Improvements on induced subgraphs of given sizes

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Abstract

Given integers m and f, let $S_n(m, f)$ consist of all integers e such that every n-vertex graph with e edges contains an m-vertex induced subgraph with f edges, and let $\sigma(m, f) = \limsup_{n\to\infty} |S_n(m, f)|/\binom{n}{2}$. As a natural extension of an extremal problem of Erdős, this was investigated by Erdős, Füredi, Rothschild and Sós twenty years ago. Their main result indicates that integers in $S_n(m, f)$ are rare for most pairs (m, f), though they also found infinitely many pairs (m, f) whose $\sigma(m, f)$ is a fixed positive constant. Here we aim to provide some improvements on this study. Our first result shows that $\sigma(m, f) \le \frac{1}{2}$ holds for all but finitely many pairs (m, f) and the constant $\frac{1}{2}$ cannot be improved. This answers a question of Erdős et. al. Our second result considers infinitely many pairs (m, f) of special forms, whose exact values of $\sigma(m, f)$ were conjectured by Erdős et. al. We partially solve this conjecture (only leaving two open cases) by making progress on some constructions which are related to number theory. Our proofs are based on the research of Erdős et. al and involve different arguments in number theory. We also discuss some related problems.

1 Introduction

The *Turán number* $ex(n, H)$ of a graph H denotes the maximum number of edges in an n-vertex graph which does not contain H as a subgraph. Since the seminal work of P. Turán, the study of Turán numbers has been a central theme in extremal graph theory (see the survey [4]). A natural generalization, which was proposed by Erdős [2] in 1963 to reduce the structure of forbidden subgraphs to one parameter (namely, their size), asks the maximum number of edges in an n-vertex graph where every m-vertex subgraph spans less than f edges.[1] This density problem and its notorious hypergraph version (initialed in [1]) are related to difficult problems in number theory (e.g. the work of Ruzsa-Szemerédi [10] on Roth's theorem [9]) and remain unsolved in general.

Here we consider another natural extremal problem, which can be viewed as an "induced subgraph" analogue of the above Erdős' problem in [2]. This was first investigated by Erdős, Füredi, Rothschild and Sós [3]. Following their notation, we say $(n, e) \to (m, f)$ if every n-vertex graph with e edges contains an induced m-vertex subgraph with exactly f edges. Taking an example, if $t_p(n)$ denotes the number of edges in the complete balanced p-partite graph on n vertices, then Turán's theorem can be equivalently stated as that $(n, e) \to (m, \binom{m}{2})$ if and only if $e > t_{m-1}(n)$. For a fixed pair (m, f), let $S_n(m, f) = \{e : (n, e) \to (m, f)\}$ and let

$$\sigma(m, f) = \limsup_{n\to\infty} \frac{|S_n(m, f)|}{\binom{n}{2}}. \tag{1}$$

Since $S_n(m, f)$ is a subset of $\left\{0, 1, ..., \binom{n}{2}\right\}$ which cannot contain 0 and $\binom{n}{2}$ simultaneously, the fraction on the right hand of (1) is at most 1 and thus any pair (m, f) satisfies $0 \le \sigma(m, f) \le 1$. In [3], Erdős, Füredi, Rothschild and Sós gave a number of constructions arising from extremal graph theory, which reveal that for most pairs (m, f), the pairs (n, e) satisfying $e \in S_n(m, f)$ are relatively rare (in sense of the measure $\sigma(m, f)$). Their main result is as follows. Throughout the rest, let $\mathcal{A} = \{(2, 0), (2, 1), (4, 3), (5, 4), (5, 6)\}$.

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[1] Here and throughout this paper, m and f are integers satisfying $m \ge 2$ and $0 \le f \le \binom{m}{2}$.

Theorem 1.1 ([3]). *If $(m,f)\notin\mathcal{A}$, then $\sigma(m,f)\le\frac{2}{3}$; otherwise, $\sigma(m,f)=1$.*

Motivated by this result, the authors [3] raised the question if

$$\text{“}\sigma(m,f)>\tfrac{1}{2}\text{ holds for only finitely many pairs”.} \tag{2}$$

Along the way to prove Theorem 1.1, an interesting intermediate result in [3] (see Construction 8 therein) says that the majority of the pairs (m,f) satisfy $\sigma(m,f)=0$. On the other hand, they [3] also proved the following “positive” result by bounding $\sigma(m,f)$ below by a positive constant for infinitely many pairs (m,f).

Theorem 1.2 ([3]). *Let m,f be integers such that there exist positive integers a,b,c satisfying $f=\binom{a}{2}=\binom{m}{2}-\binom{b}{2}=c(m-c)$. Suppose that r is the smallest integer such that f can be written in the following form*

$$f=\sum_{i=1}^{r+1}\binom{x_i}{2},\ \textit{where integers } x_i\ge 1 \textit{ satisfy } \sum_{i=1}^{r+1}x_i=m. \tag{3}$$

Then $\sigma(m,f)\ge\frac{1}{r}$. Moreover, for $r\ge 9$ it holds that $\sigma(m,f)=\frac{1}{r}$.

Erdős, Füredi, Rothschild and Sós [3] further conjectured that the inequality in Theorem 1.2 should be an equality for any r.[2] Note that for any integers m,f from Theorem 1.2, we have $(m,f)\notin\mathcal{A}$ and thus by Theorem 1.1, any integer r chosen from (3) must satisfy $r\ge 2$. We summarize this as the following.

Conjecture 1.3 ([3]). *Let m,f be integers from Theorem 1.2 and let $r\ge 2$ be from* (3)*. Then $\sigma(m,f)=\frac{1}{r}$.*

This remains open for $2\le r\le 8$. It is worth pointing out that in addition to powerful results in extremal graph theory, the proof of each of the above two theorems in [3] used tools from number theory.

In this paper we provide improvements on the above two results of [3]. We would like to emphasize that part of our proofs is based on elementary and analytic methods in number theory. Our first result answers the question of (2) in the affirmative. In addition, we show that the constant $1/2$ cannot be lower and thus is sharp.

Theorem 1.4. *Any pair $(m,f)\notin\mathcal{A}$ satisfies $\sigma(m,f)\le\frac{1}{2}$. On the other hand, there are infinitely many pairs (m,f) with $\sigma(m,f)=\frac{1}{2}$.*

The proof of Theorem 1.4 uses bounds on $\sigma(m,f)$ established by Erdős et. al in [3]. As a corollary, Theorem 1.4, together with the lower bound of Theorem 1.2, implies the verification of Conjecture 1.3 for the case $r=2$. Our second result confirms more cases of Conjecture 1.3.

Theorem 1.5. *Conjecture 1.3 holds whenever $r=2$ or $r\ge 5$.*

This is mainly built on a key concept introduced in [3] (see (7) in Section 4), which is related to number theory. We will present two proofs of Theorem 1.5, one for $r\ge 7$ using elementary arguments and another for $r\ge 5$ using analytic arguments. We defer a more detailed discussion on Theorem 1.5 to Section 4.

The rest of the paper is organized as follows. In Section 2, we collect some results from the literature. In Section 3, we prove Theorem 1.4. In Section 4, we prove Theorem 1.5. In Section 5, we consider a conjecture of Erdős et. al [3] on the existence of $\lim_{n\to\infty}|S_n(m,f)|/\binom{n}{2}$ and conclude with some remarks.

2 Preliminaries

We now prepare some results needed in later sections. The following lemma on $\sigma(m,f)$ is collected from Sections 4 and 8 of [3] (see the equations (4.1), (4.4), (8.1), (8.3), (8.4) and (8.6) therein, respectively).

Lemma 2.1 ([3]). *(i). $\sigma(m,f)=\sigma(m,\binom{m}{2}-f)$.*

(ii). If $\sigma(m,f)>\frac{1}{2}$, then $\lfloor\frac{(m-1)^2}{4}\rfloor\le f\le\lfloor\frac{m^2}{4}\rfloor$.

[2] See the paragraph before the proof of Theorem 3 in [3].

(iii). Write $f = \binom{b}{2} - b'$ for integers b, b' with $0 \le b' < b - 1$. If $\frac{1}{2}b < b' < b - 1$, then $\sigma(m, f) \le \frac{1}{2}$.

(iv). Write $f = \binom{\ell}{2} + \ell'$ for integers ℓ, ℓ' with $0 \le \ell' < \ell < m$. If $\ell' \ge m - \ell$, then $\sigma(m, f) = 0$.

(v). Let $D(m)$ denote the set of integers $xy + z$, where x, y, z are nonnegative integers satisfying that $x + y \le m$ and if $z \ge 1$ then $x + y + z \le m - 1$. If $\sigma(m, f) > \frac{1}{2}$, then $f \in D(m)$.

The following concentration inequality can be found in [5] (see its Corollary 2.2).

Lemma 2.2 ([5])**.** *Let $\binom{[N]}{n}$ be the set of n-subset of $\{1, 2, ..., N\}$ and let $h : \binom{[N]}{n} \to \mathbb{R}$ be a given function. Let C be a uniformly random element of $\binom{[N]}{n}$. Suppose that there exists $\alpha > 0$ such that $|h(A) - h(A')| \le \alpha$ for any $A, A' \in \binom{[N]}{n}$ with $|A \cap A'| = n - 1$. Then for any real $t > 0$, $P(|h(C) - \mathbb{E}[h(C)]| \ge t) \le 2\exp\left(-\frac{2t^2}{\min\{n, N-n\}\alpha^2}\right)$.*

We also need two lemmas from number theory. Let $\mathbb{N}$ denote the set of non-negative integers.

Lemma 2.3 (Bennett, see [3])**.** *The equation $2\binom{x}{2} = \binom{y^2}{2}$ has a unique solution $(x, y) = (3, 2)$ in positive integers.*

Lemma 2.4 (Gauss)**.** *Define $\mathcal{G} = \{x^2 + y^2 + z^2 : \forall x, y, z \in \mathbb{N}\}$. Then we have $\mathbb{N} \backslash \mathcal{G} = \{4^a(8b + 7) : a, b \in \mathbb{N}\}$.*

3 Proof of Theorem 1.4

Recall that $\mathcal{A} = \{(2, 0), (2, 1), (4, 3), (5, 4), (5, 6)\}$. Our first goal is to show that any pair $(m, f) \notin \mathcal{A}$ satisfies $\sigma(m, f) \le \frac{1}{2}$. Suppose for a contradiction that there exists some pair $(m, f) \notin \mathcal{A}$ with $\sigma(m, f) > \frac{1}{2}$.

By Lemma 2.1 (ii), we have

$$\left\lfloor \frac{(m-1)^2}{4} \right\rfloor \le f \le \left\lfloor \frac{m^2}{4} \right\rfloor. \tag{4}$$

We first assert that $m \ge 8$. Considering $m = 7$, by (4) it suffices to consider $9 \le f \le 12$. By Lemma 2.1 (i), we see $\sigma(7, 10) = \sigma(7, 11)$ and $\sigma(7, 9) = \sigma(7, 12)$. Using Lemma 2.1 (iii) (with $f = 7, b = 6, b' = 4$), it follows that $\sigma(7, 10) = \sigma(7, 11) \le \frac{1}{2}$; on the other hand, Lemma 2.1 (iv) (with $f = 12, \ell = 5, \ell' = 2$) implies that $\sigma(7, 9) = \sigma(7, 12) = 0$. The cases for $m \le 6$ can be similarly verified as follows. By (4), Lemma 2.1 (i) and the fact $(m, f) \notin \mathcal{A}$, it suffices to consider: $\sigma(3, 2)$ for $m \le 3$, $\sigma(4, 2) = \sigma(4, 4)$ for $m = 4$, $\sigma(5, 5)$ for $m = 5$, and $\sigma(6, 6) = \sigma(6, 9)$ and $\sigma(6, 7) = \sigma(6, 8)$ for $m = 6$. Then one can apply Lemma 2.1 (iv) to show that each of these above pairs (m, f) satisfies $\sigma(m, f) = 0$. So we have $m \ge 8$.

From now on, we express $m \in \{2k, 2k + 1\}$ for some integer $k \ge 4$, and write f in the form of $f = \binom{\ell}{2} + \ell'$ for the unique integers ℓ, ℓ' with $0 \le \ell' < \ell$. Since $f < \binom{m}{2}$, we have $\ell < m$. By Lemma 2.1 (iv), we can derive that $0 \le \ell' \le m - \ell - 1$.

We claim that $\ell' = 0$ and thus $f = \binom{\ell}{2}$. Suppose on the contrary that $\ell' \ge 1$. We can also write f in the form $f = \binom{\ell+1}{2} - (\ell - \ell')$ where $0 < \ell - \ell' < \ell$. By Lemma 2.1 (iii), if $\frac{\ell+1}{2} < \ell - \ell' < \ell$, then $\sigma(m, f) \le \frac{1}{2}$, a contradiction. Thus, we have $\ell - \ell' \le \frac{\ell+1}{2}$. This together with $\ell' \le m - \ell - 1$ implies that $\ell \le \frac{2m-1}{3}$. We discuss according to the parity of m. First, let us consider when $m = 2k$. Then (4) implies $k^2 - k \le f = \binom{\ell}{2} + \ell' \le \binom{\ell}{2} + 2k - \ell - 1$, and solving this, we can obtain

$$\frac{3 + \sqrt{8k^2 - 24k + 17}}{2} \le \ell \le \frac{2m - 1}{3} = \frac{4k - 1}{3}.$$

Rearranging both sides, it gives $(k - 1)(k - 4) \le 0$ and thus $1 \le k \le 4$. Note that $k \ge 4$. So we have $k = 4$ and further, we can derive that $m = 8, \ell = 5, \ell' = 2$ and $f = 12$. Since $16 = \binom{7}{2} - 5$, by Lemma 2.1 (i) and (iii), $\sigma(m, f) = \sigma(8, 12) = \sigma(8, 16) \le \frac{1}{2}$, a contradiction. Hence, we may assume that $m = 2k + 1$. In this case, (4) infers that $k^2 \le f = \binom{\ell}{2} + \ell' \le \binom{\ell}{2} + 2k - \ell$, which implies that

$$\frac{3 + \sqrt{8k^2 - 16k + 9}}{2} \le \ell \le \frac{2m - 1}{3} = \frac{4k + 1}{3}.$$

The above inequality gives $k=2$, a contradiction to that $k\geq 4$. This proves the claim that $\ell'=0$.

By (4), we have $k^2-k\leq f=\binom{\ell}{2}\leq k^2$ for $m=2k$ and $k^2\leq f=\binom{\ell}{2}\leq k^2+k$ for $m=2k+1$. This leads to

$$\sqrt{2}k-1<\frac{1+\sqrt{8k^2-8k+1}}{2}\leq \ell\leq\frac{1+\sqrt{8k^2+1}}{2}<\sqrt{2}k+1\ \text{ for } m=2k,\ \text{and}$$
$$\sqrt{2}k<\frac{1+\sqrt{8k^2+1}}{2}\leq \ell\leq\frac{1+\sqrt{8k^2+8k+1}}{2}<\sqrt{2}k+2\ \text{ for } m=2k+1.$$

Since $\sqrt{2}k\notin\mathbb{N}$ for $k\in\mathbb{N}\backslash\{0\}$, we see that

$$\ell\in\{\lfloor\sqrt{2}k\rfloor,\lfloor\sqrt{2}k\rfloor+1\}\text{ for } m=2k \text{ and } \ell\in\{\lfloor\sqrt{2}k\rfloor+1,\lfloor\sqrt{2}k\rfloor+2\}\text{ for } m=2k+1. \tag{5}$$

Next we prove that $f=\binom{\ell}{2}=\frac{1}{2}\binom{m}{2}$. Suppose on the contrary that $f':=\binom{m}{2}-f\neq f$. Since $\sigma(m,f')=\sigma(m,f)>\frac{1}{2}$, repeating the above arguments, we can conclude that $f'=\binom{g}{2}$, where g is a positive integer instead of ℓ satisfying (5). Since $\lfloor\frac{(m-1)^2}{4}\rfloor\leq f\neq f'\leq\lfloor\frac{m^2}{4}\rfloor$, it follows that $|f-f'|\leq\lfloor\frac{m^2}{4}\rfloor-\lfloor\frac{(m-1)^2}{4}\rfloor=k$. As $g\neq\ell$ and both g,ℓ satisfy (5), we can derive that

$$k\geq|f-f'|=\left|\binom{\ell}{2}-\binom{g}{2}\right|\geq\min\{\ell,g\}\geq\left\lfloor\sqrt{2}k\right\rfloor,$$

which is a contradiction for $k\geq 4$. This proves that $2\binom{\ell}{2}=2f=\binom{m}{2}$, as desired.

Finally we show that m must be a perfect square. By Lemma 2.1 (v), we see $f\in D(m)$. That is, f can be written as $f=xy+z$ for nonnegative integers x,y and z satisfying that $x+y\leq m$ and if $z\geq 1$ then $x+y+z\leq m-1$. If $z\geq 1$, as $m\geq 8$, we can infer that $f=xy+z\leq\lfloor\frac{(m-1-z)^2}{4}\rfloor+z\leq\lfloor\frac{(m-2)^2}{4}\rfloor+1<\lfloor\frac{(m-1)^2}{4}\rfloor$, a contradiction to (4). Thus we have $z=0$ and $f=xy$ where $x+y\leq m$. If $x+y\leq m-1$, then we have $f=xy\leq\lfloor\frac{(m-1)^2}{4}\rfloor$ which contradicts that $f=\frac{1}{2}\binom{m}{2}$. So we must have $f=x(m-x)$ for some nonnegative integer x. Solving $x(m-x)=f=\frac{1}{2}\binom{m}{2}$, we get $x=\frac{1}{2}(m\pm\sqrt{m})$ which implies that m is a perfect square.

Therefore, we have $2\binom{\ell}{2}=\binom{m}{2}$ where m is a perfect square. By Lemma 2.3, this equation has the unique solution $(\ell,m)=(3,4)$ in positive integers, which contradicts that $m\geq 8$. This completes the proof of the first assertion of Theorem1.4 that any $(m,f)\notin\mathcal{A}$ satisfies $\sigma(m,f)\leq\frac{1}{2}$.

To show the second assertion of Theorem 1.4, we construct an infinite sequence of pairs (m,f) with $\sigma(m,f)=\frac{1}{2}$. In view of the first assertion, it is enough to show infinitely many pairs (m,f) with $\sigma(m,f)\geq\frac{1}{2}$. Using Theorem 1.2, we can further reduce to find infinitely many (m,f) satisfying the following properties:

(A). f can be expressed as $f=\binom{a}{2}=\binom{m}{2}-\binom{b}{2}=c(m-c)$ for some positive integers a,b,c,

(B). f can be expressed as $f=\binom{x_1}{2}+\binom{x_2}{2}+\binom{x_3}{2}$ for integers $x_i\geq 1$ with $x_1+x_2+x_3=m$, and

(C). f cannot be expressed as $f=\binom{y_1}{2}+\binom{y_2}{2}$ for integers $y_i\geq 1$ with $y_1+y_2=m$.

To do so, we will make use of the coming two equations, which can be easily verified for any integer $t\geq 1$:

$$\binom{5t+2}{2}=\binom{3t+1}{2}+\binom{4t+2}{2}\ \text{ and }\ \binom{3t+1}{2}=\binom{2t+1}{2}+\binom{2t+1}{2}+\binom{t}{2}.$$

We define $m=5t+2$ and $f=\binom{3t+1}{2}$. By the above equations, we see that (B) automatically holds for such (m,f), and one can choose integers $a=3t+1$ and $b=4t+2$ in (A). Next, we show that such (m,f) also satisfies (C); as otherwise, by Jensen's inequality, we can derive that for any $t\geq 1$,

$$f=\binom{y_1}{2}+\binom{5t+2-y_1}{2}\geq\binom{\lfloor 2.5t\rfloor+1}{2}+\binom{\lceil 2.5t\rceil+1}{2}>\binom{3t+1}{2}=f,$$

a contradiction. We are left to find a positive integer c such that $\binom{3t+1}{2} = c(5t+2-c)$, which implies $c = \frac{1}{2}(5t+2-\sqrt{7t^2+14t+4})$. Note that for even integers $t \geq 1$, if $\sqrt{7t^2+14t+4}$ is an integer, then c must be a positive integer. Hence, it suffices to find infinitely many even integers $t \geq 1$ such that $\sqrt{7t^2+14t+4}$ is an integer. By letting $x = \sqrt{7t^2+14t+4}$ and $y = t+1$, our task is now to find infinitely many positive integer solutions (x, y) to the Pell's equation

$$x^2 - 7y^2 = -3, \tag{6}$$

where x is even and y is odd. Note that $(x, y) = (2, 1)$ is a positive solution to (6). We also observe that $(x + y\sqrt{7})(8 + 3\sqrt{7}) = (8x + 21y) + (3x + 8y)\sqrt{7}$ and if (x, y) is a positive integer solution to (6), then so is $(8x + 21y, 3x + 8y)$. Combining these facts together, we obtain infinite positive integer solutions (x_k, y_k) to (6), where $x_k + y_k\sqrt{7} = (2 + \sqrt{7})(8 + 3\sqrt{7})^k$ for all $k \geq 0$. Using the recurrences that $x_{k+1} = 8x_k + 21y_k$ and $y_{k+1} = 3x_k + 8y_k$, it is easy to see that x_{2k} is even and y_{2k} is odd.

To give an explicit formula for the above construction, let $t_k = y_{2k} - 1$ and we can derive that

$$t_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 8 & 21 \\ 3 & 8 \end{pmatrix}^{2k} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 1.$$

Now let $m_k = 5t_k + 2$ and $f_k = \binom{3t_k+1}{2}$. By the above analysis, we have $\sigma(m_k, f_k) = \frac{1}{2}$ for all $k \geq 1$. This finishes the proof of Theorem 1.4. ■

4 Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5. First, let us prove the case $r = 2$.

Proof of Theorem 1.5 for $r = 2$. Let m, f be integers from Theorem 1.2 with $r = 2$. Then $(m, f) \notin \mathcal{A}$ and by Theorem 1.4, we have $\sigma(m, f) \leq \frac{1}{2}$. By Theorem 1.2, we get $\sigma(m, f) \geq \frac{1}{2}$. Thus $\sigma(m, f) = \frac{1}{2}$. ■

To prove other cases of Theorem 1.5, we introduce the following concept given in [3]. Let

$$C(n, r) = \left\{ \sum_{i=1}^{r} \binom{n_i}{2} : \sum_{i=1}^{r} n_i = n \text{ and } n_i \in \mathbb{N} \text{ for } 1 \leq i \leq r \right\}. \tag{7}$$

So $C(n, r)$ consists of all possible numbers of edges in an n-vertex graph formed by at most r cliques. A direct application of the Cauchy-Schwarz inequality shows that the minimum element in $C(n, r)$ is at least $n^2/2r - n/2$, thus implying $|C(n, r)| \leq \frac{n^2}{2} - \frac{n^2}{2r}$. The following result was given in [3] implicitly, which reveals the importance of $C(n, r)$ for Conjecture 1.3, that is, if $C(n, r)$ is almost full, then Conjecture 1.3 holds for such r.

Lemma 4.1 ([3])**.** *Let $r \geq 2$. If $|C(n, r)| = \frac{n^2}{2} - \frac{n^2}{2r} + o(n^2)$, then Conjecture 1.3 holds for the case r.*

Proof. Let m, f be integers from Theorem 1.2 and let $r \geq 2$ be from (3). Suppose $|C(n, r)| = \frac{n^2}{2} - \frac{n^2}{2r} + o(n^2)$. Take any $e \in C(n, r)$. Then there exists an n-vertex graph G with e edges formed by at most r cliques. Clearly, any m-vertex subgraph of G is a subgraph consisting of at most r cliques. By the choice of (m, f), $f \in C(m, r+1) \backslash C(m, r)$. This shows that any m-vertex induced subgraph of G cannot have f edges. So $e \notin S_n(m, f)$. That also says, $S_n(m, f) \cap C(n, r) = \emptyset$. Therefore, we have $\sigma(m, f) \leq \limsup_{n\to\infty} \left(1 - |C(n, r)|/\binom{n}{2}\right) = \limsup_{n\to\infty} \left(\frac{1}{r} + o(1)\right) = \frac{1}{r}$, and thus the equality holds for the case r of Conjecture 1.3. □

Brueggeman and Hildebrand (unpublished, see [3]) showed that there exists a constant $c_r > 0$ such that

$$\left[\frac{n^2}{2r} + c_r n, \frac{n^2 - n}{2} - c_r n^{3/2}\right] \subseteq C(n, r) \text{ for each } r \geq 9.$$

This, together with Lemma 4.1, was applied by Erdős et. al in [3] to derive the equality $\sigma(m, f) = \frac{1}{r}$ for $r \geq 9$ in Theorem 1.2. By the above discussion, to complete the proof of Theorem 1.5, it suffices to prove the following.

Theorem 4.2. *Let $r \geq 5$. Then we have $|C(n, r)| = \frac{n^2}{2} - \frac{n^2}{2r} + o(n^2)$.*

We remark (as we shall see later in the proof of Theorem 4.3) that if Theorem 4.2 holds for the case s, then it holds for any $r \geq s$.

4.1 An elementary proof for $r \geq 7$

We first prove the following weak version of Theorem 4.2 by elementary arguments.

Theorem 4.3. *Let $r \geq 7$. Then for some constant $c_r > 0$, we have $\left[\frac{n^2}{2r} + c_r n, \frac{n^2-n}{2} - c_r n^{3/2}\right] \subseteq C(n, r)$.*

Proof. We first show that it suffices to handle the case $r = 7$. Suppose that $\left[\frac{n^2}{2r} + c_r n, \frac{n^2-n}{2} - c_r n^{3/2}\right] \subseteq C(n, r)$ holds. We claim that this will lead to the analog statement for the case $r + 1$. For any $n_0 \in C\left(n - \lfloor \frac{n}{r+1} \rfloor, r\right)$, we have $n_0 + \binom{\lfloor \frac{n}{r+1} \rfloor}{2} \in C(n, r + 1)$, that is, $C\left(n - \lfloor \frac{n}{r+1} \rfloor, r\right) + \binom{\lfloor \frac{n}{r+1} \rfloor}{2} \subseteq C(n, r + 1)$. Also because $C(n, r) \subseteq C(n, r + 1)$, a careful calculation would give $\left[\frac{n^2}{2(r+1)} + c_{r+1} n, \frac{n^2-n}{2} - c_{r+1} n^{3/2}\right] \subseteq C(n, r + 1)$ for some $c_{r+1} > 0$, as claimed.

For $r = 7$, we will prove that

$$\left[\frac{n^2}{14} + \frac{n}{2} + 2100, \frac{n^2 - n}{2} - 66n^{3/2}\right] \subseteq C(n, 7).$$

Let $m \in \left[\frac{n^2}{14} + \frac{n}{2} + 2100, \frac{n^2-n}{2} - 66n^{3/2}\right]$ be an integer. Note that $m \in C(n, 7)$ if and only if there exist non-negative integers $n_1, n_2, \ldots, n_7$ such that $\sum_{i=1}^{7} n_i^2 = 2m + n$ and $\sum_{i=1}^{7} n_i = n$. Therefore, in order to prove $m \in C(n, 7)$, we only need to find integer solutions to

$$\sum_{i=1}^{3} (x_i^2 + (2t - x_i)^2) + (n - 6t)^2 = 2m + n, \tag{8}$$

where $0 \leq x_i \leq 2t$ $(1 \leq i \leq 3)$ and $6t \leq n$. The equation (8) is equivalent to

$$2\sum_{i=1}^{3}(x_i - t)^2 = 2m + n - (n - 6t)^2 - 6t^2.$$

In view of Lemma 2.4, the equation (8) is solvable if there exists $t \in \mathbb{N}$ such that

$$6t \leq n,\ 0 \leq 2m + n - (n - 6t)^2 - 6t^2 \leq t^2 \text{ and } 2m + n - (n - 6t)^2 - 6t^2 \in 2\mathcal{G}, \tag{9}$$

where $\mathcal{G}$ is from Lemma 2.4 and the second inequality insure that $0 \leq x_i \leq 2t$ for $1 \leq i \leq 3$.

We denote $f(t) := f(t; m, n) = 2m+n-(n-6t)^2-6t^2 = -42(t-\frac{n}{7})^2 - \frac{n}{7}^2 + n + 2m$. When $0 \leq t \leq \frac{n}{7}$, $f(t)$ is an increasing function with $f(0) = -n^2+n+2m$ and $f(\frac{n}{7}) = -\frac{n}{7}^2+n+2m$. Since $\frac{n^2}{14} + \frac{n}{2} + 2100 \leq m \leq \frac{n^2-n}{2} - 66n^{3/2}$, we have $f(0) < 0$ and $f(\frac{n}{7}) > f(\frac{n}{7} - 10) = -4200 - \frac{n}{7}^2 + n + 2m \geq 0$. Now we can choose an integer t_0 with $0 \leq t_0 \leq \frac{n}{7} - 10$ such that $f(t_0) \leq 0$ and $f(t_0 + 1) > 0$. Next we show there exists an integer $t_0 + 1 \leq t \leq t_0 + 10$, satisfying (9).

We first claim that $t_0 > 11\sqrt{n} - 1$. Indeed, suppose that $t_0 \leq 11\sqrt{n} - 1$, then we can see that $0 < f(t_0 + 1) \leq f(11\sqrt{n}) = -5082n + 132n^{3/2} - n^2 + n + 2m \leq -5082n < 0$, which is a contradiction. It is easy to see that $f(t+1)-f(t) = -84t-42+12n$, and if $t > 11\sqrt{n}-1$, then $f(t+10)-f(t) = -840t-4200+120n < t^2$. Thus, for any $t_0+1 \leq t \leq t_0+10 \leq n/7$, we have $6t \leq 6n/7 < n$ and $0 < f(t_0+1) \leq f(t) \leq f(t_0+10) \leq f(t_0)+t_0^2 \leq t_0^2 < t^2$. It is left to explain that one can choose t with $t_0 + 1 \leq t \leq t_0 + 10$ such that $f(t) \in 2\mathcal{G}$.

Note that $f(t)$ is alwalys an even integer. Also note that $8b + 7 \equiv 7, 15\,(mod\ 16)$, $4(8b + 7) \equiv 12\,(mod\ 16)$ and $4^a(8b + 7) \equiv 0\,(mod\ 16)$ for all $a \geq 2$. Thus by Lemma 2.4, it suffices to find $f(t)$ such that $\frac{f(t)}{2} \not\equiv 0, 7, 12, 15\,(mod\ 16)$. It is clear that we can choose t' with $t_0 + 1 \leq t' \leq t_0 + 8$, such that $t' + n \equiv 0\,(mod\ 8)$. Then

$$\frac{f(t' + 1) - f(t')}{2} = -42t' - 21 + 6n \equiv 11\,(mod\ 16) \quad \text{and}$$

$$\frac{f(t' + 2) - f(t' + 1)}{2} = -42(t' + 1) - 21 + 6n \equiv 1\,(mod\ 16).$$

We can easily see from above that there exists $t \in \{t', t' + 1, t' + 2\}$ such that $\frac{f(t)}{2} \not\equiv 0, 7, 12, 15\,(mod\ 16)$. This completes the proof of Theorem 4.3. ■

4.2 Proof of Theorem 4.2

In this subsection, we will use analytic method to prove Theorem 4.2. Let f and g be two functions on the same domain, where f takes complex values and g takes non-negative real values. We use the Vinogradov symbols $f \ll g$, if there exists a constant $C > 0$ such that $|f| \le Cg$.[3]

As the previous analysis in the proof of Theorem 4.3, we may assume that $r = 5$ in the following proof. Indeed, since $C(n,r) \subseteq C(n,r+1)$ and $C\Big(n - \lfloor \frac{n}{r+1} \rfloor, r\Big) + \binom{\lfloor \frac{n}{r+1} \rfloor}{2} \subseteq C(n, r+1)$, then by the definition of $C(n,r)$ and induction on r, we can get the desired result for $r+1$.

Let $\mathcal{E}(n) = \Big\{\frac{n^2}{10} + \frac{n^2}{\log n} \le m \le \frac{n^2-n}{2} - \frac{n^2}{\log n} : \mathcal{R}(m) = 0\Big\}$, with

$$\mathcal{R}(m) = \sum_{\substack{1 \le x_1, x_2, x_3, x_4 \le \frac{n}{5} - \frac{n}{\log n} \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1+x_2+x_3+x_4-n)^2 = 2m+n}} 1, \tag{10}$$

where x_1, x_2, x_3, x_4 are integers. Note that $\mathcal{R}(m) > 0$ implies that $m \in C(n,5)$. Therefore, to prove Theorem 4.2, it is enough to prove $|\mathcal{E}(n)| = o(n^2)$. We will show the following stronger result. Let $\mathbb{Z}$ be the set of integers.

Theorem 4.4. $|\mathcal{E}(n)| = O(n^{2-\frac{1}{50}})$.

We denote $N = \frac{n}{5} - \frac{n}{\log n}$. For positive integers x_1, x_2, x_3, x_4, we use $\vec{\mathbf{x}}$={x_1, x_2, x_3, x_4} to denote vectors in $\mathbb{Z}^4$. Let $Q(\vec{\mathbf{x}}) = Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 + x_2 + x_3 + x_4 - n)^2$. For a vector $\vec{\mathbf{y}}$={y_1, y_2, y_3, y_4} $\in \mathbb{Z}^4$, the notation $\vec{\mathbf{x}} \le \vec{\mathbf{y}}$ means $x_i \le y_i$ for all $1 \le i \le 4$, and $1 \le \vec{\mathbf{x}} \le C$ means $1 \le x_i \le C$ for all $1 \le i \le 4$. For convenience, we write $\mathbf{e}(\alpha) = e^{2\pi i \alpha}$. Define

$$f(\alpha) = \sum_{1 \le \vec{\mathbf{x}} \le N} \mathbf{e}(\alpha Q(\vec{\mathbf{x}})). \tag{11}$$

Then by (10) and (11), we have

$$\mathcal{R}(m) = \int_{\frac{1}{n}}^{1+\frac{1}{n}} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha. \tag{12}$$

We define

$$\mathfrak{M}(X) = \bigcup_{1 \le q \le X} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{X}{qn^2}, \frac{a}{q} + \frac{X}{qn^2}\right] \text{ and } \mathfrak{m}(X) = \left[\frac{1}{n}, 1 + \frac{1}{n}\right] \backslash \mathfrak{M}(X).$$

Note that the above union is pairwise disjoint for $X \le \frac{n}{2}$. Also note that both $\int_{\mathfrak{M}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha$ and $\int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha$ take real values. We first estimate the integral on $\mathfrak{m}(X)$ in the following lemma.

Lemma 4.5. *Let $L \le \frac{n}{2}$. Then* $\sum_{\frac{n^2}{10} + \frac{n^2}{\log n} \le m \le \frac{n^2-n}{2} - \frac{n^2}{\log n}} \left|\int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha\right|^2 = O\Big(\frac{n^6(\log n)^5}{L^2}\Big)$.

To show this lemma, we need the following claim.

Claim 4.6. *Suppose that $|\alpha - \frac{a}{q}| \le \frac{1}{q^2}$ with $(a,q) = 1$. Then*

$$|f(\alpha)|^2 \ll n^8(\log n)^4 \left(\frac{1}{q} + \frac{1}{n} + \frac{q}{n^2}\right)^4.$$

Proof. This can be proved by the standard difference argument. Note that

$$|f(\alpha)|^2 = \sum_{1 \le \vec{\mathbf{x}},\, \vec{\mathbf{y}} \le N} \mathbf{e}\Big(\alpha\Big(Q(\vec{\mathbf{x}}) - Q(\vec{\mathbf{y}})\Big)\Big) = \sum_{-N+1 \le \vec{\mathbf{h}} \le N-1} \sum_{\substack{1 \le \vec{\mathbf{y}} \le N \\ 1 \le \vec{\mathbf{y}} + \vec{\mathbf{h}} \le N}} \mathbf{e}\Big(\alpha\Big(Q(\vec{\mathbf{y}} + \vec{\mathbf{h}}) - Q(\vec{\mathbf{y}})\Big)\Big).$$

[3] Here and in the rest, $|f|$ denotes the modulus of a complex number f.

By triangle inequality, we deduce that

$$|f(\alpha)|^2 \le \sum_{-N+1\le\vec{\mathbf{h}}\le N-1} \Big| \sum_{\substack{1\le\vec{\mathbf{y}}\le N\\ 1\le\vec{\mathbf{y}}+\vec{\mathbf{h}}\le N}} \mathbf{e}\Big(\alpha\Big(Q(\vec{\mathbf{y}}+\vec{\mathbf{h}}) - Q(\vec{\mathbf{y}})\Big)\Big)\Big|.$$

Fix $\vec{\mathbf{h}}$ and let $h = \sum_{i=1}^4 h_i$. Note that

$$\Big| \sum_{\substack{1\le\vec{\mathbf{y}}\le N\\ 1\le\vec{\mathbf{y}}+\vec{\mathbf{h}}\le N}} \mathbf{e}\Big(\alpha\Big(Q(\vec{\mathbf{y}}+\vec{\mathbf{h}}) - Q(\vec{\mathbf{y}})\Big)\Big)\Big| \le \Big| \sum_{\substack{1\le\vec{\mathbf{y}}\le N\\ 1\le\vec{\mathbf{y}}+\vec{\mathbf{h}}\le N}} \mathbf{e}\Big(2\alpha\sum_{j=1}^4 (h+h_j)y_j\Big)\Big| \le \prod_{j=1}^4 \Big| \sum_{\substack{1\le y_j\le N\\ 1-h_j\le y_j\le N-h_j}} \mathbf{e}\Big(2\alpha(h+h_j)y_j\Big)\Big|.$$

Write $\|\beta\| = \min_{m\in\mathbb{Z}} |\beta - m|$. Since $\sum_{\substack{1\le y_j\le N\\ 1-h_j\le y_j\le N-h_j}} \mathbf{e}\Big(2\alpha(h+h_j)y_j\Big) \ll \min(N, \|2\alpha(h+h_j)\|^{-1})$, we obtain

$$|f(\alpha)|^2 \ll \sum_{-N+1\le\vec{\mathbf{h}}\le N-1} \prod_{j=1}^4 \min(N, \|2\alpha(h+h_j)\|^{-1}) \ll \sum_{-10(N-1)\le\vec{\mathbf{t}}\le 10(N-1)} w(\vec{\mathbf{t}}) \prod_{j=1}^4 \min(N, \|\alpha t_j\|^{-1}),$$

where $w(\vec{\mathbf{t}}) = \sum_{-N+1\le\vec{\mathbf{h}}\le N-1,\ 2(h+h_j)=t_j,\ \forall 1\le j\le 4} 1$. It is easy to see that $w(\vec{\mathbf{t}}) \in \{0, 1\}$, and thus we have

$$|f(\alpha)|^2 \ll \sum_{-10(N-1)\le\vec{\mathbf{t}}\le 10(N-1)} \prod_{j=1}^4 \min(N, \|\alpha t_j\|^{-1}) = \Bigg(\sum_{-10(N-1)\le t\le 10(N-1)} \min(N, \|\alpha t\|^{-1}) \Bigg)^4 \tag{13}$$

As $|\alpha - \frac{a}{q}| \le \frac{1}{q^2}$ and $(a, q) = 1$, by Lemma 2.2 in [11] (with $X = 10N, Y = N/10$), we have

$$\sum_{-10(N-1)\le t\le 10(N-1)} \min(N, \|\alpha t\|^{-1}) \ll N^2(\log N)\left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^2}\right). \tag{14}$$

This, together with (13), (14) and the definition of N, completes the proof of Claim 4.6. □

Proof of Lemma 4.5. We can see that $\int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha$ is the Fourier coefficient of the function which is $f(\alpha)$ on $\mathfrak{m}(L)$ and 0 otherwise. Hence, by Bessel's inequality, we have

$$\sum_{\frac{n^2}{10}+\frac{n^2}{\log n}\le m\le \frac{n^2-n}{2}-\frac{n^2}{\log n}} \Big| \int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha \Big|^2 \le \int_{\mathfrak{m}(L)} |f(\alpha)|^2\,d\alpha.$$

For $X < \frac{n}{2}$, we define $\mathfrak{n}(X) = \mathfrak{M}(2X)\backslash\mathfrak{M}(X)$, and for $X = \frac{n}{2}$, we write $\mathfrak{n}(X) = \left[\frac{1}{n}, 1+\frac{1}{n}\right]\backslash\mathfrak{M}(X)$. Let $t = \lceil \log_2 \frac{n}{2L} \rceil$. By the dyadic argument and the definition of $\mathfrak{n}(X)$, we can see that $\mathfrak{m}(L) \subseteq \mathfrak{n}(L)\cup\mathfrak{n}(2L)\cup\cdots\cup\mathfrak{n}(2^{t-1}L)\cup\mathfrak{n}(\frac{n}{2})$. Thus we only need to prove that for $L \le X \le \frac{n}{2}$, $\int_{\mathfrak{n}(X)} |f(\alpha)|^2\,d\alpha \ll \frac{n^6(\log n)^4}{X^2}$.

By Dirichlet's approximation theorem (see Lemma 2.1 in [11]), for $\alpha \in \mathfrak{n}(X)$, there exist $a, q \in \mathbb{N}$ such that $|\alpha - \frac{a}{q}| \le \frac{X}{qn^2}$, $1 \le a \le q \le \frac{n^2}{X}$ and $(a, q) = 1$. Since $\alpha \notin \mathfrak{M}(X)$, we further have $q > X$. Now it follows from Claim 4.6 that $\sup_{\alpha\in\mathfrak{n}(X)} |f(\alpha)|^2 \ll \frac{n^8(\log n)^4}{X^4}$. Note that the measure of $\mathfrak{n}(X)$ is $|\mathfrak{n}(X)| \ll \frac{X^2}{n^2}$, so we obtain $\int_{\mathfrak{n}(X)} |f(\alpha)|^2\,d\alpha \ll \frac{n^6(\log n)^4}{X^2}$, which completes the proof of Lemma 4.5. ■

In order to estimate the contribution from $\mathfrak{M}(X)$ in (12), we define

$$S(q, a) = \sum_{1\le\vec{\mathbf{x}}\le q} \mathbf{e}\left(\frac{a}{q}Q(\vec{\mathbf{x}})\right) \quad\text{and}\quad T(q; m) = \frac{1}{q^4}\sum_{\substack{1\le a\le q\\ (a,q)=1}} S(q, a)\mathbf{e}\left(-\frac{a}{q}(2m+n)\right). \tag{15}$$

The following claim can be proved by some standard argument (see Lemmas 2.10 and 2.11 in [11]).

Claim 4.7. *$T(q; m)$ is multiplicative as a function of q.*

Also $T(q; m)$ takes real values. In the next claim we bound $|T(q; m)|$ from above for prime powers q.

Claim 4.8. *Assume that p is a prime and $k \in \mathbb{Z}^+$. Then $|T(p^k; m)| \le c_p \cdot p^{-k}\left(1 - \frac{1}{p}\right)$, where $c_2 = 4$, $c_5 = \sqrt{5}$ and $c_p = 1$ if $p \nmid 10$.*

Proof. We can deduce that

$$|S(q, a)|^2 = \sum_{1 \le \vec{\mathbf{x}},\, \vec{\mathbf{h}} \le q} \mathbf{e}\left(\frac{a}{q}\left(Q(\vec{\mathbf{x}} + \vec{\mathbf{h}}) - Q(\vec{\mathbf{x}})\right)\right) \le \sum_{1 \le \vec{\mathbf{h}} \le q} \left| \sum_{1 \le \vec{\mathbf{x}} \le q} \mathbf{e}\left(\frac{a}{q}\left(2 \sum_{j=1}^{4} (h_1 + \cdots + h_4 + h_j) x_j\right)\right)\right| \le q^4 S_q,$$

where S_q is the number of solutions to $2(h_1 + \cdots + h_4 + h_j) \equiv 0 \pmod q$ $(1 \le j \le 4)$ with $1 \le h_1, ..., h_4 \le q$. The last inequality holds by the fact that $\sum_{j=1}^{q} \mathbf{e}\left(\frac{a}{q} t j\right) = q$, when $t \equiv 0 \pmod q$ and $\sum_{j=1}^{q} \mathbf{e}\left(\frac{a}{q} t j\right) = 0$ otherwise. Solving the congruence equations, we can get that $S_{2^k} = 16$, $S_{5^k} = 5$ and $S_{p^k} = 1$ if $p \nmid 10$, which implies that $|S(2^k, a)| \le 4 \cdot 2^{2k}$, $|S(5^k, a)| \le \sqrt{5} \cdot 5^{2k}$, and $|S(p^k, a)| \le p^{2k}$ if $p \nmid 10$. Note that $|\{1 \le a \le p^k : (a, p^k) = 1\}| = p^k\left(1 - \frac{1}{p}\right)$. This completes the proof of Claim 4.8 by (15). □

In case $(q, 10) = 1$, we can get the following better bound for $|T(q; m)|$.

Claim 4.9. *Assume that $(q, 10) = 1$. Then $|T(q; m)| \le \frac{1}{q^2}\left(q, n^2 - 5(2m + n)\right)$.*

Proof. By Claim 4.7, we only need to prove the claim for q as a power of some prime p. So we may assume that $q = p^k$, where p is a prime with $(p, 10) = 1$ and $k \ge 1$ is an integer. Let $\bar{r}$ denotes an integer r' satisfying $rr' \equiv 1 \pmod q$. Let $\mathbf{A} = I_{4\times4} + J_{4\times4}$, where I is the identity matrix, and all the entries in J are 1. Note that $\det(\mathbf{A}) = 5$ and $Q(\vec{\mathbf{x}}) = \vec{\mathbf{x}}^T \mathbf{A} \vec{\mathbf{x}} - 2n(x_1 + x_2 + x_3 + x_4) + n^2$. Let $\vec{\mathbf{b}} = \bar{5} n \mathbf{A}^* \vec{\mathbf{1}}$, where $\mathbf{A}^*$ is the adjugate matrix of $\mathbf{A}$. We get that $Q(\vec{\mathbf{y}} + \vec{\mathbf{b}}) = \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}} + \bar{5} n^2$, and $S(q, a) = \sum_{1 \le \vec{\mathbf{y}} \le q} \mathbf{e}\left(\frac{a}{q} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) \mathbf{e}\left(\frac{a}{q} \bar{5} n^2\right) = \sum_{1 \le \vec{\mathbf{y}} \le p^k} \mathbf{e}\left(\frac{a}{p^k} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) \mathbf{e}\left(\frac{a}{p^k} \bar{5} n^2\right)$.

Let $\vec{\mathbf{y}} = \vec{\mathbf{u}} p^{k-1} + \vec{\mathbf{v}}$. When $k \ge 2$, we have that

$$\sum_{1 \le \vec{\mathbf{y}} \le p^k} \mathbf{e}\left(\frac{a}{p^k} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) = \sum_{1 \le \vec{\mathbf{u}} \le p} \sum_{1 \le \vec{\mathbf{v}} \le p^{k-1}} \mathbf{e}\left(\frac{a}{p^k} (2 \vec{\mathbf{u}}^T p^{k-1} \mathbf{A} \vec{\mathbf{v}} + \vec{\mathbf{v}}^T \mathbf{A} \vec{\mathbf{v}})\right) = p^2 \sum_{\substack{1 \le \vec{\mathbf{v}} \le p^{k-1} \\ 2\mathbf{A}\vec{\mathbf{v}} \equiv \vec{\mathbf{0}} \ (\mathrm{mod}\ p)}} \mathbf{e}\left(\frac{a}{p^k} \vec{\mathbf{v}}^T \mathbf{A} \vec{\mathbf{v}}\right).$$

The last equality holds since $\sum_{1 \le \vec{\mathbf{u}} \le p} \mathbf{e}\left(\frac{a}{p} 2 \vec{\mathbf{u}}^T \mathbf{A} \vec{\mathbf{v}}\right) = p^2$, when $2\mathbf{A}\vec{\mathbf{v}} \equiv \vec{\mathbf{0}} \pmod p$ and $\sum_{1 \le \vec{\mathbf{u}} \le p} \mathbf{e}\left(\frac{a}{p} 2 \vec{\mathbf{u}}^T \mathbf{A} \vec{\mathbf{v}}\right) = 0$, otherwise. For $(p, 10) = 1$, $2\mathbf{A}\vec{\mathbf{v}} \equiv \vec{\mathbf{0}} \pmod p$ if and only if $\vec{\mathbf{v}} \equiv \vec{\mathbf{0}} \pmod p$. Thus we have $\sum_{1 \le \vec{\mathbf{y}} \le p^k} \mathbf{e}\left(\frac{a}{p^k} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) = p^2 \sum_{1 \le \vec{\mathbf{y}} \le p^{k-2}} \mathbf{e}\left(\frac{a}{p^{k-2}} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right)$, for $k \ge 2$. By Lemma 26 in [6] and its proof, we have $\sum_{1 \le \vec{\mathbf{y}} \le p} \mathbf{e}\left(\frac{a}{p} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) = \left(\frac{5}{p}\right) p^2$, where $\left(\frac{5}{p}\right)$ denotes the Jacobi symbol. Thus we get that $\sum_{1 \le \vec{\mathbf{y}} \le p^k} \mathbf{e}\left(\frac{a}{p^k} \vec{\mathbf{y}}^T \mathbf{A} \vec{\mathbf{y}}\right) = \left(\frac{5}{p^k}\right) p^{2k}$ for all $k \ge 1$, and $S(q, a) = \left(\frac{5}{q}\right) q^2 \mathbf{e}\left(\frac{a}{q} \bar{5} n^2\right)$. Then we can see that $|T(q; m)| = \frac{1}{q^2} \left| \sum_{\substack{1 \le a \le q \\ (a,q)=1}} \mathbf{e}\left(\frac{a}{q}(\bar{5} n^2 - (2m+n))\right)\right| \le \frac{1}{q^2}\left(q, n^2 - 5(2m+n)\right)$. The last inequality is a standard upper bound of the Ramanujan sum. □

Define $G(m) = \sum_{q=1}^{\infty} T(q; m)$. By Claims 4.7, 4.8 and 4.9, $G(m)$ is absolutely convergent.

Claim 4.10. *Suppose that $n^2 - 5(2m+n) \ne 0$. Then there is an absolute constant $c > 0$ such that $G(m) \ge \frac{c}{\log \log n}$.*

Proof. Let $\Delta = n^2 - 5(2m + n)$, $P_1 = \{\text{prime } p : p \nmid 10\Delta\}$, $P_2 = \{\text{prime } p : p \mid \Delta \text{ and } p \nmid 10\}$. By Claim 4.7, we have $G(m) = \prod_{p\ prime} \left(1 + \sum_{k=1}^{\infty} T(p^k; m)\right)$. Combining Claims 4.8 and 4.9, we get that

$$1 + \sum_{k=1}^{\infty} T(p^k; m) \ge \begin{cases} 1 - \sum_{k=1}^{\infty} \frac{1}{p^{2k}} = 1 - \frac{1}{p^2 - 1} \ge \left(1 - \frac{1}{p^2}\right)^2 & \text{if } p \in P_1, \\ 1 - \sum_{k=1}^{\infty} \frac{1}{p^k} = 1 - \frac{1}{p-1} & \text{if } p \in P_2, \\ 1 - \sum_{k=1}^{\infty} p^{-k+\frac{1}{2}}\left(1 - \frac{1}{p}\right) = 1 - p^{-\frac{1}{2}} > 0 & \text{if } p = 5, \\ 1 + T(p; m) - \sum_{k=2}^{\infty} p^{-k+2}\left(1 - \frac{1}{p}\right) = T(2; m) = 1 & \text{if } p = 2. \end{cases}$$

We first see that $\prod_{p\in P_1}\left(1-\frac{1}{p^2}\right)^2 \geq \prod_p\left(1-\frac{1}{p^2}\right)^2 = \zeta(2)^{-2}$. Then there exist constants $c_1, c_2 > 0$ such that

$$G(m) \geq c_1 \cdot \prod_{p\in P_2}\left(1-\frac{1}{p-1}\right) = c_1 e^{\sum_{p\in P_2}\log\left(1-\frac{1}{p-1}\right)} \geq c_1 e^{-\sum_{p\in P_2}\frac{1}{p-1}} \geq c_2 e^{-\sum_{p|\Delta}\frac{1}{p}}.$$

Let $Y = \log\Delta$. From Theorem 6.16 [8], we have $\sum_{p\leq Y}\frac{1}{p} \leq c_3 \log\log Y$ for some constant $c_3 > 1$ and thus there exists constant $c_4 > 0$ such that

$$\sum_{\substack{p \text{ prime} \\ p|\Delta}} \frac{1}{p} = \sum_{\substack{p|\Delta \\ p\leq Y}} \frac{1}{p} + \sum_{\substack{p|\Delta \\ p>Y}} \frac{1}{p} \leq \sum_{p\leq Y}\frac{1}{p} + \frac{1}{Y}\sum_{\substack{p|\Delta \\ p>Y}} 1 \leq c_3 \log\log Y + \frac{1}{Y}\cdot\frac{\log\Delta}{\log Y} \leq c_4 \log\log\log\Delta,$$

which implies that $G(m) \geq c/\log\log\Delta$ and completes the proof of Claim 4.10. □

Define $I(\beta) = \int_{[0,N]^4} \mathbf{e}\big(\beta Q(\vec{\mathbf{y}})\big)\, d\vec{\mathbf{y}}$ and $\mathfrak{I}(m) = \int_{-\infty}^{+\infty} I(\beta)\mathbf{e}(-\beta(2m+n))\, d\beta$. Note that $\mathfrak{I}(m)$ takes real values. Using integration by parts, we have $I(\beta) \ll |\beta|^{-2}$. By Fourier inverse formula (see [7], p460), we have $n^2(\log n)^{-10} \ll \mathfrak{I}(m) \ll n^2$. For the contribution from $\mathfrak{M}(X)$, we have the following.

Lemma 4.11. *Let $L = n^{\frac{1}{50}}$. Then $\int_{\mathfrak{M}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\, d\alpha = G(m)\mathfrak{I}(m) + O(n^{2-\frac{1}{100}})$.*

Proof. Let $f^*(\alpha) = \frac{1}{q^4}S(q,a)I(\alpha - \frac{a}{q})$. Suppose that $|\alpha - \frac{a}{q}| \leq \frac{L}{qn^2}$ with $1 \leq a \leq q \leq L$ and $(a,q) = 1$, then the partial summation formular (see Lemma 2.6 in [11] with $c_i = 1$, $X = N$) implies $f(\alpha) = f^*(\alpha) + O(N^3L)$. Thus we get that

$$\int_{\mathfrak{M}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\, d\alpha = \int_{\mathfrak{M}(L)} f^*(\alpha)\mathbf{e}(-\alpha(2m+n))\, d\alpha + O(N^3L\cdot|\mathfrak{M}(L)|) = R^* + O(N^3L^3n^{-2}),$$

where $|\mathfrak{M}(L)|$ denotes the measure of $\mathfrak{M}(L)$. And we can deduce from above that

$$R^* = \sum_{q\leq L}\sum_{\substack{1\leq a\leq q \\ (a,q)=1}} \int_{|\alpha-\frac{a}{q}|\leq\frac{L}{qn^2}} f^*(\alpha)\mathbf{e}(-\alpha(2m+n))\, d\alpha = \sum_{q\leq L} T(q;m)\cdot\mathfrak{I}^*(m),$$

where $\mathfrak{I}^*(m) = \int_{|\beta|\leq\frac{L}{qn^2}} I(\beta)\mathbf{e}(-\beta(2m+n))\, d\beta = \mathfrak{I}(m) + O(\frac{qn^2}{L})$ using the definition of $\mathfrak{I}(m)$ and $I(\beta) \ll |\beta|^{-2}$.

It is well-known (see Theorem 6.25 [8]) that for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that $\tau(n) \leq c_\epsilon n^\epsilon$, where $\tau(n)$ denotes the number of factors of n. Let $\Delta = n^2 - 5(2m+n)$ and $q = 2^a5^bq_1$ with $a, b \geq 0$ and $(q_1, 10) = 1$. By Claims 4.7, 4.8 and 4.9, we have $q|T(q;m)| \leq \frac{4(q_1,\Delta)}{q_1}$. Thus $\sum_{q\leq L} T(q;m)O(\frac{qn^2}{L}) \leq \sum_{q\leq L}\frac{(q,\Delta)}{q}O\Big(\frac{(\log L)^2n^2}{L}\Big)$. Since

$$\sum_{q\leq L}\frac{1}{q}(q,\Delta) = \sum_{\substack{d|\Delta \\ d\leq L}}\sum_{\substack{q\leq L \\ (q,\Delta)=d}}\frac{d}{q} \leq \sum_{\substack{d|\Delta \\ d\leq L}}\sum_{k\leq\frac{L}{d}}\frac{1}{k} \leq \log L\cdot\sum_{\substack{d|\Delta \\ d\leq L}} 1 \leq c_\epsilon\Delta^\epsilon\log L = O\Big((\log L)n^{2\epsilon}\Big),$$

we can get that $\sum_{q\leq L} T(q;m)O(\frac{qn^2}{L}) = O\Big(\frac{(\log L)^3n^{2+2\epsilon}}{L}\Big)$. Since $\sum_{q\leq L} T(q;m) = G(m) - \sum_{q>L} T(q;m)$ and $\mathfrak{I}(m) = O(n^2)$, by the same method above, we can get that $\sum_{q>L} T(q;m)\mathfrak{I}(m) = O\Big(\frac{(\log L)^3n^{2+2\epsilon}}{L}\Big)$. Combining the above two bounds, we get that $R^* = G(m)\mathfrak{I}(m) + O\Big(\frac{(\log L)^3n^{2+2\epsilon}}{L}\Big)$. By choosing $\epsilon = \frac{1}{400}$, we finish the proof of Lemma 4.11. □

Now we are ready to prove Theorem 4.4, which would complete the proof of Theorem 1.5.

Proof of Theorem 4.4. Let $L = n^{\frac{1}{50}}$. Let W denote the set of integers $m \in \left[\frac{n^2}{10} + \frac{n^2}{\log n}, \frac{n^2-n}{2} - \frac{n^2}{\log n}\right]$ such that $\left|\int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n))\, d\alpha\right| \geq n^2(\log n)^{-11}$. Since $L = n^{\frac{1}{50}} \leq \frac{n}{2}$, by Lemma 4.5, we have

$$|W| = O\left(\frac{n^6(\log n)^5L^{-2}}{n^4(\log n)^{-22}}\right) = O(n^2(\log n)^{27}L^{-2}).$$

For integers $m \in \left[\frac{n^2}{10} + \frac{n^2}{\log n}, \frac{n^2-n}{2} - \frac{n^2}{\log n}\right] \backslash W$, by Claim 4.10, Lemma 4.11 and $\mathfrak{J}(m) \gg n^2(\log n)^{-10}$, we have

$$\begin{aligned}\mathcal{R}(m) &= \int_{\frac{1}{n}}^{1+\frac{1}{n}} f(\alpha)\mathbf{e}(-\alpha(2m+n))\,d\alpha = \int_{\mathfrak{m}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n)) + \int_{\mathfrak{M}(L)} f(\alpha)\mathbf{e}(-\alpha(2m+n)) \\ &= G(m)\mathfrak{J}(m) + O(n^{2-\frac{1}{100}}) + O(n^2(\log n)^{-11}) \gg n^2(\log n)^{-10}(\log\log n)^{-1} > 0.\end{aligned}$$

In particular, we see that $\mathcal{E}(n) \subseteq W$ and $|\mathcal{E}(n)| \le |W| = O(n^2(\log n)^{27}L^{-2}) = O(n^{2-\frac{1}{50}})$. This completes the proof of Theorem 4.4. ■

5 Concluding remarks

We now discuss some open problems related to the study of induced subgraphs of given sizes here.

Erdős, Füredi, Rothschild and Sós [3] also conjectured that the limsup in (1) is actually a limit. That says, the limit $\sigma^*(m, f) = \lim_{n\to\infty} |S_n(m, f)|/\binom{n}{2}$ exists for every pair (m, f). As mentioned earlier, a result of [3] implies that the majority of pairs (m, f) satisfy $\sigma(m, f) = 0$, and in these cases, it is clear that $\sigma^*(m, f)$ exists (for being zero). For the pairs (m, f) from Theorem 1.2 with $r = 2$ or $r \ge 5$, by Theorem 1.5 we have $\sigma(m, f) = \frac{1}{r}$, and in fact, it also holds that $\sigma^*(m, f) = \frac{1}{r}$. Despite of these supportive results, it seems to be a challenging problem to determine the existence of this limit for any pair (m, f) with $\sigma(m, f) > 0$. We prove the following lemma towards this conjecture, which says that it would suffice to find some large $S_n(m, f)$ with most of its integers appearing in few consecutive intervals.

Lemma 5.1. *Let (m, f) be a pair with $\sigma = \sigma(m, f) > 0$. If for any $\epsilon > 0$, there exists some integer n such that at least $(\sigma - \epsilon)\binom{n}{2}$ integers in $S_n(m, f)$ belong to a union of at most $\epsilon\sqrt{n}$ intervals of consecutive integers, then the limit $\sigma^*(m, f)$ exists.*

Proof. Fix any $\epsilon > 0$. Then there exists some n such that $S_n(m, f)$ contains disjoint intervals $I_1, I_2, ..., I_k$ of consecutive integers, where $k \le \epsilon\sqrt{n}$ and $\sum_{j=1}^k |I_j| \ge (\sigma - \epsilon)\binom{n}{2}$. Let $I_j = [c_j\binom{n}{2}, d_j\binom{n}{2}]$ for each $j \in [k]$, where $\sum_{j=1}^k (d_j - c_j) \ge \sigma - \epsilon$. Let N be any sufficiently large integer and let $\mathcal{E} = \bigcup_{j=1}^k I'_j$, where $I'_j = \left[(c_j + 2n^{-\frac{1}{2}})\binom{N}{2}, (d_j - 2n^{-\frac{1}{2}})\binom{N}{2}\right]$ for $j \in [k]$.

We claim that $\mathcal{E} \subseteq S_N(m, f)$. To see this, consider any $E \in \mathcal{E}$ (say $E \in I'_j$ for some $j \in [k]$) and any N-vertex graph G with E edges. We aim to show that G contains an n-vertex induced subgraph with e edges, where $e \in I_j$. For an n-subset A of $V(G)$, let $e(A)$ be the number of edges in the induced subgraph $G[A]$. Then for any $A, A' \in \binom{V(G)}{n}$ with $|A \cap A'| = n - 1$, we can easily check that $|e(A) - e(A')| \le n - 1$. Let C be a uniformly random element of $\binom{V(G)}{n}$. It is easy to see that $\mathbb{E}[e(C)] = E \cdot \binom{N-2}{n-2}/\binom{N}{n} = \frac{E}{\binom{N}{2}}\binom{n}{2}$. By Lemma 2.2 (with $\alpha = n - 1$ and $t = n^{\frac{1}{2}}(n - 1)$), we get that

$$P\left(\left|e(C) - \mathbb{E}[e(C)]\right| \ge n^{\frac{1}{2}}(n-1)\right) \le 2\exp\left(-\frac{2n(n-1)^2}{\min\{n, N-n\}(n-1)^2}\right) \le 2e^{-2} < 1.$$

Therefore, there must exist an n-subset B of $V(G)$ such that $\left|e(B) - \frac{E}{\binom{N}{2}}\binom{n}{2}\right| < n^{\frac{1}{2}}(n-1)$. This gives that

$$c_j\binom{n}{2} \le \frac{E}{\binom{N}{2}}\binom{n}{2} - n^{\frac{1}{2}}(n-1) \le e(B) \le \frac{E}{\binom{N}{2}}\binom{n}{2} + n^{\frac{1}{2}}(n-1) \le d_j\binom{n}{2},$$

where $E \in I'_j = \left[(c_j + 2n^{-\frac{1}{2}})\binom{N}{2}, (d_j - 2n^{-\frac{1}{2}})\binom{N}{2}\right]$. So $G[B]$ is an n-vertex induced subgraph of G, where $e(B) \in I_j \subseteq S_n(m, f)$. Then $G[B]$ (and thus G) contains an m-vertex induced subgraph with f edges. This shows that $E \in S_N(m, f)$ and thus $\mathcal{E} \subseteq S_N(m, f)$, proving the claim.

Since $\sum_{j=1}^k (d_j - c_j) \ge \sigma - \epsilon$ and $k \le \epsilon\sqrt{n}$, we see that for sufficiently large N,

$$|S_N(m, f)| \ge |\mathcal{E}| = \sum_{j=1}^k |I'_j| = \sum_{j=1}^k \left(d_j - c_j - 4n^{-\frac{1}{2}}\right)\binom{N}{2} \ge \left(\sigma - \epsilon - 4kn^{-\frac{1}{2}}\right)\binom{N}{2} \ge (\sigma - 5\epsilon)\binom{N}{2}.$$

This shows that for any $\epsilon > 0$, we have $\liminf_{N\to\infty} \frac{|S_N(m,f)|}{\binom{N}{2}} \geq \sigma - 5\epsilon$. Hence $\liminf_{N\to\infty} \frac{|S_N(m,f)|}{\binom{N}{2}} \geq \sigma = \limsup_{N\to\infty} \frac{|S_N(m,f)|}{\binom{N}{2}}$, which implies that $\sigma^*(m, f) = \lim_{N\to\infty} \frac{|S_N(m,f)|}{\binom{N}{2}}$ exists. This completes the proof of Lemma 5.1. □

It also seems natural to consider the same problem for hypergraphs. Let $r \geq 3$ and m, f be integers satisfying $0 \leq f \leq \binom{m}{r}$. Let $S^r_n(m, f)$ consist of all integers e such that every n-vertex r-graph with e edges contains an m-vertex induced subhypergraph with f edges, and let $\sigma_r(m, f) = \limsup_{n\to\infty} |S^r_n(m, f)|/\binom{n}{r}$. Compared with the graph case (i.e., Theorem 1.4), it is much easier to show the following.

Lemma 5.2. *Let $r \geq 3$. If $(m, f) \notin \{(r, 0), (r, 1)\}$, then $\sigma_r(m, f) \leq 1 - r!/r^r$; otherwise, $\sigma_r(m, f) = 1$.*

In fact, it is enough to use the construction of complete r-partite r-graphs for all but finitely many pairs (m, f).

The authors of [3] mentioned the question if $\lim_{n\to\infty} \left(\max_{0\leq f\leq\binom{m}{2}} \sigma(m, f) \right) = 0$. It was not, as answered by their Theorem 1.2. A direct corollary of Theorem 1.4 shows that $\limsup_{m\to\infty} \left(\max_{0\leq f\leq\binom{m}{2}} \sigma(m, f) \right) = \frac{1}{2}$, but it is not known yet if $\lim_{m\to\infty} \left(\max_{0\leq f\leq\binom{m}{2}} \sigma(m, f) \right)$ exists or not.

We conclude this paper with a remark that by a more careful refinement of the arguments in Subsection 4.2, one can show that $|C(n, r)| = \frac{n^2}{2} - \frac{n^2}{2r} + o(n^2)$ holds for $r = 4$ and *odd* integers n. It would be interesting to understand the remaining cases $r = 3, 4$ in Conjecture 1.3.

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