thus we have $d_P(x_1) \geq k - \delta - 1$ (note that $x_1 \in K$ and $|K| \geq k - \delta$) and $d_P(x_m) \geq \delta$. Similarly as the proofs in Lemma 3.2, we only need consider the case that there exist i and j with $2 \leq i < j \leq m - 1$ such that $x_j \in N_P(x_1)$ and $x_i \in N_P(x_m)$. Moreover, by the proofs of Lemma 3.2 (since there is a clique of size $k - \delta$ in $G[V(P)]$, the proofs here are much easier), J contains a copy of $F \in \mathcal{F}(m, k, r, \delta)$ for $1 \leq r \leq \delta - 1$, where each $F \in \mathcal{F}(m, k, r, \delta)$ with a partition $V(F) = A \cup B \cup C \cup D$ on m vertices satisfies the following:

- $F[A]$ and $F[B]$ are complete graphs with $|A| = r$ and $|B| = k - 2\delta + r - 1$;
- $F[C]$ is empty with $|C| = \delta - r + 1$;
- $F(A, C)$ and $F(B, C)$ are complete bipartite graphs;
- $F[D]$ is empty or a path when $m \geq k + 1$ and $|C| = 2$;
- and $F[C \cup D]$ is a C -path.

The graph family $\mathcal{F}(m, k, r, \delta)$ plays the same role as the graph family $\mathcal{F}(m, k, r)$, that is, if a 2-connected graph with $c(J) < k$ containing a copy of $F \in \mathcal{F}(m, k, r, \delta)$, then each component of $J - V(F)$ can only be adjacent to C of $V(F)$. Thus, if $|C| \geq 3$, then $J - V(F)$ is an independent set. Since $\delta(J) \geq \delta$, we have either $m = k$, $r = 1$ and $|C| = \delta$, or $r = \delta - 1$ and $|C| = 2$. If $m = k$, $r = 1$ and $|C| = \delta$, then J and hence G are subgraphs of $H(n, k, \delta)$. Let $r = \delta - 1$ and $|C| = 2$. Note that J is 2-connected with $c(J) < k$. Each path starting from C ending at C is on at most $\delta + 1$ vertices. Since $\delta(J) \geq \delta$, it follows from a result of Erdős and Gallai [2] (a minimum degree version of Theorem 5.1) that $J - A \cup B \cup C$ is a union of copies of $K_{\delta-1}$. Thus J and hence G are subgraphs of $Z(n, k, \delta)$, i.e., the union of a clique $K_{k-\delta}$ and some cliques $K_{\delta+1}$'s, where any two cliques share the same two vertices.

Now we suppose that $J + x_1x_m$ contains a copy of $\mathcal{K}_{k,1}$, then it is easy (as in the proof of Theorem 4.1) to find a path on at least k vertices starting from K and ending at $J - K$. The result follows by applying the previous argument. \square

Now we are ready to deduce Theorem 1.1 from Theorem 4.1.

Proof of Theorem 1.1. Let $2 \leq s \leq \max\{2, \ell - 1\}$ and G be an n -vertex 2-connected *maximal* (in the sense that adding any edge will create a cycle of length at least k) graph with $c(G) < k$ and $N_s(G) > h_s(n, k, \ell - 1)$. Note that if G satisfies the conclusion of Theorem 1.1, then any subgraph of G also satisfies the conclusion of Theorem 1.1. Thus, it suffices to prove Theorem 1.1 for the maximal graph G . Similarly as the proof of Corollary 5.2, we have $|V(H(G, \ell - 1))| \geq k - \ell + 1$. Since G is 2-connected with $c(G) < k$, we have $\omega(G) \leq k - 2$. For $\ell - 1 \geq 2$, i.e., $k \geq 7$ and $\ell \geq 3$, basic calculations show that there is a constant n_0 such that if $n > n_0$, then $h_s(n, k, \ell - 1) \geq h_s(n, k, 2)$. Let $n \geq n_0$ be sufficiently large and $m \geq k$. Combining the above arguments and applying Theorem 4.1 with $\alpha = 1$ and $\beta = 2$ when $\ell \geq 3$, and with $\alpha = 1$ and $\beta = 1$ when $\ell = 2$, G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $r \in \{1, \ell - 1, \ell\}$ or $F \in \{F_2(k, k, 2), F_5(k, k, 2), F_0(10, 10, 2), F_4(11, 10, 2), F(\ell)\}$.

If $F \notin \mathcal{F}(m, k, r)$ with $r = \ell - 1, \ell$, then similarly as the proof of Proposition 4.3(iv), it is easy to check that there is an $X \subseteq V(G)$ of order at most ℓ such that $G - X$ is a star forest. Moreover, if k is odd, then $G - X$ is an independent set and each vertex of it can only be adjacent to X , where $X \subseteq V(F)$ is of size ℓ .

Assume that G contains a copy of $F \in \mathcal{F}(m, k, r)$ for $r \in \{\ell - 1, \ell\}$. For $\ell \leq 3$, again, the result follows similarly as the proof of Proposition 4.3(iv). Furthermore, if $k = 5$, then there is an $X \subseteq V(G)$ of order two such that $G - X$ is an independent set; if $k = 7$, then there is an $X \subseteq V(G)$ of order two such that $G - X$ is a star forest. Let $\ell \geq 5$ when $s = 3$ and $\ell \geq 4$ otherwise. We need the following fact that

$$(\ell - 1) \binom{\ell - 1}{s - 1} \geq \binom{\ell + 1}{s} - \binom{2}{s} \text{ holds for } 2 \leq s \leq \ell - 1.$$

To see this, first let $s = 2$. Since $\ell \geq 4$, we have $(\ell - 1)^2 \geq \binom{\ell + 1}{s} - 1$. Now let $3 \leq s \leq \ell - 1$. Then it is enough to show that $s(\ell - 1)(\ell - s + 1) \geq (\ell + 1)\ell$. It is easy to see that $s(\ell - 1)(\ell - s + 1)$ attains its minimum when $s = 3$ or $s = \ell - 1$. If $s = 3$, then by $\ell \geq 5$, we have $3(\ell - 1)(\ell - 2) \geq (\ell + 1)\ell$. If $s = \ell - 1 > 3$, then $\ell \geq 5$, and hence we have $2(\ell - 1)^2 > (\ell + 1)\ell$, proving this fact.

Let $n \geq n_0 + t\ell - t$, where t is a large constant. Since G contain a copy of $F \in \mathcal{F}(m, k, r)$ for $r \in \{\ell - 1, \ell\}$. In both cases, F contains a clique with size $\ell - 1$ such that after deleting the vertices of it, the resulting graph G_1 is 2-connected. Moreover, each vertex of the clique is only adjacent to two vertices of F (C when $F \in \mathcal{F}(k, k, \ell - 1)$, and w, w_1 or w, w_2 when $F \in \mathcal{F}(k, k, \ell)$). Thus, by $s \leq \ell - 1$ and the above fact we have

$$N_s(G_1) \geq h_s(n, k, \ell - 1) - \left(\binom{\ell + 1}{s} - \binom{2}{s} \right) > h_s(n - \ell + 1, k, \ell - 1).$$

Repeat this progress $t - 1$ times. Since t is sufficiently large, we have

$$N_s(G_t) \geq h_s(n, k, \ell - 1) - t \left(\binom{\ell + 1}{s} - \binom{2}{s} \right) > N_s(K_{n_0 + \ell - 1}),$$

a contradiction. The proof of Theorem 1.1 now is complete. \square

We remark that for the case $s = 3$ and $k \in \{9, 10\}$ in the conclusion of Theorem 1.1, one can obtain a refined structural description by deleting at most four vertices so that the resulting graph is very close to a star forest (there may be some triangles in the resulting graph).

Next, we show how to use the above results to deduce some of the main results in [4, 5, 8] (in equivalent forms). We need the following lemma.

Lemma 5.3. *For $n \geq k \geq 9$, let G be an n -vertex 2-connected graph with $c(G) < k$. If G contains a copy of $F \in \mathcal{F}(m, k, r)$ with $r \in \{\ell - 1, \ell\}$, then $e(G) \leq h_2(n, k, \ell - 1)$.*

Proof. Let $F \in \mathcal{F}(m, k, \ell - 1)$ and $C = \{c_1, c_2\}$. Let $k = 2\ell + 1$ be odd. Then it is easy to see that the longest path starting from c_1 ending at c_2 is on at most $\ell + 1$ vertices. Since $\ell \geq 4$ and $n \geq k$, by Theorem 5.1,

$$e(G) \leq \frac{(\ell - 2)(n - 2)}{2} + 2n - 3 < \binom{\ell + 2}{2} + (\ell - 1)(n - \ell - 2) = h_2(n, k, \ell - 1),$$

as desired. Let $k = 2\ell + 2$ be even and $\ell \geq 5$. Note that the longest path starting from c_1 ending at c_2 in G is on $\ell + 2$ vertices (if there is a path starting from c_1 ending at c_2 in G on $\ell + 3$ vertices, then one may easily check that $c(G) \geq k$ by G contains a copy of F , a contradiction). Then by Theorem 5.1 and $\ell \geq 5$, we have

$$e(G) \leq \frac{(\ell - 1)(n - 2)}{2} + 2n - 3 < \binom{\ell + 3}{2} + (\ell - 1)(n - \ell - 3) = h_2(n, k, \ell - 1).$$

Thus we may suppose $k = 10$. If the longest path starting from c_1 and ending in c_2 has at most five vertices, then by Theorem 5.1, we have

$$e(G) \leq \frac{2(n-2)}{2} + 2n - 3 < 3n - 3 = h_2(n, 10, 3).$$

Thus we may assume that there is a longest path $P = c_1x_1x_2x_3x_4c_2$ in G . Let $G' = G - \{c_1, c_2\}$. Let X be the component of G' contains $\{x_1, x_2, x_3, x_4\}$. Let $X' = X \setminus \{x_1, x_2, x_3, x_4\}$. Let C be a component of $G[X']$ and $P^* = y_1y_2 \dots y_s$ be a longest path in C such that y_1 and y_s are adjacent to distinct vertices of P respectively. First we show that $s \leq 3$. Assume that $s \geq 4$. Since the longest path starting from c_1 and ending in c_2 has at most six vertices, y_1 and y_s are adjacent to c_1 and c_2 , respectively. Note that P^* is connected to $\{x_1, x_2, x_3, x_4\}$. We can easily find a path starting from c_1 and ending in c_2 has at least seven vertices, a contradiction. If $s = 1$, then C is an isolated vertex. Hence, the number of edges incident with C in G is at most three. If $s = 2$, then y_1 and y_2 is adjacent to two vertices of P^* with distant at least three respectively. As the proof Proposition 4.3(iv), we can show that C is a star and hence the number of edges incident with C in G is at most $2|C|$. At last, let $s = 3$. Tedious analysis shows that C is $K_{2,|C|-2}$, a star or a triangle. In all of the above cases, the number of edges incident with C in G is at most $3|C|$.

For any other component Y of G' , since $c(G) < 10$, by Theorem 5.1, the number edges incident with it is at most $3|Y|$. Summing all the above edges, we have $e(G) \leq \binom{6}{2} + 3(n-6) = 3n - 3 = h_2(n, 10, 3)$. We finish the proof when $F \in \mathcal{F}(m, k, \ell - 1)$.

Let $F \in \mathcal{F}(m, k, \ell)$. Then k is even and $n \geq 10$. Let w, w_1 and w_2 be the vertices of F as in Section 2. Since $c(G) < k$, the longest path starting from w_1 or w_2 through $G - \{w, w_1, w_2\}$ ending at w is on at most $\ell + 1$ vertices and each component of $G - \{w, w_1, w_2\}$ can only be adjacent to w_1, w or w_2, w . Let G_i be the induced subgraph of G containing $\{w, w_i\}$ and all components of $G - \{w, w_1, w_2\}$ which is adjacent to w_i for $i = 1, 2$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. Then $n = n_1 + n_2 - 1$. Since $n \geq 10$ and $\ell \geq 4$, by Theorem 5.1, we have

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) + 1 \leq \frac{(\ell-2)(n_1-2)}{2} + 2n_1 - 3 + \frac{(\ell-2)(n_2-2)}{2} + 2n_2 - 3 + 1 \\ &= \frac{(\ell-2)(n-3)}{2} + 2(n+1) - 3 - 2 < \binom{\ell+3}{2} + (\ell-1)(n-\ell-3) = h_2(n, k, \ell-1). \end{aligned}$$

This finishes the proof of the lemma. \square

Now we have the following immediate corollary, which can imply some of the main results in [4,5,8].

Corollary 5.4. *Let G be an n -vertex 2-connected graph with $c(G) < k$ and minimum degree $\delta(G) = \delta$. Let $k \geq 9$ and $\ell - 1 \geq \delta + 1$. If*

$$e(G) > \max\{h_2(n, k, \ell - 1), h_2(n, k, \delta + 1)\},$$

then one of the following holds:

- (a) G contains a copy of $F \in \mathcal{F}(k, k, 1) \cup \{F_2(k, k, 2), F_5(k, k, 2), F_0(10, 10, 2), F_4(11, 10, 2), F(\ell)\}$;
- (b) G is a subgraph of $Z(n, k, \delta)$;
- (c) G is a subgraph of $H(n, k, \delta)$.

Proof. This result follows directly from Corollary 5.2 with $s = 2$ and Lemma 5.3. \square

We would like to briefly explain how this corollary can imply the main result in [8] (i.e., Theorem 1.7). In our setting, Theorem 1.7 of [8] states that under the same conditions, if in addition $k \geq 11$, then one of the following four cases holds: (1) G is a subgraph of $H(n, k, \delta)$; (2) G is a subgraph of $H(n, k, \ell)$; (3) if $\delta = 2$ and k is even, then G is a subgraph of a member of two well-characterized families of graphs; (4) if $\delta \geq 3$, then G is a subgraph of $Z(n, k, \delta)$. It is clear that (c) and (b) in Corollary 5.2 correspond to the above (1) and (4), respectively. Without referring to uncomplicated details, we point out that if G contains a copy of $F \in \mathcal{F}(k, k, 1) \cup \{F_2(k, k, 2), F_5(k, k, 2), F(\ell)\}$, then this would lead to the above (2) and (3).¹⁴

¹⁴Here, as $k \geq 11$, the copy F cannot be $F_0(10, 10, 2)$ or $F_4(11, 10, 2)$.

We also can extend our results to connected graphs without paths of a given length. Let G be an n -vertex connected graph without containing a path of length $k - 2$. Let G^* be the graph obtained from G by adding a new vertex v and joining v to all vertices of G . Then G^* is an $(n + 1)$ -vertex 2-connected graph containing no cycle of length at least k . Now using a similar argument as in [7] (i.e., consider the unlabeled s -cliques without containing v), one can prove the following result as an analogous path version of Theorem 1.1.

Corollary 5.5. *Let $k \geq 5$, $2 \leq s \leq \max\{2, \ell - 2\}$ and $n \geq n_0(\ell)$, where $n_0(\ell)$ is a large constant depending on ℓ . Let G be an n -vertex connected graph without containing path of length $k - 2$. Then $N_s(G) \leq h_s(n, k - 2, \ell - 2)$ unless*

- (a) $s = 3$ and $k \in \{9, 10\}$,
- (b) $k = 2\ell + 1$, $k \neq 7$, and $G \subseteq H(n, k - 2, \ell - 1)$, or
- (c) $k = 2\ell + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most $\ell - 1$.

To conclude this paper, we would like to propose the following conjecture. This (if true) would give a strengthening of Theorem 1.1 (to cover all ranges of n similarly as in Corollary 5.4).

Conjecture 5.6. *Let G be a 2-connected graph on n vertices and let ab be an edge in G . Let $r \geq 4$ and $s \geq 2$ be integers, and let $n - 2 = x(r - 3) + t$ for some $0 \leq t \leq r - 4$. If $N_s(G) > x \binom{r-1}{s} + \binom{t+2}{s}$, then there is a cycle on at least r vertices containing the edge ab .*

This also can be viewed as a clique version of Theorem 5.1 of Fan [3].

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