Intersections of longest cycles in vertex-transitive and highly connected graphs

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Abstract

Motivated by the classical conjectures of Lovász, Thomassen, and Smith, recent work has renewed interest in the study of longest cycles in important graph families, such as vertex-transitive and highly connected graphs. In particular, Groenland et al. proved that if two longest cycles and in a graph share m vertices, then there exists a vertex cut of size $O(m^{8/5})$ separating them, yielding improved bounds toward these conjectures. Their proof combines Turán-type arguments with computer-assisted search.

We prove two results addressing problems of Babai (1979) and Smith (1984) on intersections of longest cycles in vertex-transitive and highly connected graphs. First, we strengthen the bound of Groenland et al. by showing that if two longest cycles and in a graph share m vertices, then there exists a vertex cut of size $O(m^{3/2})$ separating them. As a consequence, we show that in every k-connected graph, any two longest cycles intersect in at least $\Omega(k^{2/3})$ vertices, improving the best known bound toward Smith's conjecture. Our proof is purely combinatorial, employing supersaturation-type estimates beyond the existing Turán-type approach. Second, we prove that in every connected vertex-transitive graph on n vertices, any two longest cycles intersect in at least $f(n)$ vertices for some function $f(n) \to \infty$ as $n \to \infty$, thereby resolving a problem of Babai (1979) for the class of vertex-transitive graphs central to his original motivation. In doing so, we introduce a new method for constructing longer cycles in vertex-transitive graphs based on a given cycle, which may be of independent interest.

1   Introduction

The study of longest cycles is a central theme in both extremal and structural graph theory. Our work is motivated by several longstanding problems in this area, including two conjectures of Lovász [18] and Thomassen (cf. [2]) on vertex-transitive graphs, a conjecture of Smith (cf. [3]) on highly connected graphs, and a problem of Babai [1], originally motivated by vertex-transitive graphs but posed implicitly for 3-connected cubic graphs. The first two conjectures concern Hamiltonicity, while the latter two address the intersection sizes of longest cycles. Our main results make progress toward these problems and highlight closer connections between them. Throughout, we denote by $c(G)$ the length of a longest cycle in the graph G.

1.1   Background

A graph G is said to be *vertex-transitive* if, for any two vertices $u, v \in V(G)$, there exists an automorphism of G that maps u to v. The classical conjectures of Lovász [18] and Thomassen (cf. [2]) assert that every connected vertex-transitive graph on $n \geq 3$ vertices contains a Hamiltonian cycle (resp. path), with only finitely many exceptions. These conjectures remain widely open

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despite extensive study (see the survey [16], more recent papers [5,9], and the references therein). In 1979, Babai [1] proved that every connected vertex-transitive graph G on $n \geq 3$ vertices contains a cycle of length at least $\Omega(\sqrt{n})$. His elegant proof proceeds by double-counting the number T of triples $\{C, D, v\}$, where C and D are distinct longest cycles in G and $v \in V(C) \cap V(D)$ is a common vertex. Since G is vertex-transitive, every vertex lies in exactly $d \geq 1$ longest cycles, so G contains exactly $nd/c(G)$ distinct longest cycles. We may assume that G is 3-connected.[1] Since any two longest cycles in every 3-connected graph intersect in at least three vertices,[2] we have

$$n\binom{d}{2} = T \geq 3 \cdot \binom{nd/c(G)}{2},$$

which implies that $c(G) \geq \sqrt{3n}$. Observe that if any two longest cycles in such a graph G intersect in at least $f(n)$ vertices, then one immediately obtains the stronger bound $c(G) \geq \sqrt{n \cdot f(n)}$. Motivated by this observation, Babai posed the following problem in the same paper [1], restricting attention to 3-connected cubic graphs as an initial case.

Problem 1.1 (Babai [1], Problem 2). *Is there a function $f(n) \to \infty$ as $n \to \infty$ such that in every 3-connected cubic graph on n vertices, any two longest cycles intersect in at least $f(n)$ vertices?*

It is therefore natural to ask the analogous question for all connected vertex-transitive graphs.

Problem 1.2. *Is there a function $f(n) \to \infty$ as $n \to \infty$ such that in every connected vertex-transitive graph on n vertices, any two longest cycles intersect in at least $f(n)$ vertices?*

The $\Omega(n^{1/2})$ bound on the length of the longest cycle in connected vertex-transitive graphs stood for over four decades, until DeVos [6] improved it in 2023. Using the Orbit-Stabilizer Theorem, DeVos [6] established a notable double-counting lemma relating the problem to the *transversal* of longest cycles, a concept studied extensively (see [10,15,17,20]); a subset of vertices in a graph G is called a *t-transversal* if it contains at least t vertices from every longest cycle in G. His lemma shows that if a connected vertex-transitive graph G on n vertices contains a t-transversal A, then

$$c(G) \geq tn/|A|. \tag{1}$$

Combining this with the simple fact that if two longest cycles and in a graph share m vertices, then there exists a vertex cut of size at most m^2 separating them (which also forms a 1-transversal; see Proposition 2.1), DeVos [6] showed that $c(G) = \Omega(n^{3/5})$ for every connected vertex-transitive graph G on n vertices. Recently, Groenland, Longbrake, Steiner, Turcotte, and Yepremyan [12] gave a unified treatment concerning longest cycles in vertex-transitive and highly connected graphs. Their work extends DeVos' approach and shows that for any connected vertex-transitive graph G, $c(G) = \Omega(n^{13/21})$. The central result underlying their unified approach is the following, whose proof combines Turán-type arguments with computer-assisted search and linear programming.

Theorem 1.3 (Groenland et al. [12], Lemma 1.4). *If two longest cycles in a graph G share m vertices, then there exists a vertex cut of size $O(m^{8/5})$ separating these two cycles.*

More recently, Norin, Steiner, Thomassé, and Wollan [19] achieved a further improvement, showing that $c(G) = \Omega(n^{9/14})$ for every connected vertex-transitive graph G on n vertices. Using equation (1), their approach is to upper-bound the size of certain 1-transversals A as a function of $c(G)$, in contrast to [6,12], where the bound is expressed in terms of the intersection size of two longest cycles (cf. Theorem 1.3).

Another motivating problem is a longstanding conjecture on the intersection size of two longest cycles in highly connected graphs, attributed to S. Smith and traceable to at least 1984. As noted earlier in Babai's work [1], such insights can be valuable for addressing other problems on cycles.

Conjecture 1.4 (Smith's Conjecture; see, e.g., Conjecture 4.15 in [3]). *For every $k \geq 2$, any two longest cycles in a k-connected graph intersect in at least k vertices.*

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[1] Unless G is 2-regular (see [11]), in which case it is simply the cycle C_n and the result of Babai holds trivially.
[2] We will elaborate on this fact and its generalizations shortly; see Conjecture 1.4.

This conjecture has attracted considerable attention (see the survey [21]). It has been confirmed for $k \leq 8$ in [13, 22] and for $k \geq (n + 16)/7$ in [14]. For the general case, Burr and Zamfirescu (see [3]) proved that any two longest cycles in a k-connected graph intersect in at least $\sqrt{k} - 1$ vertices. This was later improved in 1998 by Chen, Faudree, and Gould [4] to $\Omega(k^{3/5})$ using a Turán-type argument. Recently, Groenland et al. [12], based on Theorem 1.3, established the best-known bound to date and proved that the intersection size of any two longest cycles in a k-connected graph is $\Omega(k^{5/8})$.

1.2 Main results

In this paper, motivated by the aforementioned conjectures and problems, we present two results on the intersection size of longest cycles in vertex-transitive and highly connected graphs.

Our first result strengthens Theorem 1.3 in the following quantitative form.

Theorem 1.5. *If two longest cycles in a graph G share m vertices, then there exists a vertex cut of size $O(m^{\frac{3}{2}})$ separating these two cycles.*[3]

Our proof is purely combinatorial and relies on supersaturation-type estimates, extending beyond the existing Turán-type approaches (e.g., [4,12]). Using this result, Menger's Theorem immediately yields the following bound for Conjecture 1.4, improving upon the current record, namely the $\Omega(k^{5/8})$ bound of Groenland et al. [12].

Corollary 1.6. *Let $k \geq 2$ be any integer. Every two longest cycles in a k-connected graph intersect in $\Omega(k^{2/3})$ vertices.*

Another implication of Theorem 1.5 concerns the length $c(G)$ of a longest cycle in a connected vertex-transitive graph G on n vertices. To see this, let m denote the minimum size of the intersection of two longest cycles in G. Then, evidently the vertex set of every longest cycle forms an m-transversal, and by Theorem 1.5, there exists a 1-transversal of size $O(m^{3/2})$. Applying equation (1) twice, we obtain

$$c(G) \geq \min_{m \in \mathbb{N}} \max \left\{ \sqrt{mn}, \frac{n}{O(m^{3/2})} \right\} = \Omega(n^{5/8}).$$

This improves the $\Omega(n^{13/21})$ bound of Groenland et al. [12], but falls short of the very recent $\Omega(n^{9/14})$ bound of Norin et al. [19]. On the other hand, since every connected d-regular vertex-transitive graph is $\Omega(d)$-connected (see [11]), Theorem 1.5 outperforms the $\Omega(n^{9/14})$ bound for connected d-regular vertex-transitive graphs when $d = \Omega(n^{3/7})$, as follows.

Corollary 1.7. *Let $n > d \geq 2$. Every connected d-regular vertex-transitive graph on n vertices contains a cycle of length $\Omega(d^{\frac{1}{3}} n^{\frac{1}{2}})$.*

Our second result is motivated by Babai's work [1]. We prove the following, showing that any two longest cycles in a connected vertex-transitive graph intersect in sufficiently many vertices.

Theorem 1.8. *Let $n > d \geq 2$. Every two longest cycles in a connected d-regular vertex-transitive graph on n vertices intersect in $\Omega\big((\log_d n)^{1/3}\big)$ vertices.*

Combining this with Corollary 1.6, we obtain the following corollary, which provides an affirmative answer to Problem 1.2. Throughout, we use $\ln x$ to denote the natural logarithm.

Corollary 1.9. *Every two longest cycles in a connected vertex-transitive graph on n vertices intersect in $\Omega\big((\ln n / \ln \ln n)^{1/3}\big)$ vertices.*

This can be seen as evidence (albeit arguably weak) supporting the conjecture of Lovász [18].

The rest of this paper is organized as follows. In Section 2, we provide the necessary preliminaries, including the crucial relation that reduces the proof of Theorem 1.5 to a problem in extremal graph theory. In Section 3, we present the proofs of Theorem 1.5 and Corollary 1.7. In Section 4, we prove Theorem 1.8 and Corollary 1.9. Finally, in Section 5, we conclude with several remarks.

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[3]We emphasize again that such a vertex cut forms a 1-transversal in G; see Proposition 2.1.

2 Preliminaries

In this section, we first introduce the notations used throughout this paper, and then define a useful auxiliary graph based on two longest cycles for the proof of Theorem 1.5. We also explain how this auxiliary graph plays a key role in the proof of Theorem 1.5.

2.1 Notations

We mostly follow the standard graph-theoretic notation from [8]. All graphs are finite unless otherwise specified, and the term *disjoint* always refers to *vertex-disjoint*. Let G be a graph. As mentioned earlier, we write $c(G)$ for the length of a longest cycle in G. Let $A, B \subseteq V(G)$ be two vertex subsets, not necessarily disjoint. A path $P = x_1 \cdots x_t$ in G is called an (A, B)-*path*, if $V(P) \cap A = \{x_1\}$ and $V(P) \cap B = \{x_t\}$. If $A = \{a\}$, we abbreviate $(\{a\}, B)$-paths by (a, B)-paths. For a subgraph H of G, we abbreviate $(V(H), B)$-paths by (H, B)-paths. We say that a vertex subset $S \subseteq V(G)$ *separates* A and B if every (A, B)-path intersects S. In this case, we call S an (A, B)-*separator*. Note that necessarily $A \cap B \subseteq S$. We also write $G - A$ for the subgraph of G induced by $V(G) \setminus A$.

For a path P (or a cycle C), we write $|P|$ (or $|C|$) for the number of its edges. Given $u, v \in V(P)$, let $P[u, v]$ denote the unique subpath of P with endpoints u and v. If P has endpoints a, b, and we fix a direction from a to b, we define the associated linear order $\prec$ on $V(P)$ as follows: for $u, v \in V(P)$, we write $u \prec v$ if a, u, v, b occur on P in the given direction.

For integers $a \leq b$, we denote $[a, b] = \{i \in \mathbb{Z} : a \leq i \leq b\}$. If a is positive, we write $[a]$ for the set $[1, a]$.

2.2 An auxiliary graph associated with two longest cycles

In this subsection, let G be a 2-connected graph, and let X, Y be any two longest cycles in G. Here, we discuss some basic propositions and define an auxiliary graph arising from this setting.

We begin with a simple fact. Recall the definition of a t-transversal given before equation (1). We refer to a 1-transversal simply as a *transversal*.

Proposition 2.1. *We have $V(X) \cap V(Y) \neq \emptyset$, and any vertex-cut separating X and Y is a transversal of G.*

Proof. Suppose that $V(X) \cap V(Y) = \emptyset$. Since G is 2-connected, there exist two disjoint (X, Y)-paths. Together with $X \cup Y$, these paths would form two cycles X' and Y' satisfying $|X'| + |Y'| > |X| + |Y|$, which contradicts the maximality of X and Y.

Now consider any vertex-cut A separating X and Y. Clearly, A contains all vertices in $V(X) \cap V(Y)$, which is non-empty. Let Z be any other longest cycle in G. Since $V(Z) \cap V(X) \neq \emptyset$ and $V(Z) \cap V(Y) \neq \emptyset$, there exists a subpath of Z which is a $\big(V(Z) \cap V(X), V(Z) \cap V(Y)\big)$-path. This path must intersect A, implying $V(Z) \cap A \neq \emptyset$. Hence, A is a transversal of G. □

Let $M = V(X) \cap V(Y)$ and $m = |M|$. Then $X - M$ and $Y - M$ decompose into m subpaths $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$, respectively (allowing empty subpaths). In the remainder of the paper, we refer to these subpaths X_i or Y_i as *segments*. The following proposition establishes a useful property for any collection of disjoint $(X - M, Y - M)$-paths in G.

Proposition 2.2. *For any $1 \leq i, j \leq m$, there do not exist two disjoint $(X - M, Y - M)$-paths in G whose endpoints belong to X_i and Y_j.*

Proof. Assume to the contrary that such paths L_1 and L_2 exist. Let $V(L_t) \cap V(X_i) = \{u_t\}$ and $V(L_t) \cap V(Y_j) = \{v_t\}$ for $t \in [2]$. Without loss of generality, we can assume that $|X_i[u_1, u_2]| \leq |Y_j[v_1, v_2]|$. Replacing the subpath $X_i[u_1, u_2]$ in X_i by the new (u_1, u_2)-path $L_1 \cup Y_j[v_1, v_2] \cup L_2$, we obtain a cycle in G with more than $|X|$ edges, contradicting the maximality of X. □

With these propositions in place, we now formalize the notion of the auxiliary graph associated with X and Y. It is defined for a given collection $\mathcal{P}$ of disjoint $(X - M, Y - M)$-paths and will play a crucial role in the proof of Theorem 1.5.

Definition 2.3. *Let X, Y be two longest cycles in a graph G. Let $M = V(X) \cap V(Y)$ and $m = |M|$. Let $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_m\}$ denote the path decompositions of $X - M$ and $Y - M$, respectively. Consider any collection $\mathcal{P}$ of disjoint $(X - M, Y - M)$-paths in G. Then the auxiliary graph $F := F(X, Y, \mathcal{P})$ is constructed as follows:*

(1) *The graph F is bipartite with two parts $A = \{x_1, \cdots, x_m\}$ and $B = \{y_1, \cdots, y_m\}$, where the vertex x_i (resp. y_i) corresponds to the segment X_i (resp. Y_i) in G.*

(2) *For $i, j \in [m]$, define the edge $x_i y_j \in E(F)$ if and only if $\mathcal{P}$ contains a (X_i, Y_j)-path.*

We point out that by Proposition 2.2, the graph F is simple and has exactly $2m$ vertices and $|\mathcal{P}|$ edges. The auxiliary graph F was explicitly considered in [4, 12], where the main idea in both papers is to show that F does not contain a fixed bipartite graph H as a subgraph (that is, F is H-*free*). Applying known estimates for the Turán number of H then yields an upper bound on $|\mathcal{P}| = e(F)$ as a function of m. In particular, [4] proves that F is $K_{3,257}$-free, while [12] shows that F is Q_3^+-free, where Q_3^+ denotes the graph obtained from $K_{4,4}$ by deleting a matching of size three. In the next section, we employ supersaturation arguments, extending beyond the earlier approaches that rely solely on Turán-type bounds. Specifically, we establish Theorem 1.5 by proving that the graph F contains only $O(m)$ copies of $K_{2,7}$, which implies $|\mathcal{P}| = e(F) = O(m^{3/2})$. Menger's Theorem then provides the required vertex cut for Theorem 1.5.

3 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. Throughout, let G be a graph and X, Y be two longest cycles in G with $M = V(X) \cap V(Y)$ and $m = |M|$. Our goal is to show that there exists a vertex cut of size $O(m^{3/2})$ separating X and Y. This is evident if X and Y lie in different blocks of G. Hence, we may assume that G is 2-connected.

We fix a cyclic ordering for each of the cycles X and Y. Let $\{X_1, \ldots, X_m\}$ and $\{Y_1, \ldots, Y_m\}$ denote the path decompositions of $X - M$ and $Y - M$, respectively, such that $X_1, \ldots, X_m$ (resp. $Y_1, \ldots, Y_m$) appear sequentially according to the chosen cyclic ordering of X (resp. Y).

Consider any collection $\mathcal{P}$ of disjoint $(X - M, Y - M)$-paths, and let $F := F(X, Y, \mathcal{P})$ be the bipartite graph defined in Definition 2.3. Our strategy is to use certain local structural conditions of F to find two cycles Q_1, Q_2 in G such that

$$E(Q_1) \cup E(Q_2) \supseteq E(X) \cup E(Y) \quad \text{and} \quad |Q_1| + |Q_2| > |X| + |Y|,$$

thereby contradicting the maximality of X and Y. We call such a pair $\{Q_1, Q_2\}$ of cycles a **winning certificate**.

We now introduce a key concept – the *types* of 4-cycles in F, determined by the relative positions in both X and Y of the endpoints of the four paths in $\mathcal{P}$ corresponding to its edges. Consider a 4-cycle $x_i y_k x_j y_\ell$ in F, where, throughout this section, we always assume that $i < j$ and $k < \ell$. By Definition 2.3, the edge $x_i y_k$ in F uniquely corresponds to an (X_i, Y_k)-path $P_{ik} \in \mathcal{P}$ in G; similarly, the edges $x_i y_\ell$, $x_j y_k$, and $x_j y_\ell$ correspond to the paths $P_{i\ell}$, P_{jk}, and $P_{j\ell}$ in $\mathcal{P}$, respectively. For convenience, we set

$$\mathcal{P}' = \{P_{ik}, P_{i\ell}, P_{jk}, P_{j\ell}\}.$$

We denote by $u_{ik}, u_{i\ell}, u_{jk}, u_{j\ell}$ the endpoints of $\mathcal{P}'$ in X, where $u_{ik} = V(X) \cap V(P_{ik})$, and the remaining three vertices are defined similarly. Similarly, let $v_{ik}, v_{i\ell}, v_{jk}, v_{j\ell}$ be the corresponding endpoints of $\mathcal{P}'$ in Y. See Figure 1 for an illustration. Let $<_X$ and $<_Y$ denote the cyclic ordering of X and Y, respectively. Each segment X_i (resp. Y_i) inherits the ordering $<_X$ (resp. $<_Y$).

Definition 3.1. *A 4-cycle $x_i y_k x_j y_\ell$ in F is said to be of **type** (α, β), where $\alpha, \beta \in \{0, 1\}$ are defined as follows:*

(1) *If both $u_{ik} <_X u_{i\ell}$ and $u_{jk} <_X u_{j\ell}$ hold, or both $u_{ik} >_X u_{i\ell}$ and $u_{jk} >_X u_{j\ell}$ hold, then let $\alpha = 0$; otherwise, $\alpha = 1$.*

(2) *If both $v_{ik} <_Y v_{jk}$ and $v_{i\ell} <_Y v_{j\ell}$ hold, or both $v_{ik} >_Y v_{jk}$ and $v_{i\ell} >_Y v_{j\ell}$ hold, then let $\beta = 0$; otherwise, $\beta = 1$.*
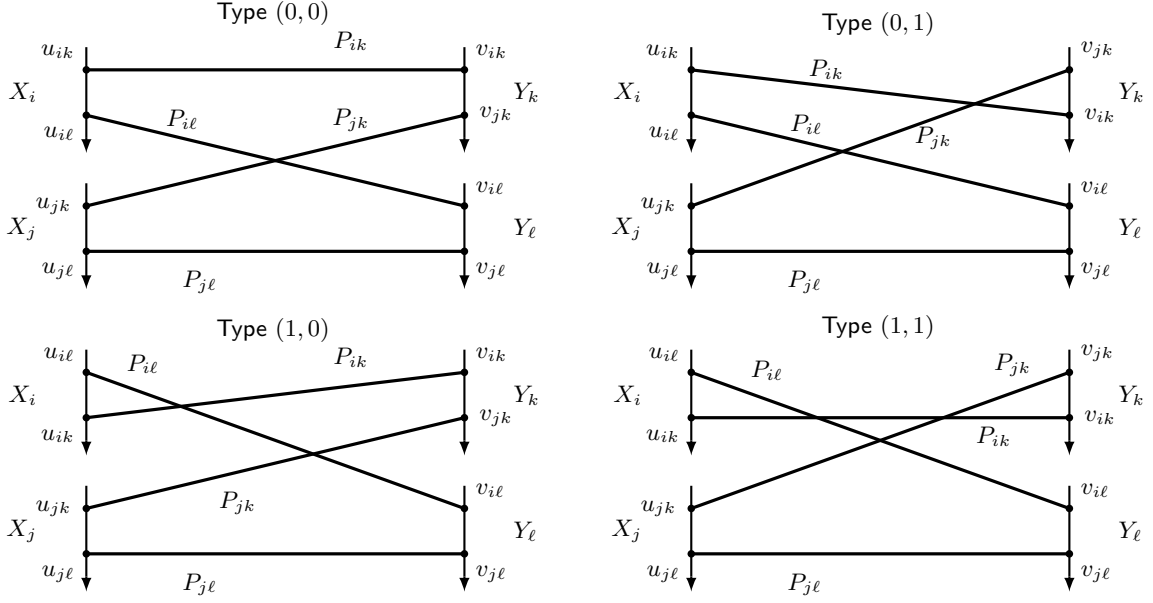
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Figure 1: Corresponding paths in G for all types of 4-cycles in F.

Figure 1 illustrates the four possible types of 4-cycles in F. Intuitively, $\alpha = 0$ if and only if the order of $u_{ik}, u_{i\ell}$ on X_i coincides with the order of $u_{jk}, u_{j\ell}$ on X_j with respect to $<_X$; similarly for β. We note that the type of a 4-cycle is invariant under the choice of the cyclic ordering $<_X$ on X. Indeed, reversing the ordering $<_X$ on X simultaneously reverses the relative orderings of both $\{u_{ik}, u_{i\ell}\}$ and $\{u_{jk}, u_{j\ell}\}$. Therefore, reversing $<_X$ (and likewise $<_Y$) does not affect the type of the 4-cycle. Below we examine the existence of 4-cycles of each type.

Our first lemma shows that, in our setting, F does not contain any 4-cycle of type $(0,0)$. An equivalent statement was proved in [4], where it played a central role in showing that F is $K_{3,257}$-free with the aid of the Erdős–Szekeres theorem.

Lemma 3.2. *The graph F contains no 4-cycle of type* $(0,0)$.

Proof. Suppose for a contradiction that $x_i y_k x_j y_\ell$ is a 4-cycle of type $(0,0)$. Keeping the previous notation for paths and endpoints, we can assume without loss of generality that

$$u_{ik} <_X u_{i\ell}, \quad u_{jk} <_X u_{j\ell}, \quad v_{ik} <_Y v_{jk}, \quad \text{and} \quad v_{i\ell} <_Y v_{j\ell}.$$

For any distinct vertices $u, v \in V(X)$, let $\overrightarrow{X}[u,v]$ denote the path from u to v along the fixed cyclic ordering $<_X$ of X. The notation $\overrightarrow{Y}[u,v]$ is defined analogously for Y. Construct two cycles Q_1 and Q_2 in G as follows (Figure 2 depicts Q_1 in red and Q_2 in blue):

$$Q_1 = P_{ik}[u_{ik}, v_{ik}] \cup \overrightarrow{Y}[v_{j\ell}, v_{ik}] \cup P_{j\ell}[v_{j\ell}, u_{j\ell}] \cup X_j[u_{j\ell}, u_{jk}]$$
$$\cup P_{jk}[u_{jk}, v_{jk}] \cup \overrightarrow{Y}[v_{jk}, v_{i\ell}] \cup P_{i\ell}[v_{i\ell}, u_{i\ell}] \cup X_i[u_{i\ell}, u_{ik}],$$
$$Q_2 = P_{ik}[u_{ik}, v_{ik}] \cup Y_k[v_{ik}, v_{jk}] \cup P_{jk}[v_{jk}, u_{jk}] \cup \overrightarrow{X}[u_{i\ell}, u_{jk}]$$
$$\cup P_{i\ell}[u_{i\ell}, v_{i\ell}] \cup Y_\ell[v_{i\ell}, v_{j\ell}] \cup P_{j\ell}[v_{j\ell}, u_{j\ell}] \cup \overrightarrow{X}[u_{j\ell}, u_{ik}].$$

It is straightforward to verify that (Q_1, Q_2) forms a winning certificate, since

$$E(Q_1) \cup E(Q_2) \supseteq E(X) \cup E(Y) \quad \text{and} \quad |Q_1| + |Q_2| = |X| + |Y| + 2 \sum_{P \in \mathcal{P}'} |P| > |X| + |Y|,$$

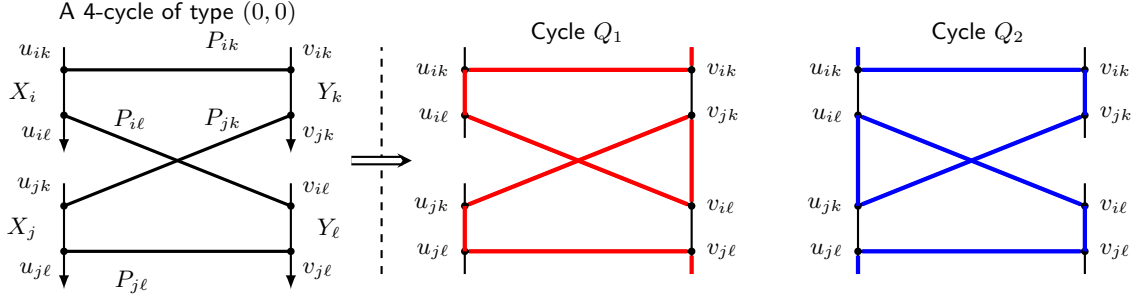contradicting the maximality of X and Y. $\qquad\square$

Figure 2: A winning certificate (Q_1, Q_2) corresponding to a 4-cycle of type $(0,0)$.

Next, we consider two 4-cycles in F of type $(1,0)$, which correspond to eight paths from $\mathcal{P}$. To streamline our arguments, we introduce the following notation.

Definition 3.3. *For indices $1 \le i_1 < j_1 \le m$ and $1 \le i_2 < j_2 \le m$, the pairs (i_1, j_1) and (i_2, j_2) are **crossing**, if i_1, j_1, i_2, j_2 are pairwise distinct and the interval $[i_1, j_1]$ contains exactly one element in $\{i_2, j_2\}$; otherwise, we say that (i_1, j_1) and (i_2, j_2) are **non-crossing**.*

We first show that a particular arrangement of the endpoints of these eight paths on X and Y guarantees the existence of a winning certificate.

Lemma 3.4. *Let $x_{i_1} y_{k_1} x_{j_1} y_{\ell_1}$ and $x_{i_2} y_{k_2} x_{j_2} y_{\ell_2}$ be two 4-cycles of type $(1,0)$ in F. If (i_1, j_1) and (i_2, j_2) are crossing, then either $\{k_1, \ell_1\} \cap \{k_2, \ell_2\} \ne \emptyset$, or (k_1, ℓ_1) and (k_2, ℓ_2) are crossing.*

Proof. Let (i_1, j_1) and (i_2, j_2) be crossing. Suppose, for the sake of contradiction, that $\{k_1, \ell_1\} \cap \{k_2, \ell_2\} = \emptyset$ and that (k_1, ℓ_1) and (k_2, ℓ_2) are non-crossing. Let $\mathcal{P}'$ denote the collection of the eight disjoint paths in $\mathcal{P}$ corresponding to the edges in the two 4-cycles.

We may assume, without loss of generality, that the four segments $X_{i_1}, X_{i_2}, X_{j_1}, X_{j_2}$ are cyclically arranged on X, and that $Y_{k_1}, Y_{\ell_1}, Y_{k_2}, Y_{\ell_2}$ are cyclically arranged on Y; see Figure 3.

By considering all possible relative orderings of the endpoints of the paths in $\mathcal{P}'$, we claim that there are four non-isomorphic cases that need to be examined in general. To see this, we first consider the four pairs of endpoints corresponding to the 4-cycle $x_{i_1} y_{k_1} x_{j_1} y_{\ell_1}$, which are highlighted by the ellipse-shaped shadows in Figure 3. Since this 4-cycle is of type $(\alpha, \beta) = (0, 1)$, we can select the initial cyclic orderings of X and Y such that $u_{i_1 k_1} <_X u_{i_1 \ell_1}$ and $v_{i_1 k_1} <_Y v_{j_1 k_1}$. Then since $\alpha = 0$ and $\beta = 1$, it follows that $u_{j_1 \ell_1} <_X u_{j_1 k_1}$ and $v_{i_1 \ell_1} <_Y v_{j_1 \ell_1}$. In this way, we could assume that the four pairs covered by the ellipse-shaped shadows are fixed. It remains to consider the four pairs of endpoints corresponding to the second 4-cycle $x_{i_2} y_{k_2} x_{j_2} y_{\ell_2}$, where the pairs $\{u_{i_2 k_2}, u_{i_2 \ell_2}\}$ and $\{u_{j_2 \ell_2}, u_{j_2 k_2}\}$ are highlighted by the triangle-shaped shadows and the pairs $\{v_{i_2 k_2}, v_{j_2 k_2}\}$ and $\{v_{i_2 \ell_2}, v_{j_2 \ell_2}\}$ are highlighted by the rectangle-shaped shadows in Figure 3. Since this cycle has type $(0, 1)$, the relative orderings in the two pairs covered by the triangle-shaped shadows are consistent (as $\alpha = 0$), yielding two possibilities (see the differences between Figures 3a and 3b). Similarly, the relative orderings in the two pairs covered by the rectangle-shaped shadows are inconsistent (as $\beta = 1$), also giving two possibilities. This proves the claim.

Therefore, it suffices to discuss the four cases, under the alternative orderings of the pairs covered by the triangle-shaped and rectangle-shaped shadows. Consider one such instance in Figure 3a. In this case, there exist two cycles Q_1 and Q_2 (depicted by red and blue, respectively) satisfying

$$E(Q_1) \cup E(Q_2) \supseteq E(X) \cup E(Y) \quad \text{and} \quad |Q_1| + |Q_2| = |X| + |Y| + 2 \sum_{P \in \mathcal{P}'} |P| > |X| + |Y|,$$

contradicting the maximality of X and Y. This forms a winning certificate (Q_1, Q_2). Figure 3b considers another instance, where we preserve the relative orderings of pairs covered by the rectangle-shaped shadows, and reverse that covered by the triangle-shaped shadows. It is clear from the figure that the red and blue lines yield another winning certificate. The verifications for the remaining two cases are similar, which we omit here. □
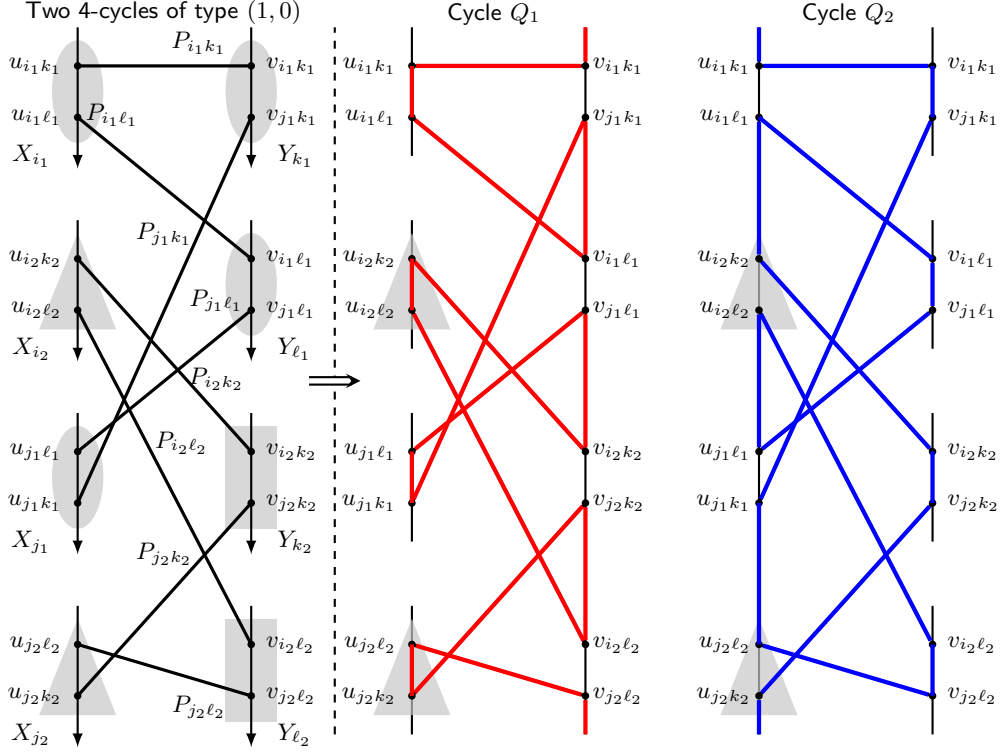
(a) A winning certificate (Q_1, Q_2) corresponding to a configuration of two 4-cycles of type $(1, 0)$.
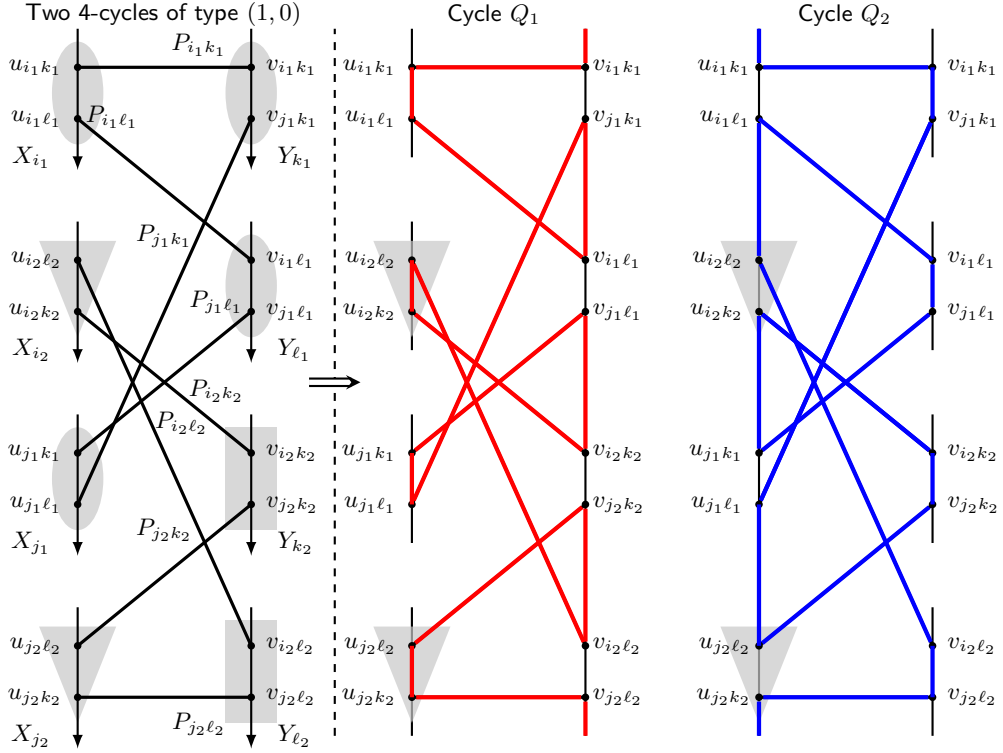
(b) The counter configuration from (a) with reversed order of pairs in triangle-shaped shadows.

Figure 3: Key steps in the proof of Lemma 3.4.

The following lemma provides a key tool for obtaining an upper bound on the number of copies of $K_{2,7}$ in F. For $x \in V(F)$, let $N_F(x)$ denote its neighborhood. For $1 \le i < j \le m$, we define $a_{ij} = \left| N_F(x_i) \cap N_F(x_j) \right|$.

Lemma 3.5. *Let $L = \{(i,j) : a_{ij} \ge 7, \ 1 \le i < j \le m\}$. Then any two pairs (i_1, j_1) and (i_2, j_2) in L are non-crossing.*

Proof. Consider the pair (i_1, j_1) and a copy of $K_{2,7}$ in F with vertex set $\{x_{i_1}, x_{j_1}\} \cup \{y_{k_t} : t \in [7]\}$. Let $\mathcal{Q}$ be the set of paths in $\mathcal{P}$ that correspond to the edges of this $K_{2,7}$. Each segment Y_{k_t} contains two endpoints $v_{i_1 k_t}, v_{j_1 k_t}$ from two paths in $\mathcal{Q}$ whose relative ordering can be either $v_{i_1 k_t} <_Y v_{j_1 k_t}$ or $v_{i_1 k_t} >_Y v_{j_1 k_t}$. By the pigeonhole principle, at least $\lceil 7/2 \rceil = 4$ distinct Y-segments share the same endpoint ordering. By Lemma 3.2, there is no 4-cycle of type $(0,0)$ in F. Therefore, we obtain a subgraph $K_{2,4}$ that only contains 4-cycles of type $(1,0)$.

Applying above arguments to pairs (i_1, j_1) and (i_2, j_2), we obtain indices $1 \le k_1 < \cdots < k_4 \le m$ and $1 \le \ell_1 < \cdots < \ell_4 \le m$ such that both $F[\{x_{i_1}, x_{j_1}, y_{k_1}, \cdots, y_{k_4}\}]$ and $F[\{x_{i_2}, x_{j_2}, y_{\ell_1}, \cdots, y_{\ell_4}\}]$ are isomorphic to $K_{2,4}$, and any 4-cycle in these subgraphs has type $(1,0)$.

We claim that there are two indices in $\{k_1, \cdots, k_4\}$ and two indices in $\{\ell_1, \cdots, \ell_4\}$ that are pairwise distinct and form non-crossing pairs. Suppose there are three indices $\{h_1, h_2, h_3\} \subseteq \{\ell_1, ..., \ell_4\}$ distinct from $\{k_1, k_2\}$. By the pigeonhole principle, two of h_1, h_2, h_3 (say h_1, h_2) must both lie in the interval (k_1, k_2) or both in $[1, k_1) \cup (k_2, m]$. Then the two disjoint pairs (k_1, k_2) and (h_1, h_2) are non-crossing, as claimed. Otherwise, we have $\{k_1, k_2\} \subseteq \{\ell_1, ..., \ell_4\}$; similarly, we may assume that $\{k_3, k_4\} \subseteq \{\ell_1, ..., \ell_4\}$. This implies $k_t = \ell_t$ for $t \in [4]$. Thus, the two pairs (k_1, k_2) and (ℓ_3, ℓ_4) are non-crossing and share no common element, which proves the claim.

Using this claim, we can deduce from Lemma 3.4 that (i_1, j_1) and (i_2, j_2) are non-crossing, completing the proof. $\square$

We need one more lemma that provides an upper bound on the number of pairs that are pairwise non-crossing.

Lemma 3.6. *Let $m \ge 2$ be an integer, and let $\{(i_t, j_t)\}_{t=1}^r$ be a collection of r distinct pairs with $1 \le i_t < j_t \le m$ for all $t \in [r]$, such that any two pairs are non-crossing. Then $r \le 2m - 3$.*

Proof. Let $f(m)$ denote the maximum number of such pairs. We proceed by induction on m to show that $f(m) \le 2m - 3$. First observe that $f(2) = 1$ and $f(m)$ is non-decreasing in m. Assume that $f(k) \le 2k - 3$ holds for all $2 \le k \le m - 1$. Consider a collection of pairs $\{(i_t, j_t)\}_{t=1}^{f(m)}$ of size $f(m)$. Define $\mathcal{I} = \{[i_t, j_t] : t \in [f(m)]\} \setminus \{[1, m]\}$ to be a collection of close intervals. An interval in $\mathcal{I}$ is called *maximal*, if it is maximal under interval inclusion. Without loss of generality, we can assume that $\mathcal{I}' = \{[i_t, j_t]\}_{t=1}^s$ contains all maximal intervals in $\mathcal{I}$. We point out that these maximal intervals must be pairwise internally-disjoint, because any two pairs $(i_t, j_t), (i_{t'}, j_{t'})$ are non-crossing. Moreover, each interval in $\mathcal{I}$ belongs to exactly one maximal interval.

If $|\mathcal{I}'| = 1$, then each pair (i_t, j_t) except for $(1, m)$ satisfies $i_1 \le i_t < j_t \le j_1$. This implies

$$f(m) \le 1 + f(j_1 - i_1 + 1) \le 1 + f(m - 1) \le 2m - 4.$$

Now we may assume $s = |\mathcal{I}'| \ge 2$. Then, using properties mentioned above, for each pair $(i_t, j_t) \in \mathcal{I}$ except for $(1, m)$, there is exactly one index $u \in [s]$ such that $i_u \le i_t < j_t \le j_u$. This yields

$$f(m) \le 1 + \sum_{u \in [s]} f(j_u - i_u + 1) \le 1 + 2 \sum_{u \in [s]} \left(j_u - i_u \right) - s \le 1 + 2(m - 1) - s \le 2m - 3,$$

where the second inequality follows by induction and the third inequality holds because $[i_u, j_u]$ for all $u \in [s]$ are pairwise internally-disjoint. The proof is complete. $\square$

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let X, Y be two longest cycles in a graph G which shares m vertices. Consider any collection $\mathcal{P}$ of disjoint $(X - M, Y - M)$-paths in G, where $M = V(X) \cap V(Y)$. Let $F := F(X, Y, \mathcal{P})$ be defined from Definition 2.3, and let $\mathcal{E} = e(F) = |\mathcal{P}|$. We aim to show that

$$|\mathcal{P}| = \mathcal{E} \le \sqrt{10} m^{\frac{3}{2}} + \frac{m}{2}.$$

We may assume $\mathcal{E} \geq m$. Under this assumption, the convexity of the function $\frac{x(x-1)}{2}$ yields

$$\sum_{1 \leq i < j \leq m} a_{ij} = \sum_{i \in [m]} \binom{\deg_F(y_i)}{2} \geq m \cdot \binom{\sum_{i \in [m]} \deg_F(y_i)/m}{2} = m\binom{\mathcal{E}/m}{2} = \frac{\mathcal{E}(\mathcal{E}-m)}{2m}.$$

Let $\mathbf{1}_A$ denote the indicator function of event A. Then we have

$$\sum_{1 \leq i < j \leq m} a_{ij} \cdot \mathbf{1}_{\{a_{ij} \geq 7\}} \geq \sum_{1 \leq i < j \leq m} a_{ij} - 6\binom{m}{2} \geq \frac{\mathcal{E}(\mathcal{E}-m)}{2m} - 3m(m-1).$$

Recall that $L = \{(i,j) : a_{ij} \geq 7, \ 1 \leq i < j \leq m\}$. Evidently $a_{ij} = \left|N_F(x_i) \cap N_F(x_j)\right| \leq m$ for all $1 \leq i < j \leq m$, so we can deduce that

$$|L| = \sum_{1 \leq i < j \leq m} \mathbf{1}_{\{a_{ij} \geq 7\}} \geq \sum_{1 \leq i < j \leq m} a_{ij} \cdot \mathbf{1}_{\{a_{ij} \geq 7\}} / m \geq \frac{\mathcal{E}(\mathcal{E}-m)}{2m^2} - 3(m-1).$$

By Lemma 3.5, the pairs in L are pairwise non-crossing. Applying Lemma 3.6, we obtain

$$\frac{\mathcal{E}(\mathcal{E}-m)}{2m^2} - 3(m-1) \leq |L| \leq 2m - 3.$$

Solving this inequality gives $\mathcal{E}^2 - m\mathcal{E} - (10m^3 - 12m^2) \leq 0$, which implies $|\mathcal{P}| = \mathcal{E} \leq \sqrt{10}m^{\frac{3}{2}} + \frac{m}{2}$, as desired. Thus, there are at most $\sqrt{10}\,m^{3/2} + \frac{m}{2}$ disjoint paths in G between $X - M$ and $Y - M$. By Menger's Theorem, this implies the existence of a vertex cut of size at most $\sqrt{10}\,m^{3/2} + \frac{m}{2}$ separating $X - M$ and $Y - M$. Adding the set M of size m, we obtain a vertex cut of size at most $\sqrt{10}\,m^{3/2} + \frac{3m}{2}$ separating X and Y in G, completing the proof. $\qquad\square$

We conclude this section with a proof of Corollary 1.7.

Proof of Corollary 1.7. Let G be a connected d-regular vertex-transitive graph on n vertices, where $n > d \geq 2$. Then G is $\Omega(d)$-connected (see [11]). By Corollary 1.6, every two longest cycles in G intersect in $\Omega(d^{2/3})$ vertices. Let A denote the set of vertices in a fixed longest cycle. Then A is a t-transversal in G for some $t = \Omega(d^{2/3})$. Applying equation (1), we obtain

$$c(G) \geq \sqrt{tn} = \sqrt{\Omega(d^{2/3})n} = \Omega(d^{\frac{1}{3}} \cdot n^{\frac{1}{2}}).$$

This provides the existence of a desired cycle in G. $\qquad\square$

4 Proof of Theorem 1.8

In this section we prove Theorem 1.8, which asserts that any two longest cycles in a connected vertex-transitive graph must share a sufficiently large set of vertices.

To proceed, we employ a structural lemma on connected vertex-transitive graphs due to DeVos and Mohar [7]. It asserts that in a connected vertex-transitive graph, a small separator cannot disconnect two large sets unless the graph possesses a special global structure. For a formal statement, we first introduce some notation. Let G be a connected vertex-transitive graph. For any $x, y \in V(G)$, let $\mathrm{dist}(x,y)$ denote the length of a shortest (x,y)-path in G, and define

$$\mathrm{diam}(G) = \max_{x,y \in V(G)} \mathrm{dist}(x,y).$$

The *neighborhood* $N(A)$ of a subset $A \subseteq V(G)$ denotes the set of all vertices in $V(G) \setminus A$ adjacent to some vertex in A. A partition $\sigma = \{B_1, \ldots, B_k\}$ of $V(G)$ is called a *system of imprimitivity* if for every automorphism g of G and every set $B \in \sigma$, the image $B^g := g(B)$ remains in σ. These vertex sets $B \in \sigma$ are called *blocks*. A *cyclic system* $\vec{\sigma}$ on G is a system of imprimitivity σ equipped with a cyclic ordering of the blocks that is preserved by the automorphism group $\mathrm{Aut}(G)$ of G. For positive integers s, t, we say G is (s,t)-*ring-like* with respect to $\vec{\sigma}$, if each block of $\vec{\sigma}$ has size s and any two adjacent vertices in G belong to blocks that are at distance at most t in the cyclic ordering of $\vec{\sigma}$.

Theorem 4.1 (DeVos and Mohar, Theorem 1.9 in [7]). *For every integer $k \geq 1$ and every connected vertex-transitive graph G with $\operatorname{diam}(G) \geq 31k^2$, if a non-empty set $A \subseteq V(G)$ satisfies $|A| \leq |V(G)|/2$, $|N(A)| = k$, and $G[A \cup N(A)]$ is connected, then one of the following holds.*

(1) $|A| \leq 2k^3 + k^2$;

(2) There exist positive integers s, t with $st \leq k/2$ and a cyclic system $\vec{\sigma}$ on G such that G is (s,t)-ring-like with respect to $\vec{\sigma}$, and there exists an interval J of $\vec{\sigma}$ with $Q := \bigcup_{B \in J} B$ satisfying

$$Q \supseteq A \quad and \quad |Q \setminus A| \leq \frac{k^3}{2} + k^2.$$

The following observation can be verified directly.

Observation 4.2. *Let X and Y be two cycles in a graph G. Let $\{P_1, \ldots, P_r\}$ be a collection of pairwise disjoint subpaths of X, and let $\{Q_1, \ldots, Q_r\}$ be a collection of pairwise disjoint subpaths of Y such that, for each $i \in [r]$,*

- *P_i and Q_i have the same endpoints x_{2i-1}, x_{2i},*

- *the vertices $x_1, x_2, \ldots, x_{2r}$ appear in the same cyclic order along both X and Y, and*

- *each Q_i could only intersect X at vertices of the subpaths P_j.*

Then the subgraph obtained from X by replacing P_i with Q_i for each $i \in [r]$ (i.e., deleting all edges of P_i and adding all edges of Q_i) forms a cycle in G.

We now present the proof of Theorem 1.8, which uses the bound of Theorem 1.5. For a family $\mathcal{H}$ of subgraphs, we define $V(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} V(H)$ and $E(\mathcal{H}) := \bigcup_{H \in \mathcal{H}} E(H)$.

Proof of Theorem 1.8. Let G be a connected d-regular vertex-transitive graph on n vertices. Let X, Y be any two longest cycles in G, and let $m := |V(X) \cap V(Y)| \geq 1$. Our goal is to show $m \geq \left(\log_d n\right)^{1/3}/100$. It suffices to assume that $d \geq 3$.

Let k denote the size of a minimum (X, Y)-separator in G. Then there exists a subset $A \subseteq V(G)$ with $|A| \leq n/2$ such that $N(A)$ is a vertex cut of G of size k separating X and Y, and $G[A \cup N(A)]$ contains exactly one of X and Y (without loss of generality, say X). This implies

$$|A| \geq |V(X)| - |N(A)| = c(G) - k. \tag{2}$$

Furthermore, we can assume that $G[A \cup N(A)]$ is connected, as otherwise we instead consider the intersection of A with the component of $G[A \cup N(A)]$ that contains X. By the proof of Theorem 1.5, we have $k \leq 10m^{3/2} + 3m/2$. If $k \geq \frac{\log_d^{1/2} n}{10}$, then $m \geq \frac{\log_d^{1/3} n}{100}$, which gives the desired bound. We may therefore assume $k < \frac{\log_d^{1/2} n}{10}$ and equivalently,

$$n > d^{100k^2} \geq 3^{100k^2}. \tag{3}$$

Since G is d-regular, we have $\operatorname{diam}(G) \geq \log_{d-1}(n/d) \geq 31k^2$. Applying Theorem 4.1 for G and A, we conclude that one of the items (1) or (2) holds.

First, suppose that item (1) holds; that is, $|A| \leq 2k^3 + k^2$. By Babai [1] we have $c(G) \geq 3\sqrt{n}$. Together with (2), this yields $3\sqrt{n} - k \leq c(G) - k \leq |A| \leq 2k^3 + k^2$. Hence, $n \leq \frac{1}{9}(2k^3 + k^2 + k)^2 < 3^{100k^2}$, which contradicts (3). Therefore, from now on, we may assume that item (2) holds.

Then there exists a cyclic system $\vec{\sigma}$ on G such that G is (s,t)-ring-like with respect to $\vec{\sigma}$, where $st \leq k/2$, and there exists an interval J of $\vec{\sigma}$ satisfying that $Q := \bigcup_{B \in J} B \supseteq A$ and $|Q \setminus A| \leq \frac{k^3}{2} + k^2$. Let $B_1, \ldots, B_{n/s}$ be all blocks in the cyclic system $\vec{\sigma}$. Without loss of generality, we may assume that for some $j \in [n/s]$, $V(X) \subseteq \bigcup_{i \in [j]} B_i$, where $V(X) \cap V(B_1) \neq \emptyset$ and $V(X) \cap V(B_j) \neq \emptyset$. Since $V(X) \subseteq A \cup N(A) \subseteq \left(\bigcup_{B \in J} B\right) \cup N(A)$ and $|N(A)| = k$, we deduce that $|J| \geq j - k$.

We first claim that $j \geq n/(2s)$. Suppose for a contradiction that $j < n/(2s)$. By the vertex-transitivity of G, there is an automorphism $g \in \operatorname{Aut}(G)$ that maps B_1 to B_{j+1}. Then the image

X^g is also a longest cycle in G, whose vertices belong to $\bigcup_{i \in [j+1, 2j+1]} B_i$. Since $2j + 1 \leq n/s$, evidently we have $V(X) \cap V(X^g) = \emptyset$, which contradicts Proposition 2.1. This proves our claim that $j \geq n/(2s)$. In particular, $j \geq n/(2s) \geq 3^{100k^2}/k \geq 10^{k^2}$.

We now proceed to establish the key step of the proof, namely that

$$j \geq n/s - 10^{k^2}. \tag{4}$$

Suppose, for the sake of contradiction, that this is not the case. Then $10^{k^2} < 3^{100k^2}/k \leq n/(2s) \leq j < n/s - 10^{k^2}$. We aim to construct a cycle in G which is strictly longer than X.

For each $\ell \in [n/s - t + 1]$, let E_ℓ denote the set of edges in X incident to some vertex in $\bigcup_{i \in [\ell, \ell+t-1]} B_i$. Note that the edges of E_ℓ are decomposed into disjoint subpaths of X with length at least two. By the vertex-transitivity of G, for every $i \in [j]$, there is an automorphism $g_i \in \mathrm{Aut}(G)$ that maps B_i to B_1. For each $\ell \in [j]$, we write $E_\ell^{g_\ell}$ for the image of E_ℓ under g_ℓ, so that

$$\text{every edge in } E_\ell^{g_\ell} \text{ is incident to some vertex in } \bigcup_{i \in [t]} B_i. \tag{5}$$

We define an equivalence relation $\sim$ among these automorphisms $g_1, \cdots, g_j$ by writing $g_u \sim g_v$ for some $u, v \in [j]$, if the following hold:

(a). $E_u^{g_u} = E_v^{g_v}$ (as two edge sets incident to $\bigcup_{i \in [t]} B_i$);

(b). The edges in $E_u^{g_u}$ appear in the same cyclic ordering along both longest cycles X^{g_u} and X^{g_v}.

Now we calculate the number of equivalence classes under $\sim$. By (5), every edge in $E_u^{g_u}$ has both endpoints in $B^* := \bigcup_{i \in [-t+1, 2t]} B_i$, where $B_{-i} := B_{n/s-i}$ for $i \in [0, n/s - 1]$. This implies $|E_u^{g_u}| \leq \binom{|B^*|}{2} \leq \binom{3st}{2}$ for every $u \in [j]$. We then consider the possible cyclic orderings for edges in $E_u^{g_u}$. As noted above, the edges in $E_u^{g_u}$ are decomposed into disjoint subpaths with length at least two, say p such paths. Then $p \leq |B^*|/2 = 3st/2$. As each such path can be placed in 2 ways, there are exactly $2^p(p-1)! < (2p)! \leq (3st)!$ possible cyclic orderings of $E_u^{g_u}$ in longest cycles containing these edges. Therefore, the number of equivalence classes under $\sim$ is at most

$$\sum_{i=0}^{\binom{3st}{2}} \binom{\binom{3st}{2}}{i} \cdot (3st)! = 2^{\binom{3st}{2}} \cdot (3st)! \leq 2^{\binom{3k/2}{2}} \cdot (3k/2)! < 10^{k^2}.$$

By the pigeonhole principle, there exist indices $1 \leq u < v \leq 10^{k^2} \leq j$ such that $g_u \sim g_v$. Define the automorphism $g := (g_v)^{-1} \circ g_u \in \mathrm{Aut}(G)$. Then we have

$$B_u^g = B_v \quad \text{and} \quad E_u^g = E_v.$$

Since $j < n/s - 10^{k^2}$, no edge connects $\bigcup_{i \in [u-1]} B_i$ and $\bigcup_{i \in [u+t, j]} B_i$. Hence, every edge in $E(X) \setminus E_u$ is entirely contained in either $G\big[\bigcup_{i \in [u-1]} B_i\big]$ or $G\big[\bigcup_{i \in [u+t, j]} B_i\big]$. These edges form several disjoint subpaths of X. Define $\mathcal{R}$ to be the collection of disjoint paths formed by $E(X) \setminus E_u$ within $G[\bigcup_{i \in [u+t, j]} B_i]$, and define $\mathcal{R}'$ to be the collection of disjoint paths formed by $E(X) \setminus E_v$ within $G[\bigcup_{i \in [v+t, j]} B_i]$; see Figure 4 for an illustration. Then we have $E(\mathcal{R}') \subseteq E(\mathcal{R})$.

We claim that $E(\mathcal{R}) \setminus E(\mathcal{R}') \neq \emptyset$. Select an edge $e \in E_u$ whose endpoints belong to B_α and B_β respectively, such that $\min\{\alpha, \beta\}$ is maximized. Then the edge $e^g \in E_u^g = E_v$, hence $e^g \notin E(\mathcal{R}')$. Note that e^g is incident to blocks $B_{\alpha+v-u}$ and $B_{\beta+v-u}$. We assert that $e^g \notin E_u$ and thus, $e^g \in E(\mathcal{R})$. Indeed, if $e^g \in E_u$, then $\min\{\alpha + v - u, \beta + v - u\} > \min\{\alpha, \beta\}$ would contradict the maximality of $\min\{\alpha, \beta\}$ for our choice of e. We have demonstrated that $e^g \in E(\mathcal{R}) \setminus E(\mathcal{R}')$, and the claim follows. This implies $|E(\mathcal{R})| > |E(\mathcal{R}')|$.

It is clear that the endpoints of subpaths (of X) formed by E_v is the same as that of $E(X) \setminus E_v$;[4] similarly, the endpoints of subpaths (of X) formed by E_u is the same as that of $E(X) \setminus E_u$, which says equivalently, the endpoints of subpaths (of X^g) formed by $E_v = E_u^g$ is the same as that

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[4]Note that these subpaths may be viewed in different cycles, but these endpoints are determined solely by the edge set E_v, and the same holds for other edge sets.
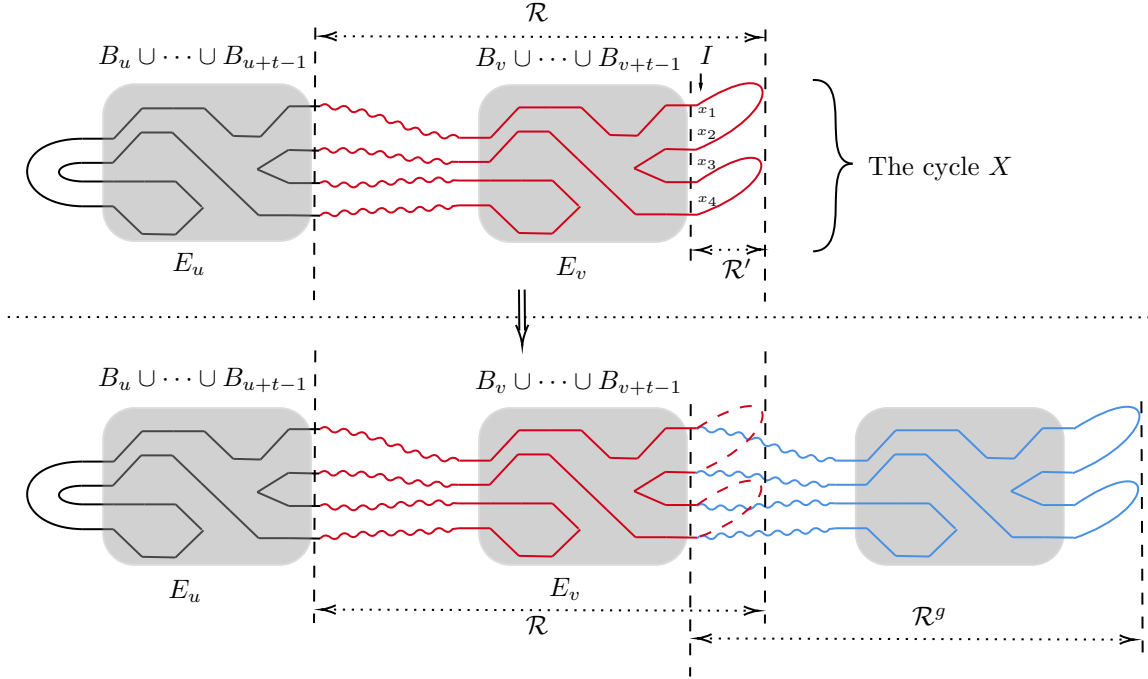
Figure 4: Finding a longer cycle than the given cycle X in vertex-transitive graphs G.

of $(E(X) \setminus E_u)^g$. Summarizing, the endpoints of subpaths formed by E_v, by $E(X) \setminus E_v$, and by $(E(X) \setminus E_u)^g$ are all the same. Let I denote the set of these endpoints that belong to $\bigcup_{i \in [v+t,j]} B_i$.

Recall the definitions of $\mathcal{R}$ and $\mathcal{R}'$. Let $\mathcal{R}^g$ be the image of $\mathcal{R}$ under the automorphism g. Putting everything together, we can derive that the endpoints of the paths in $\mathcal{R}'$ are the same as those of $\mathcal{R}^g$, that is, the vertices of I. Moreover, since $g_u \sim g_v$, these vertices of I appear in the same cyclic order along both X and X^g. Therefore, we can label the vertices as $I = \{x_1, x_2, \cdots, x_{2r}\}$ and write $\mathcal{R}' = \{P_1, \cdots, P_r\}$ and $\mathcal{R}^g = \{Q_1, \cdots, Q_r\}$, such that P_i and Q_i share the same endpoints $\{x_{2i-1}, x_{2i}\}$ for every $i \in [r]$.

We view all $P_i \in \mathcal{R}'$ as subpaths of the cycle X and all $Q_i \in \mathcal{R}^g$ as subpaths of the cycle X^g. To apply Observation 4.2, it remains to verify that each Q_i could only intersect X at vertices of some P_j. This follows since all vertices of $Q_i \in \mathcal{R}^g$ are entirely contained in $\bigcup_{j \in [v+t, j+v-u]} B_j$, whereas the vertices in $V(X) \setminus V(\mathcal{R}')$ must belong to $\bigcup_{j \in [v+t-1]} B_j$. Since $j + (v-u) < n/s - 10^{k^2} + (v-u) < n/s$, these two block intervals are disjoint, which completes the verification. Finally, we can apply Observation 4.2 to conclude that the edges in $E(X) \cup E(\mathcal{R}^g) \setminus E(\mathcal{R}')$ form a cycle in G of length $|X| + |E(\mathcal{R})| - |E(\mathcal{R}')| > |X|$, which contradicts the maximality of X. This proves (4), namely, $j > n/s - 10^{k^2}$. Hence, we have $|J| \geq j - k \geq n/s - 10^{k^2} - k$.

Recall that $Q = \bigcup_{B \in J} B$. Since $|Q \setminus A| \leq \frac{k^3}{2} + k^2$, we obtain the following inequality

$$\frac{n}{2} \geq |A| \geq |Q| - \left(\frac{k^3}{2} + k^2\right) = s|J| - \left(\frac{k^3}{2} + k^2\right) \geq s(n/s - 10^{k^2} - k) - \left(\frac{k^3}{2} + k^2\right) > n - 10^{k^2} \cdot k,$$

which gives $n < 2k \cdot 10^{k^2} < 3^{100k^2}$, contradicting (3), and completes the proof of Theorem 1.8. $\square$

To conclude this section, we derive Corollary 1.9 from Theorem 1.8.

Proof of Corollary 1.9. By Corollary 1.6 and Theorem 1.8, every two longest cycles in a connected d-regular vertex-transitive graph on n vertices intersect in at least

$$\min_{d \geq 2} \max\left\{\Omega\left((\log_d n)^{1/3}\right), \Omega(d^{2/3})\right\} = \Omega\left((\ln n / \ln \ln n)^{1/3}\right)$$

vertices, where the minimum is achieved when $d = \Theta\left((\ln n / \ln \ln n)^{1/2}\right)$. $\square$

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5 Concluding remarks

In this paper, we obtain two results on the intersection of longest cycles in graphs G, which are closely related to the problems of estimating the size of transversals and the parameter $c(G)$.

In the proof of Theorem 1.8, given a cycle X, we exploit the "block structure" of vertex-transitive graphs to construct a longer cycle by assembling certain subpaths of X and their images under automorphisms. It would be interesting to see whether this approach can be further improved and applied in other contexts.

Next we discuss a natural extension of longest cycles to the so-called *maximum R-subdivisions*, a concept studied in [15, 17], and we present a corresponding generalization of Theorem 1.5.

Definition 5.1. *Let R be a multigraph. A* **maximum R-subdivision** *in a graph G is a subdivision of R with the maximum number of vertices in G.*

Let P_2 be a single edge and C_1 a single vertex with a loop. Then a longest path (respectively, cycle) in a graph G is a maximum P_2-subdivision (respectively, C_1-subdivision) in G.

The following result extends Theorem 1.5 to maximum R-subdivisions for any multigraph R. Notably, the analogous statement of Theorem 1.5 also holds for longest paths.

Theorem 5.2. *For any multigraph R, if two maximum R-subdivisions in a graph G share m vertices, then there exists a vertex cut of size $O_R(m^{\frac{3}{2}})$ separating them.*

Proof sketch. Let X and Y be two maximum R-subdivisions in G, and define $M = V(X) \cap V(Y)$ and $m = |M|$. For each $e \in E(R)$, let $P_e(X)$ denote the subdivided path or cycle in X corresponding to e – specifically, a path if $e = P_2$, or a cycle if $e = C_1$.

By the pigeonhole principle, among the $e(R)^2$ edge pairs $(e, e') \in E(R)^2$, any collection $\mathcal{P}$ of k disjoint $(X - M, Y - M)$-paths must include at least $k/e(R)^2$ paths whose endpoints lie in $P_e(X)$ and $P_{e'}(Y)$, for some pair $(e, e') \in E(R)^2$. Decompose $P_e(X) - V(Y)$ (resp. $P_{e'}(Y) - V(X)$) into segments $X_1, \ldots, X_m$ (resp. $Y_1, \ldots, Y_m$)[5], and construct the auxiliary bipartite graph $F := F(P_e(X), P_{e'}(Y), \mathcal{P})$ as in Definition 2.3. Here, each vertex of F represents a segment X_i or Y_j, and F contains at least $k/e(R)^2$ edges, each representing an $(X - M, Y - M)$-path whose endpoints lie in the corresponding segments.

We assert that $e(F) = O_R(m^{3/2})$. The proof largely follows Section 3 (which covers the case $R = C_1$). Here, we focus on $e = e' = P_2$, highlighting only the differences and providing detailed explanations as needed; the other cases where e or e' is C_1 can be handled similarly.

In this case, $P_e(X)$ and $P_{e'}(Y)$ are two paths in G, and the definition of a winning certificate should be adapted to a pair (Q_1, Q_2) of paths satisfying the following conditions:

- Q_1 has the same endpoints as $P_{e'}(Y)$, and Q_2 has the same endpoints as $P_e(X)$.

- $V(Q_1) \cap V(Y) \subseteq V(P_{e'}(Y))$, and $V(Q_2) \cap V(X) \subseteq V(P_e(X))$.

- $E(Q_1) \cup E(Q_2) \supseteq E(P_e(X)) \cup E(P_{e'}(Y))$, and $|Q_1| + |Q_2| > |P_e(X)| + |P_{e'}(Y)|$.

Given such a winning certificate, replacing $P_e(X)$ by Q_2 in X and $P_{e'}(Y)$ by Q_1 in Y produces two R-subdivisions X' and Y' such that $e(X') + e(Y') > e(X) + e(Y)$, contradicting the maximality of X and Y. The red and blue paths in Figure 2 and Figure 3 still serve as examples of such winning certificates. The main difference, compared to the proof in Section 3, arises in the proof of Lemma 3.4, where we can no longer assume that the Y-segments are arranged as in Figure 3 and there would be 32 cases rather than 4 cases to be verified. To see this, in addition to the configuration in Figure 3, the segments on Y may also be arranged as $(Y_{k_1}, Y_{\ell_1}, Y_{\ell_2}, Y_{k_2})$. For each 4-cycle, there are two possible relative orderings of the two vertex pairs on the X-segments (and similarly for the Y-segments). This gives four distinct relative orderings per 4-cycle. In total, there are $2 \cdot 4^2 = 32$ possible configurations to consider. We omit the detailed verification that, using constructions almost identical to Lemma 3.4, each case indeed produces a winning certificate.

It follows that $k/e(R)^2 \le e(F) = O(m^{3/2})$, which implies $k = O_R(m^{3/2})$. The conclusion then follows by Menger's Theorem. $\square$

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[5]If e or e' is P_1, the number of segments becomes $m + 1$.

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