Monochromatic subgraphs in iterated triangulations

Jie Ma* and Tianyun Tang†
School of Mathematical Sciences
University of Science and Technology of China
Hefei, Anhui 230026, China

Xingxing Yu‡
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332, USA

Abstract

For integers $n \geq 0$, an iterated triangulation $\mathrm{Tr}(n)$ is defined recursively as follows: $\mathrm{Tr}(0)$ is the plane triangulation on three vertices and, for $n \geq 1$, $\mathrm{Tr}(n)$ is the plane triangulation obtained from the plane triangulation $\mathrm{Tr}(n-1)$ by, for each inner face F of $\mathrm{Tr}(n-1)$, adding inside F a new vertex and three edges joining this new vertex to the three vertices incident with F.

In this paper, we show that there exists a 2-edge-coloring of $\mathrm{Tr}(n)$ such that $\mathrm{Tr}(n)$ contains no monochromatic copy of the cycle C_k for any $k \geq 5$. As a consequence, the answer to one of two questions asked in [4] is negative. We also determine the radius two graphs H for which there exists n such that every 2-edge-coloring of $\mathrm{Tr}(n)$ contains a monochromatic copy of H, extending a result in [4] for radius two trees.

1 Introduction

For graphs G and H, we write $G \to H$ if, for any 2-edge-coloring of G, there is a monochromatic copy of H. Otherwise, we write $G \not\to H$. We say that H is *planar unavoidable* if there exists a planar graph G such that $G \to H$. This notion is introduced and studied in [4].

Deciding if $G \to H$ is clearly equivalent to asking whether a graph G admits a decomposition (i.e., an edge-decomposition) such that one of the two graphs in the decomposition contains the given graph H. The well-known Four Color Theorem [2, 3] (also see [9]) implies that every planar graph admits a decomposition to two bipartite graphs; so planar unavoidable graphs must be bipartite. A result of Goncalves [5] says that every planar

*Email: jiema@ustc.edu.cn. Partially supported by NSFC grant 11622110.
†Email: tty123@mail.ustc.edu.cn.
‡Email: yu@math.gatech.edu. Partially supported by NSF grant DMS-1600738.
AMS Subject Classification: 05C55, 05C10, 05D10
Keywords: Triangulation, Planar unavoidable, Coloring, Monochromatic subgraph

graph admits a decomposition to two outer planar graph (although we have not seen a detailed proof); so planar unavoidable graphs must be also outer planar. There are a number of interesting results about decomposing planar graphs, see [1, 6–8, 10].

For any positive integer n, let P_n denote the path on n vertices, and K_n denote the complete graph on n vertices. For integer $n \geq 3$, we use C_n to denote the cycle on n vertices. It is shown in [4] that P_n, C_4, and all trees with radius at most 2 are planar unavoidable. This is done by analyzing several sequences of graphs.

In this paper, we investigate one such sequence – the iterated triangulations, which is of particular interest as suggested in [4]. Let $n \geq 0$ be an integer. An *iterated triangulation* $\mathrm{Tr}(n)$ is a plane graph defined as follows: $\mathrm{Tr}(0) \cong K_3$ is the plane triangulation with exactly two faces. For each $i \geq 0$, let $\mathrm{Tr}(i+1)$ be obtained from the plane triangulation $\mathrm{Tr}(i)$ by adding a new vertex in each of the inner faces of $\mathrm{Tr}(i)$ and connecting this vertex with edges to the three vertices in the boundary of their respective face. The authors of [4] asked whether for any planar unavoidable graph H there is an integer n such that $\mathrm{Tr}(n) \to H$. They also asked whether there exists an integer $k \geq 3$ such that the even cycle C_{2k} is planar-unavoidable.

Our first result indicates that a positive answer to one of the above questions implies a negative answer to the other. Let H^+ be the bipartite graph obtained by adding an edge to the unique 6-vertex tree with 4 leaves and 2 vertices of degree three.

Theorem 1.1. *For all positive integer n, $\mathrm{Tr}(n) \not\to C_k$ for $k \geq 5$, $\mathrm{Tr}(n) \not\to H^+$, and $\mathrm{Tr}(n) \not\to K_{2,3}$*

As another direct consequence, we see that if B is a bipartite graph and $\mathrm{Tr}(n) \to B$ for some n then every block of B must be a C_4 or K_2. This can be used to characterize all radius two graphs B for which there exists n such that $\mathrm{Tr}(n) \to B$, generalizing a result in [4] for radius two trees. To state this characterization, we need additional notations. A *flower* F_k is a collection of k copies of C_4s sharing a common vertex, which is called the *center*. A *jellyfish* J_k is obtained from F_k and a k-ary tree of radius two by identifying the center of F_k with the root of the k-ary tree. A *bistar* B_k is obtained from one C_4 and two disjoint $K_{1,k}$s by identifying the roots of the $K_{1,k}$s with two non-adjacent vertices of C_4, respectively.

Theorem 1.2. *Let L be a graph with radius two. Then there exists n such that $\mathrm{Tr}(n) \to L$ if, and only if, L is a subgraph of a jellyfish or bistar.*

We organize this paper as follows. In Section 2, we prove $\mathrm{Tr}(n) \not\to C_k$ for $k \geq 5$ and $\mathrm{Tr}(n) \not\to H^+$ by finding a special edge coloring scheme for $\mathrm{Tr}(n)$. In Section 3, we complete the proof of Theorem 1.1 by using another edge coloring scheme on $\mathrm{Tr}(n)$. From Theorem 1.1, we can derive the following: if L has radius 2 and $\mathrm{Tr}(n) \to L$ for some n, then L is a subgraph of a jellyfish or bistar. Hence to prove Theorem 1.2, it suffices to show that for any $k \geq 1$ there exists some n such that $\mathrm{Tr}(n) \to J_k$ and $\mathrm{Tr}(n) \to B_k$. We prove the former statement in Section 4 and the latter one in Section 5 by showing that we can choose n to be linear in k.

2 H^+ and C_k for $k \geq 5$

In this section, we prove Theorem 1.1 for H^+ and C_k, with $k \geq 5$. First, we describe the 2-edge-coloring of $\text{Tr}(n)$ that we will use. Let $\sigma : E(\text{Tr}(n)) \to \{0, 1\}$ be defined inductively for all $n \geq 1$ as follows:

(i) Fix an arbitrary triangle T bounding an inner face of $\text{Tr}(1)$, and let $\sigma(e) = 0$ if $e \in E(T)$ and $\sigma(e) = 1$ if $e \in E(\text{Tr}(1)) \setminus E(T)$.

(ii) Suppose for some $1 \leq i < n$, we have defined $\sigma(e)$ for all $e \in E(\text{Tr}(i))$. We extend σ to $E(\text{Tr}(i+1))$ as following. Let $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ be arbitrary, let $v_0v_1v_2v_0$ denote the triangle bounding the inner face of $\text{Tr}(i-1)$ containing x, and fix a labeling so that $\sigma(xv_1) = \sigma(xv_2)$.

(iii) Let $x_j \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ be such that x_j is inside the face of $\text{Tr}(i)$ bounded by the triangle $xv_jv_{j+1}x$, where $j = 0, 1, 2$ and the subscripts are taken modulo 3. Define $\sigma(xv_0) = \sigma(x_0v_0) = \sigma(x_2v_0) = \sigma(x_jx)$ for all $j = 0, 1, 2$, and $\sigma(xv_1) = \sigma(x_0v_1) = \sigma(x_1v_1) = \sigma(x_1v_2) = \sigma(x_2v_2)$.

We now proceed by a sequence of claims to show that σ has no monochromatic C_k for $k \geq 5$ nor monochromatic H^+, thereby proving $\text{Tr}(n) \not\to C_k$ for $k \geq 5$ and $\text{Tr}(n) \not\to H^+$.

(1) For $1 \leq i \leq n$ and $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$, $|\{\sigma(xv) : v \in V(\text{Tr}(i-1))\}| = 2$.

We apply induction on i. The basis case $i = 1$ follows from (i) above. So assume $2 \leq i \leq n$. Let $v_0v_1v_2v_0$ be the triangle bounding the inner face of $\text{Tr}(i-1)$ containing x. Without loss of generality assume that $v_0 \in V(\text{Tr}(i-1)) \setminus V(\text{Tr}(i-2))$. Let $v_1v_2v_3v_1$ denote the triangle bounding the face of $\text{Tr}(i-2)$, with v_0 inside $v_1v_2v_3v_1$. By induction hypothesis, $|\{\sigma(v_0v_k) : k = 1, 2, 3\}| = 2$.

Suppose $\sigma(v_0v_1) = \sigma(v_0v_2)$. Then by (ii) and (iii), $\sigma(xv_0) = \sigma(v_0v_3)$ and $\sigma(xv_1) = \sigma(xv_2) = \sigma(v_0v_1)$. So $|\{\sigma(xv_k) : k = 0, 1, 2\}| = 2$.

So assume $\sigma(v_0v_1) \neq \sigma(v_0v_2)$. By symmetry, we further assume $\sigma(v_0v_2) = \sigma(v_0v_3)$. Then by (ii) and (iii), we see that $\sigma(xv_1) = \sigma(xv_0) = \sigma(v_0v_1)$ and $\sigma(xv_2) = \sigma(v_0v_2)$. So $\sigma(xv_2) \neq \sigma(xv_1)$ and hence, $|\{\sigma(xv_k) : k = 0, 1, 2\}| = 2$. □

(2) Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\text{Tr}(i)$, where $0 \leq i < n$, let $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$ with v inside $v_0v_1v_2v_0$. Then, for any $v_0w \in E(\text{Tr}(n))$ with w inside $v_0v_1v_2v_0$, $\sigma(v_0w) = \sigma(v_0v)$.

Let $v_0w \in E(\text{Tr}(n))$ with w inside $v_0v_1v_2v_0$. Then there exists $k \geq 0$ with $i + k + 1 \leq n$, such that $w \in V(\text{Tr}(i+k+1)) \setminus V(\text{Tr}(i+k))$. We prove (2) by applying induction on k. The basis case is trivial because $k = 0$ implies $w = v$.

So assume $k \geq 1$. Let $v_0v_3v_4v_0$ be the triangle bounding an inner face of $\text{Tr}(i+k-1)$ with w inside $v_0v_3v_4v_0$, and let $v_5 \in V(\text{Tr}(i+k)) \setminus V(\text{Tr}(i+k-1))$ that is inside $v_0v_3v_4v_0$. By symmetry, assume w is inside $v_0v_5v_4v_0$. By induction hypothesis, $\sigma(v_0v_5)$=$\sigma(v_0v)$.

Suppose $\sigma(v_4v_5) = \sigma(v_0v_5)$. Hence by (ii) and (iii), $\sigma(v_0w) = \sigma(wv_4) = \sigma(v_0v_5)$. Thus $\sigma(v_0w) = \sigma(v_0v)$. Now assume $\sigma(v_4v_5) \neq \sigma(v_0v_5)$. Then $\sigma(v_3v_5) = \sigma(v_0v_5)$ or $\sigma(v_3v_5)$=$\sigma(v_4v_5)$. It follows from (iii) that $\sigma(v_0w) = \sigma(v_0v_5)$. Hence, $\sigma(v_0w) = \sigma(v_0v)$. □

(3) Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\mathrm{Tr}(i)$ with $0 \le i \le n-2$, and let $v \in V(\mathrm{Tr}(i{+}1))\backslash V(\mathrm{Tr}(i))$ such that v is inside $v_0v_1v_2v_0$ and $\sigma(vv_0) \ne \sigma(vv_1) = \sigma(vv_2)$. Then for any $vw \in E(\mathrm{Tr}(n))$ with w inside $v_0v_1v_2v_0$, $\sigma(vw) = \sigma(vv_0)$.

To prove (3), let $\{w_0, w_1, w_2\} \subseteq V(\mathrm{Tr}(i{+}2))\backslash V(\mathrm{Tr}(i{+}1))$ such that w_j is inside $vv_jv_{j+1}v$ for $j = 0, 1, 2$, with subscripts modulo 3. By (ii) and (iii), $\sigma(vw_0) = \sigma(vw_2) = \sigma(vw_1) = \sigma(vv_0)$. By (2), there exists some $j \in \{0, 1, 2\}$ with $\sigma(vw) = \sigma(vw_j)$. Hence, $\sigma(vw) = \sigma(vv_0)$. $\square$

(4) Let $v_0v_1v_2v_0$ be a triangle bounding an inner face of $\mathrm{Tr}(i)$, where $0 \le i \le n-2$, and let $v \in V(\mathrm{Tr}(i{+}1))\backslash V(\mathrm{Tr}(i))$ such that v is inside $v_0v_1v_2v_0$ and $\sigma(vv_0) \in \{\sigma(vv_1), \sigma(vv_2)\}$. Then for any $w \in (N(v) \cap N(v_0)) \setminus \{v_1, v_2\}$, $\sigma(wv_0) \ne \sigma(wv)$.

To prove (4), we may assume by symmetry and (1) that $\sigma(vv_2) \ne \sigma(vv_0) = \sigma(vv_1)$. Then $\sigma(wv_0) = \sigma(vv_0)$ by (2), and $\sigma(wv) = \sigma(vv_2)$ by (3). Hence, $\sigma(wv_0) \ne \sigma(wv)$. $\square$

(5) Suppose upv is a monochromatic path of length two in $\mathrm{Tr}(n)$ with $uv \in E(\mathrm{Tr}(i+1))$ and $p \in V(\mathrm{Tr}(n)) \setminus V(\mathrm{Tr}(i+1))$. Then any monochromatic path in $\mathrm{Tr}(n)$ between u and v and of the color $\sigma(up)$ has length at most two.

Consider any monochromatic path $P = a_0a_1...a_r$ of the color $\sigma(up)$ with $a_0 = v$ and $a_r = u$. First, suppose $uv \in E(\mathrm{Tr}(0))$. Let $\mathrm{Tr}(0) = uvwu$ and $x \in V(\mathrm{Tr}(1)) \setminus V(\mathrm{Tr}(0))$. By (2), $\sigma(ux) = \sigma(up)$ and $\sigma(vx) = \sigma(vp)$; so $\sigma(xu) = \sigma(xv)$. Thus, by (i), $\sigma(wx) = \sigma(wu) = \sigma(wv) \ne \sigma(xu)$. Let $v_0v_1 \dots v_n$ be a path in $\mathrm{Tr}(n)$ with $v_0 = w$, $v_1 = x$ and for $1 \le i \le n$, $v_i \in V(\mathrm{Tr}(i)) \setminus V(\mathrm{Tr}(i-1))$ is inside $v_{i-1}uvv_{i-1}$. By (ii) and (iii), $\sigma(v_iu) = \sigma(v_iv) = \sigma(vx)$ for $1 \le i \le n$, and $\sigma(v_iv_{i+1}) = \sigma(xw)$ for $0 \le i \le n-1$. By planarity, P is contained in the closed region bounded by $uvwu$. So either $P = uv$ or there exists some $1 \le k \le r-1$ such that $a_k \in \{v_0, ..., v_n\}$. We may assume the latter case occurs. If $\{a_{k-1}, a_{k+1}\} = \{u, v\}$, then $r = 2$. Hence without loss of generality, let $a_{k-1} \notin \{u, v\}$. Then by (2) and (3), $\sigma(a_{k-1}a_k) = \sigma(v_iv_{i+1}) \ne \sigma(pu)$ for $i \in \{0, 1, ...n-1\}$, a contradiction. Hence $r \le 2$.[1]

Thus, we may assume $uv \notin E(\mathrm{Tr}(0))$. By symmetry, we may assume that $v \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$ for some $0 \le i < n$ and v is inside the triangle $u_1u_2u_3u_1$ bounding an inner face of $\mathrm{Tr}(i)$ and $u_1 = u$. By (4), $\sigma(u_1v) \ne \sigma(u_2v) = \sigma(u_3v)$.

If a_1 is inside vu_2u_3v then there exists $1 \le k < r$ such that a_k is inside vu_3u_2v and $a_{k+1} \in \{u_2, u_3\}$; so by (2), $\sigma(a_ka_{k+1}) = \sigma(vu_2) = \sigma(vu_3) \ne \sigma(u_1v) = \sigma(pu)$, a contradiction.

Therefore, suppose that $P \ne uv$, by symmetry, we may assume that a_1 is inside $u_1vu_2u_1$. Let $v_0 = u_2$ and let $v_1v_2 \dots v_{n-i-1}$ be the path in $\mathrm{Tr}(n)$ such that, for $1 \le \ell \le n-i-1$, $v_\ell \in V(\mathrm{Tr}(i+\ell+1)) \setminus V(\mathrm{Tr}(i+\ell))$ is inside $u_1v_{\ell-1}vu_1$.

By (ii) and (iii), $\sigma(v_\ell u_1) = \sigma(v_\ell v) = \sigma(u_1v)$ for $1 \le \ell \le n-i-1$, and $\sigma(v_\ell v_{\ell+1}) = \sigma(vu_2) \ne \sigma(vu_1)$ for $0 \le \ell \le n-i-2$. If a_1 is inside $v_\ell v_{\ell+1}vv_\ell$ for some ℓ with $0 \le \ell \le n-i-2$, then exists $1 \le k \le r$ such that a_k is inside $v_\ell v_{\ell+1}vv_\ell$ and $a_{k+1} \in \{v_\ell, v_{\ell+1}\}$; so by (3) $\sigma(a_ka_{k+1}) = \sigma(v_\ell v_{\ell+1})$, a contradiction. So $a_1 = v_\ell$ for some ℓ with $1 \le \ell \le n-i-1$. Then as $\sigma(a_1a_2) = \sigma(u_1v)$ and by (3), we have $a_2 = u_1$. Therefore, $r = 2$, proving (5). $\square$

(6) If C_k is monochromatic in $\mathrm{Tr}(n)$ then $k \le 4$.

[1] We remark that this paragraph also shows that such uv in $E(\mathrm{Tr}(0))$ cannot be in a monochromatic C_4.

Let $C_k = a_1a_2\dots a_ka_1$ be a monochromatic cycle in $\mathrm{Tr}(n)$. By (i), $E(C_k) \not\subseteq E(\mathrm{Tr}(0))$. So we may assume that there exists some $1 \le i \le k$ such that $a_{i+1} \in V(\mathrm{Tr}(\ell+1)) \setminus V(\mathrm{Tr}(\ell))$ is inside the triangle a_iuva_i which bounds an inner face of some $\mathrm{Tr}(\ell)$. We may further assume that $\ell \le n-2$, as otherwise, we could consider $\mathrm{Tr}(n+1)$ instead of $\mathrm{Tr}(n)$.[2]

Suppose $\sigma(a_ia_{i+1}) \in \{\sigma(a_{i+1}u), \sigma(a_{i+1}v)\}$. By symmetry, we may assume $\sigma(a_ia_{i+1}) = \sigma(a_{i+1}u)$. Then $a_{i+2} = u$ by (3). Hence, by (5), any monochromatic path in C_k between a_i and $a_{i+2} = u$ has length at most 2. So $k \le 4$.

Thus, we may assume $\sigma(a_ia_{i+1}) \notin \{\sigma(a_{i+1}u), \sigma(a_{i+1}v)\}$; hence, $\sigma(a_{i+1}u) = \sigma(a_{i+1}v)$. Let $w \in V(\mathrm{Tr}(\ell+2)) \setminus V(\mathrm{Tr}(\ell+1))$ be inside the triangle $a_iua_{i+1}a_i$. By (ii) and (iii), $\sigma(wa_i) = \sigma(wa_{i+1}) = \sigma(a_ia_{i+1})$. Hence, by (5), the monochromatic path $C_k - a_ia_{i+1}$ in $\mathrm{Tr}(n)$ of the color $\sigma(a_ia_{i+1}) = \sigma(wa_i)$ has length at most 2; so $k = 3$. □

(7) There is no monochromatic H^+ in $\mathrm{Tr}(n)$.

Suppose that there is a monochromatic copy of H^+ on $\{v_i : 1 \le i \le 6\}$ in which $v_1v_2v_3v_4v_1$ is a 4-cycle and v_1v_5, v_2v_6 are edges. If $v_1v_2 \in E(\mathrm{Tr}(0))$, then v_1v_2 satisfies the conditions of (5) and by the footnote from the proof of (5), there is no monochromatic C_4 containing v_1v_2, a contradiction. So $v_1v_2 \notin E(\mathrm{Tr}(0))$. By symmetry, we may assume that $v_2 \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$ for some i and that v_1uwv_1 is the triangle bounding the inner face of $\mathrm{Tr}(i)$ containing v_2. Again as before we may assume that $0 \le i \le n-2$.

If $\sigma(v_2u) = \sigma(v_2w)$, then there exists some $p \in V(\mathrm{Tr}(n)) \setminus V(\mathrm{Tr}(i+1))$ such that v_1pv_2 has the same color as $\sigma(v_1v_2)$. But $v_1v_4v_3v_2$ is a monochromatic path of length 3 in $\mathrm{Tr}(n)$ between v_1 and v_2 and of the color $\sigma(v_1v_2)$, a contradiction to (5).

Hence, $\sigma(v_1v_2) \in \{\sigma(v_2u), \sigma(v_2w)\}$ and by symmetry, we may assume $\sigma(v_1v_2) = \sigma(v_2u)$. Then by (1), $\sigma(v_1v_2) \ne \sigma(v_2w)$ and thus $\sigma(v_2v_3) = \sigma(v_2v_6) \ne \sigma(v_2w)$. This shows $w \notin \{v_3, v_6\}$. So there exists $y \in \{v_3, v_6\} \setminus \{u, w\}$. By (3), $\sigma(v_2y) = \sigma(v_2w)$, a contradiction. This completes the proof of this section.

3 Monochromatic $K_{2,3}$

In this section, we prove Theorem 1.1 for $K_{2,3}$ using a different coloring scheme on $\mathrm{Tr}(n)$ described below. Let $\sigma : E(\mathrm{Tr}(n)) \to \{0, 1\}$ be defined inductively as follows:

(i) Fix a triangle T bounding an inner face of $\mathrm{Tr}(1)$, and let $\sigma(e) = 0$ if $e \in E(T)$ and $\sigma(e) = 1$ if $e \in E(\mathrm{Tr}(1)) \setminus E(T)$.

(ii) Suppose for some $1 \le i < n$, we have defined $\sigma(e)$ for all $e \in E(\mathrm{Tr}(i))$. We now extend σ to $E(\mathrm{Tr}(i+1))$. Let $x \in V(\mathrm{Tr}(i)) \setminus V(\mathrm{Tr}(i-1))$ be arbitrary, let $v_0v_1v_2v_0$ denote the triangle bounding the inner face of $\mathrm{Tr}(i-1)$ containing x, with v_0, v_1, v_2 on the triangle in clockwise order, and let $\sigma(xv_1) = \sigma(xv_2)$.

(iii) Let $x_j \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$ such that x_j is inside the face of $\mathrm{Tr}(i)$ bounded by the triangle $xv_jv_{j+1}x$, where $j = 0, 1, 2$ and the subscripts are taken modulo 3. Define $\sigma(v_0x) = \sigma(v_0x_0) = \sigma(v_0x_2) = \sigma(xx_2) = \sigma(x_1v_1)$, and $\sigma(v_2x) = \sigma(v_2x_1) = \sigma(v_2x_2) = \sigma(xx_1) = \sigma(xx_0) = \sigma(x_0v_1)$.

[2] This is fair because $\mathrm{Tr}(n+1) \not\to C_k$ implies $\mathrm{Tr}(n) \not\to C_k$.

Note that in (ii) we have $|\{\sigma(xv_j) : j = 0, 1, 2\}| = 2$ and that in (iii) we have $\sigma(x_j v_j) \neq \sigma(x_j v_{j+1})$ for $j = 0, 1, 2$. Hence, inductively, we have

(1) For $1 \leq i \leq n$ and $x \in V(\mathrm{Tr}(i)) \setminus V(\mathrm{Tr}(i-1))$, $|\{\sigma(xv) : v \in V(\mathrm{Tr}(i-1))\}| = 2$.

(2) If $x_1x_2x_3x_1$ is a triangle which bounds an inner face of $\mathrm{Tr}(i)$ for some $1 \leq i \leq n-2$, and if $x \in V(\mathrm{Tr}(n)) \setminus V(\mathrm{Tr}(i+1))$ is inside $x_1x_2x_3x_1$ with $xx_1, xx_2 \in E(\mathrm{Tr}(n))$, then $\sigma(xx_1) \neq \sigma(xx_2)$.

These two claims are straightforward so we omit their proofs.

(3) For any $x_1x_2 \in E(\mathrm{Tr}(n))$, $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 2$ and $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 2$.

First, suppose $x_1x_2 \in E(\mathrm{Tr}(0))$. Then by (i) and (2), $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 1$ and $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 1$.

So we may assume that x_1vwx_1 bounds an inner face of $\mathrm{Tr}(i)$ and $x_2 \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$ inside x_1vwx_1. Let $v_1 \in \mathrm{Tr}(i+2)$ be inside $x_1vx_2x_1$ and $w_1 \in \mathrm{Tr}(i+2)$ be inside $x_1wx_2x_1$. By (iii), $\sigma(w_1x_1) \neq \sigma(w_1x_2)$ or $\sigma(v_1x_1) \neq \sigma(v_1x_2)$. By (2), for any $x \in V(\mathrm{Tr}(n)) \setminus V(\mathrm{Tr}(i+2))$ inside x_1vwx_1 with $xx_1, xx_2 \in E(\mathrm{Tr}(n))$, we have $\sigma(xx_1) \neq \sigma(xx_2)$. Hence, if (3) fails, then we may assume by symmetry between w_1 and v_1 that $\sigma(vx_1) = \sigma(vx_2) = \sigma(wx_1) = \sigma(wx_2) = \sigma(v_1x_1) = \sigma(v_1x_2)$, and $\sigma(w_1x_1) \neq \sigma(w_1x_2)$. Then, by (1), $\sigma(x_1x_2) \neq \sigma(x_2v) = \sigma(x_2w)$. Now by (iii), at least one of the two edges v_1x_1 and v_1x_2 has the same color as x_1x_2, a contradiction. This proves (3).

(4) If $x_1x_2x_3x_4x_1$ is a 4-cycle in $\mathrm{Tr}(n)$, then $x_1x_3 \in E(\mathrm{Tr}(n))$ or $x_2x_4 \in E(\mathrm{Tr}(n))$.

We may assume that $\{x_1, x_2, x_3, x_4\} \subseteq V(\mathrm{Tr}(i+1))$ and $x_j \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$ for some $0 \leq i < n$ and $j \in [4]$. Let $uvwu$ be the triangle bounding an inner face of $\mathrm{Tr}(i)$ such that x_j is inside it. Then $\{x_{j-1}, x_{j+1}\} \subseteq \{u, v, w\}$, implying that $x_{j-1}x_{j+1} \in E(\mathrm{Tr}(n))$. $\square$.

(5) There is no monochromatic $K_{2,3}$ in $\mathrm{Tr}(n)$.

For, suppose $\mathrm{Tr}(n)$ has a monochromatic copy of $K_{2,3}$ on $\{v_1, v_2, v_3, v_4, v_5\}$ with $v_4v_i, v_5v_i \in E(\mathrm{Tr}(n))$ for all $i = 1, 2, 3$. Then $v_4v_5 \notin E(\mathrm{Tr}(n))$ by (3) and, hence, it follows from (4) that $v_1v_2, v_2v_3, v_3v_1 \in E(\mathrm{Tr}(n))$. By planarity, $v_1v_2v_3v_1$ bounds an inner face of $\mathrm{Tr}(i)$ for some i with $1 \leq i < n$ and, by the symmetry between v_4 and v_5, we may assume that v_4 is inside $v_1v_2v_3v_1$. Then $v_4 \in V(\mathrm{Tr}(i+1)) \setminus V(\mathrm{Tr}(i))$. However, this contradicts (1), as $\sigma(v_4v_1) = \sigma(v_4v_2) = \sigma(v_4v_3)$. We have completed the proof of Theorem 1.1. ∎

4 Monochromatic J_k

In this section we prove that $\mathrm{Tr}(100k) \to J_k$ holds for any positive integer k.

We need the following result, which is Lemma 9 in [4]. The original statement in [4] states $\mathrm{Tr}(16) \to C_4$, but the same proof in [4] actually gives the following stronger version.

Lemma 4.1. *If $xyzx$ bounds the outer face of* $\mathrm{Tr}(16)$*, then any 2-edge-coloring of* $\mathrm{Tr}(16)$ *gives a monochromatic C_4 which intersects $\{x, y\}$.*

Note that if the triangle $xyzx$ bounds the outer face of $\mathrm{Tr}(n)$ and $v \in V(\mathrm{Tr}(1)) \backslash V(\mathrm{Tr}(0))$ then the subgraph of $\mathrm{Tr}(n)$ contained in the closed disc bounded by $vxyv$ is isomorphic to $\mathrm{Tr}(n-1)$. Hence, the following is an easy consequence of Lemma 4.1.

Corollary 4.2. *If $xyzx$ bounds the outer face of* $\mathrm{Tr}(17)$ *then any 2-edge-coloring of* $\mathrm{Tr}(17)$ *gives a monochromatic C_4 which intersects $\{x, y\}$ and avoids z.*

Lemma 4.3. *For any positive integer k,* $\mathrm{Tr}(38k) \rightarrow F_k$

Proof. Let $\sigma : E(\mathrm{Tr}(38k)) \rightarrow \{0, 1\}$ be an arbitrary 2-edge coloring. Let $uvwu$ be the triangle bounding the outer face of $\mathrm{Tr}(38k)$. Let $x_0 := w$ and, for $1 \leq \ell \leq 2k$, let $x_l \in V(\mathrm{Tr}(\ell)) \backslash V(\mathrm{Tr}(\ell-1))$ such that x_ℓ is inside $x_{\ell-1}uvx_{\ell-1}$. Let $y_{i,0} := x_i$ for $i \in \{0, 1, \ldots, 2k-1\}$ and, for $\ell \in \{1, \ldots, 36k\}$, let $y_{i,\ell} \in V(\mathrm{Tr}(i+1+\ell)) \setminus V(\mathrm{Tr}(i+\ell))$ such that $y_{i,\ell}$ is inside $y_{i,\ell-1}ux_{i+1}y_{i,\ell-1}$.

Suppose for each $0 \leq i \leq 2k-1$ there exists a monochromatic C_4 inside $x_iux_{i+1}x_i$ that contains u and avoids x_i. By pigeonhole principle, at least k of these C_4s are of the same color, which form a monochromatic F_k centered at u.

Hence, we may assume that there exists some $i \in \{0, 1, \ldots, 2k-1\}$ such that no monochromatic C_4 inside $x_iux_{i+1}x_i$ contains u and avoids x_i. Since $i \leq 2k-1$, $x_iux_{i+1}x_i$ bounds the outer face of a $\mathrm{Tr}(36k)$ that is contained in $\mathrm{Tr}(38k)$.

Now for each $h \in \{0, 1, \ldots, 2k-1\}$, we view the closed region bounded by $ux_{i+1}y_{i,18h}u$ as a $\mathrm{Tr}(17)$. Note that these copies of $\mathrm{Tr}(17)$ share u, x_{i+1} as the only common vertices. Taking $y_{i,18h}$ to be the vertex z in Corollary 4.2, we conclude from Corollary 4.2 that there is a monochromatic C_4 in the $\mathrm{Tr}(17)$ bounded by $ux_{i+1}y_{i,18h}u$ such that $x_{i+1} \in V(G_h)$ and $\{u, y_{i,18h}\} \cap V(G_h) = \emptyset$. By pigeonhole principle, at least k of these C_4s are of the same color, which clearly form a monochromatic F_k centered at x_{i+1}. ∎

Lemma 4.4. *Let k be a positive integer and let $uvwu$ bound the outer face of* $\mathrm{Tr}(9k+2)$*. Suppose $\sigma : E(\mathrm{Tr}(9k+2)) \rightarrow \{0, 1\}$ is a 2-edge-coloring such that $|\{\sigma(ux) : x \in V(\mathrm{Tr}(9k+2))\}| = 1$ and there is no monochromatic C_4 containing u. Then* $\mathrm{Tr}(9k+2)$ *contains monochromatic J_k centered at v.*

Proof. Without loss of generality, assume $\sigma(uv) = 0$. Then $\sigma(uy) = 0$ for all $y \in N(u)$. Let $x_0 := w$ and, for $1 \leq i \leq 8k+1$, let $x_i \in V(\mathrm{Tr}(i)) \setminus V(\mathrm{Tr}(i-1))$ such that x_i is inside $x_{i-1}uvx_{i-1}$. Since no monochromatic C_4 in $\mathrm{Tr}(9k+2)$ contains u, there is at most one $i \in \{0, 1, 2, ...8k+1\}$ such that $\sigma(x_iv) = 0$. Hence, there exists $i \in \{0, 1, \ldots 4k+2\}$ such that $\sigma(vx_j) = 1$ for $j \in \{i, i+1 \ldots i+4k-1\}$. We now make the following claim.

Claim. The subgraph of $\mathrm{Tr}(9k+2)$ contained in the closed disc bounded by $vx_i \ldots x_{i+3k-1}v$ has a monochromatic F_k of color 1 and centered at v, which we denote by F_v.

To show this, it suffices to show that for each r with $0 \leq r \leq k-1$, the subgraph T_r of $\mathrm{Tr}(9k+2))$ inside $vx_{i+3r}x_{i+3r+1}x_{i+3r+2}v$ (inclusive) contains a monochromatic C_4 of color 1 and containing v, as the union of such C_4 is an F_k centered at v. So fix an arbitrary r, with $0 \leq r \leq k-1$. Note that $\sigma(x_{i+3r}x_{i+3r+1}) = 1$ or $\sigma(x_{i+3r+1}x_{i+3r+2}) = 1$, for $0 \leq r \leq k-1$;

for, otherwise, $x_{i+3r}x_{i+3r+1}x_{i+3r+2}ux_{i+3r}$ is a monochromatic C_4 of color 0 and containing u, a contradiction. Without loss of generality, assume $\sigma(x_{i+3r}x_{i+3r+1}) = 1$.

Let $y \in V(\mathrm{Tr}(i+3r+2)) \setminus V(\mathrm{Tr}(i+3r+1))$ such that y is inside $x_{i+3r}x_{i+3r+1}vx_{i+3r}$. If there are two edges in $\{yx_{i+3r}, yx_{i+3r+1}, yv\}$ of color 0, then one can easily find a monochromatic C_4 of color 0 and containing u, a contradiction. Hence, at least two of $\{\sigma(yx_{i+3r}), \sigma(yx_{i+3r+1}), \sigma(yv)\}$ are 1. So $\{y, x_{i+3r}, x_{i+3r+1}, v\}$ induces a subgraph which contains a monochromatic C_4 of color 1. This proves the claim. $\square$

Note that for $i+3k \le r \le i+4k-1$, $ux_rx_{r-1}u$ bounds the outer face of a $\mathrm{Tr}(k+1)$. Let $z_{r,0} := x_{r-1}$ and, for $r \in \{i+3k, i+3k+1, \dots i+4k-1\}$ and $\ell \in \{1,2,\dots k\}$, let $z_{r,\ell} \in V(Tr(r+\ell)) \setminus V(Tr(r+\ell-1))$ such that $z_{r,\ell}$ is inside $z_{r,\ell-1}x_ruz_{r,\ell-1}$. Because $\sigma(uz_{r,j}) = 0$ (by assumption) and $\mathrm{Tr}(9k+2)$ has no monochromatic C_4 containing u, there is at most one $y \in \{z_{r,1}, z_{r,2}, \dots, z_{r,k}\}$ such that $\sigma(yx_r) = 0$. So there exists $k-1$ vertices in $\{z_{r,1}, ..., z_{r,k}\}$ which together with x_rv form a monochromatic $K_{1,k}$ of color 1 centered at x_r, which we denote by H_r. Now $H_{i+3k}, H_{i+3k+1}, \dots, H_{i+4k-1}$ form a monochromatic k-ary radius 2 tree rooted at v of color 1. This radius 2 tree and F_v form a monochromatic J_k of color 1, completing the proof of Lemma 4.4. ∎

Now we are ready to prove the main result of this section, that is $\mathrm{Tr}(100k) \to J_k$. Let $\sigma : E(\mathrm{Tr}(100k)) \to \{0,1\}$ be arbitrary. We show that σ always contains a monochromatic J_k. By Lemma 4.3, $\mathrm{Tr}(76k)$ contains monochromatic copy of F_{2k}, say F, and, without loss of generality, assume it is of color 1. Let the C_4s in F be $xa_{i,1}a_{i,2}a_{i,3}x$ for $i \in [2k]$. For $i \in [2k]$, let $b_i \in V(\mathrm{Tr}(76k+1)) \setminus V(\mathrm{Tr}(76k))$ such that b_i is inside $xa_{i,1}a_{i,2}a_{i,3}x$ and $a_{i,1}a_{i,2}b_ia_{i,1}$ bounds an inner face of $\mathrm{Tr}(76k+1)$. Let A_i be the family of all vertices $a \in N(a_{i,1})$ inside $a_{i,1}a_{i,2}b_ia_{i,1}$ and satisfying $\sigma(aa_{i,1}) = 1$.

(1) There exists some $i \in \{k+1, k+2, ...2k\}$ such that $|A_i| < k$.

Otherwise, suppose $|A_i| \ge k$ for $i \in \{k+1, k+2, ...2k\}$. Then let $Z_i := \{z_{i,1}, z_{i,2}, ...z_{i,k-1}\} \subseteq A_i$. Now, for each $i \in \{k+1, \dots, 2k\}$, $\{x, a_{i,1}\} \cup Z_i$ induces a graph containing a monochromatic $K_{1,k}$. Those $K_{1,k}$s form a monochromatic radius-two k-ary tree of color 1 and rooted at x, which we denote by T_x. Now $F_k \cup T_x$ is a monochromatic J_k. $\square$

Let $u := a_{i,1}$. By (1), there exists an edge $vw \in \mathrm{Tr}(78k)$ such that $uvwu$ bounds an inner face of $\mathrm{Tr}(78k)$ and $\sigma(uy) = 0$ for any $y \in N(u)$ in the closed disc bounded by $uvwu$.

Let G be the subgraph of $\mathrm{Tr}(n)$ contained in the closed disc bounded by $uvwu$. Clearly G is isomorphic to a copy of $\mathrm{Tr}(22k)$. In the rest of the proof, we should only discuss the graph G and all $\mathrm{Tr}(i)$ will be referred to this copy of $\mathrm{Tr}(22k)$. Let $x_0 := w$ and for $i \in [4k]$, let $x_i \in V(\mathrm{Tr}(i)) \setminus V(\mathrm{Tr}(i-1))$ such that x_i is inside $ux_{i-1}vu$.

(2) G contains a monochromatic copy of F_k, say F', which has color 0 and center u and is disjoint from the union of closed regions bounded by $ux_ix_{i+1}u$ over all $0 \le i \le 2k-1$.

If for each $i \in \{k, k+1, ..., 2k-1\}$ there exists a monochromatic C_4 inside $ux_{2i}x_{2i+1}u$ and containing u, then these k monochromatic C_4s of color 0 form a desired monochromatic F_k centered at u and thus (2) holds. Otherwise, since $ux_{2i}x_{2i+1}u$ bounds the outer face of a $\mathrm{Tr}(9k+2)$, it follows from Lemma 4.4 that there exists a monochromatic J_k in G. $\square$

For $j \in \{0, 1, ..., 2k-1\}$, let B_j be the family of all vertices $x \in N(x_j)$ inside $ux_jx_{j+1}u$ and satisfying $\sigma(xx_j) = 0$.

(3) There exists some $j \in \{0, 1, ..., k-1\}$ such that $|B_j| < k$.

Suppose on the contrary that there exist subsets $Z_j \subseteq B_j$ of size k for all $j \in \{0, 1, ..., k-1\}$. Then each $Z_j \cup \{u, x_j\}$ induces a graph containing a monochromatic $K_{1,k}$ which is centered at x_j and has color 0. These $K_{1,k}$s together with F' form a monochromatic J_k of color 0. This proves (3). $\square$

Let $p_0 := x_{j+1}$ and for $1 \leq \ell \leq 4k$, let $p_\ell \in V(\text{Tr}(j+\ell+1)) \setminus V(\text{Tr}(j+\ell))$ such that p_ℓ is inside $ux_jp_{\ell-1}u$. By (3), there exists some $0 \leq \ell \leq 4k-1$ such that $\sigma(zx_j) = 1$ for any $z \in N(x_j)$ in the closed disc bounded by $x_jp_\ell p_{\ell+1}x_j$.

(4) There is a monochromatic F_k inside $x_jp_\ell p_{\ell+1}x_j$, say F'', with color 1 and center x_j.

Let $z_0 := p_{\ell+1}$ and for $s \in [2k]$, let $z_s \in V(\text{Tr}(j+\ell+s+2)) \setminus V(\text{Tr}(j+\ell+s+1))$ such that z_s is inside $x_jz_{s-1}p_\ell x_j$. Note that each $x_jz_{2s}z_{2s+1}x_j$ bounds a $\text{Tr}(9k+3)$. If for each $s \in [k]$ there exists a monochromatic C_4 of color 1 inside $x_jz_{2s}z_{2s+1}x_j$ and containing x_j, then these monochromatic copies of C_4 form the desired monochromatic F_k centered at x_j. Otherwise, it follows from Lemma 4.4 that there exists a monochromatic J_k. $\square$

As $|B_j| < k$, there exists a subset $A \subseteq \{p_1, p_2, ..., p_{4k}\}$ of size $2k$ such that $\sigma(\alpha x_j) = 1$ for each $\alpha \in A$ and moreover, there is no neighbors of A belonging to $V(F'')$. Let $A := \{\alpha_1, ..., \alpha_{2k}\}$. Note that for each $h \in [2k]$, we have $\sigma(\alpha_h u) = 0$ and $\sigma(\alpha_h x_j) = 1$.

It is easy to see that there exist pairwise disjoint sets $N_h \subseteq N(\alpha_h)$ of size $2k$ for $h \in [2k]$. Then there exists $M_h \subseteq N_h$ such that $|M_h| = k-1$ and $\sigma(x\alpha_h)$ is the same for all $x \in M_h$. This gives $2k$ monochromatic copies of $K_{1,k-1}$ with centers α_h for $h \in [2k]$. At least k of them (say with centers α_h for $h \in [k]$) have the same color. If this color is 0, these copies together with $\{u\alpha_h : h \in [k]\}$ and F' give a monochromatic J_k with color 0 and center u. Otherwise, this color is 1. Then these copies together with $\{x_j\alpha_h : h \in [k]\}$ and F'' give a monochromatic J_k with color 1 and center u. This proves $\text{Tr}(100k) \to J_k$. ∎

5 Monochromatic bistar

In this section we prove $\text{Tr}(6k+30) \to B_k$. We first establish the following lemma.

Lemma 5.1. *Let $uvwu$ be the triangle bounding the outer face of* $\text{Tr}(k+10)$*. Let* $\sigma : E(\text{Tr}(k+10)) \to \{0,1\}$ *such that* $|\{\sigma(ux) : x \in V(\text{Tr}(k+10))\}| = 1$ *and there is no monochromatic C_4 containing u. Then* $\text{Tr}(k+10))$ *contains a monochromatic B_k.*

Proof. Without loss of generality, let $\sigma(uv) = 0$. Let $x_0 := w$ and, for $i \in [6]$, let $x_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$ such that x_i is inside $uvx_{i-1}u$.

Since $\text{Tr}(k+10)$ has no monochromatic C_4 containing u, we see that $|\{0 \leq i \leq 6 : \sigma(vx_i) = 0\}| \leq 1$. So there exists some $i \in \{0, 1, 2, 3, 4\}$ such that $\sigma(vx_i) = \sigma(vx_{i+1}) = \sigma(vx_{i+2}) = 1$. We have either $\sigma(x_ix_{i+1}) = 1$ or $\sigma(x_{i+1}x_{i+2}) = 1$; as otherwise $ux_ix_{i+1}x_{i+2}u$ is a monochromatic C_4 of color 0 and containing u, a contradiction. We consider two cases.

Case 1. $\sigma(x_ix_{i+1}) = \sigma(x_{i+1}x_{i+2})$.

In this case, we have $\sigma(x_ix_{i+1}) = \sigma(x_{i+1}x_{i+2}) = 1$. So $x_ix_{i+1}x_{i+2}vx_i$ is a monochromatic C_4 of color 1. Let $y_0 := x_{i+1}$ and for $\ell \in [k+1]$, let $y_\ell \in V(\mathrm{Tr}(i+1+\ell)) \setminus V(\mathrm{Tr}(i+\ell))$ such that y_ℓ is inside $uy_{\ell-1}x_iu$. Similarly let $z_0 := x_{i+1}$ and for $\ell \in [k+1]$, let $z_\ell \in V(\mathrm{Tr}(i+2+\ell)) \setminus V(\mathrm{Tr}(i+1+\ell))$ such that z_ℓ is inside $uz_{\ell-1}x_{i+2}u$.

Since $\mathrm{Tr}(k+10)$ has no monochromatic C_4 containing u, this shows that $|\{\ell \in [k+1] : \sigma(x_iy_\ell) = 0\}| \leq 1$ and $|\{\ell \in [k+1] : \sigma(x_{i+2}z_\ell) = 0\}| \leq 1$. Therefore, there exist $Y \subseteq \{y_\ell : \ell \in [k+1]\}$ and $Z \subseteq \{z_\ell : \ell \in [k+1]\}$ such that $|Y| = |Z| = k$, $\sigma(yx_i) = 1$ for each $y \in Y$ and $\sigma(zx_{i+2}) = 1$ for each $z \in Z$. Hence, $\mathrm{Tr}(k+10)$ has two monochromatic $K_{1,k}$s of color 1 with centers x_i, x_{i+1} and leave sets Y, Z, respectively. These two $K_{1,k}$s together with $vx_ix_{i+1}x_{i+2}v$ form a monochromatic B_k of color 1.

Case 2. $\sigma(x_ix_{i+1}) \neq \sigma(x_{i+1}x_{i+2})$.

Without loss of generality, let $\sigma(x_ix_{i+1}) = 0$ and $\sigma(x_{i+1}x_{i+2}) = 1$. Let $y \in V(\mathrm{Tr}(i+2)) \setminus V(\mathrm{Tr}(i+1))$ be inside $ux_ix_{i+1}u$. Because $\sigma(uy) = 0$ and $\mathrm{Tr}(k+10)$ has no monochromatic C_4 containing u, $\sigma(yx_i) = \sigma(yx_{i+1}) = 1$. Therefore, $yx_{i+1}vx_iy$ is a monochromatic C_4 of color 1. Let $y_0 := y$ and, for $\ell \in [k+1]$, let $y_\ell \in V(\mathrm{Tr}(i+2+\ell)) \setminus V(\mathrm{Tr}(i+1+\ell))$ such that y_ℓ is inside $uy_{\ell-1}x_iu$. Let $z_0 := y$ and, for $\ell \in [k+1]$, let $z_\ell \in V(\mathrm{Tr}(i+2+\ell)) \setminus V(\mathrm{Tr}(i+1+\ell))$ such that z_ℓ is inside $uz_{\ell-1}x_{i+1}u$.

The remaining proof is similar as in Case 1. We observe that $|\{\ell \in [k+1] : \sigma(x_iy_\ell) = 0\}| \leq 1$ and $|\{\ell \in [k+1] : \sigma(x_{i+1}z_\ell) = 0\}| \leq 1$. Therefore, there exist $Y \subseteq \{y_\ell : \ell \in [k+1]\}$ and $Z \subseteq \{z_\ell : \ell \in [k+1]\}$ such that $|Y| = |Z| = k$, $\sigma(yx_i) = 1$ for $y \in Y$, and $\sigma(zx_{i+1}) = 1$ for $z \in Z$. Hence, $\mathrm{Tr}(k+10)$ has two monochromatic $K_{1,k}$s of color 1 with centers x_i, x_{i+1} and leave sets Y, Z, respectively. These two $K_{1,k}$s together with $yx_{i+1}vx_iy$ form a monochromatic B_k of color 1. This proves Lemma 5.1. ∎

We are ready to prove $\mathrm{Tr}(6k+30) \to B_k$. Let $\sigma : E(\mathrm{Tr}(6k+30) \to \{0,1\}$. By Lemma 4.1, the copy of $\mathrm{Tr}(16)$ with the same outer face as of $\mathrm{Tr}(6k+30)$ contains a monochromatic C_4, say $u_1u_2u_3u_4u_1$ of color 1. For each $i \in \{1,3\}$, let v_iw_i be an edge in $\mathrm{Tr}(18)$ such that $u_iv_iw_iu_i$ is a triangle inside $u_1u_2u_3u_4u_1$. Note that $u_iv_iw_iu_i$ bounds the outer face of a $\mathrm{Tr}(6k+12)$. Let A_i be the family of all vertices $x \in N(u_i)$ inside $u_iv_iw_iu_i$ and satisfying $\sigma(xu_i) = 1$. If $|A_1| \geq k$ and $|A_3| \geq k$, then together with the monochromatic 4-cycle $u_1u_2u_3u_4u_1$, it is easy to form a monochromatic B_k of color 1.

Hence by symmetry, we may assume that $|A_1| < k$. Then there exists an edge vw in $\mathrm{Tr}(18+k)$ such that u_1vwu_1 bounds an inner face of $\mathrm{Tr}(18+k)$ and $\sigma(u_1x) = 0$ for all $x \in N(u_1)$ in the closed disc bounded by u_1vwu_1. We may assume that the induced subgraph contained in the closed disc bounded by u_1vwu_1 has a monochromatic C_4 say u_1xyzu_1 (as otherwise, it contains a B_k by Lemma 5.1). Furthermore, we have $\{x, y, z\} \subseteq V(\mathrm{Tr}(2k+28))$.

Let $\{p_0, q_0\} \subseteq V(\mathrm{Tr}(2k+29)) \setminus V(\mathrm{Tr}(2k+28))$ such that both xyp_0x and yzq_0y bound two inner faces of $\mathrm{Tr}(2k+29)$. For $\ell \in [3k]$, let $p_\ell \in V(\mathrm{Tr}(2k+29+\ell)) \setminus V(\mathrm{Tr}(2k+28+\ell))$ such that p_ℓ is inside $xp_{\ell-1}yx$. Similarly, for $\ell \in [3k]$, let $q_\ell \in V(\mathrm{Tr}(2k+29+\ell)) \setminus V(\mathrm{Tr}(2k+28+\ell))$ such that q_ℓ is inside $yq_{\ell-1}zy$. Moreover, let

$$B_1 := \{p \in N(x) : p \text{ is inside } xp_0yx \text{ and } \sigma(xp) = 0\},$$

$$B_2 := \{q \in N(z) : q \text{ is inside } yq_0zy \text{ and } \sigma(zq) = 0\}.$$

If $|B_1| \geq k$ and $|B_2| \geq k$, we can find two monochromatic $K_{1,k}$s of color 0, one inside xp_0yx rooted at x and one inside yq_0zy rooted at z; these two $K_{1,k}$s and u_1xyzu_1 form a monochromatic B_k of color 0. So we may assume, without loss of generality, that $|B_1| < k$.

Let $C := \{\ell \in [3k] : \sigma(yp_\ell) = 0\}$. We claim $|C| < k$. Suppose on the contrary that $|C| \geq k$. Then there is a monochromatic $K_{1,k}$ with root y and leaves in C of color 0. Since $\sigma(u_1p) = 0$ for all $p \in N(u_1)$ inside u_1vw, there is also a monochromatic $K_{1,k}$ with root u_1 and leaves inside u_1xyzu_1 of color 0. Now these two $K_{1,k}$s and u_1xyzu_1 form a monochromatic B_k of color 0.

So $|B_1| < k$ and $|C| < k$. Then there exist p_h, p_s with $h, s \in [3k]$ such that $\sigma(p_hx) = \sigma(p_hy) = \sigma(p_sx) = \sigma(p_sy) = 1$. Because xp_0p_1x bounds an inner face of $\mathrm{Tr}(2k + 30)$, it also bounds the outer face of a $\mathrm{Tr}(4k)$. As $|B_1| < k$, there exists a monochromatic $K_{1,k}$ of color 1 with the root x and k leavers inside xp_0p_1x. Similarly, as $|C| < k$, there exists a monochromatic $K_{1,k}$ of color 1 with root y and k leavers inside yp_0p_1y. Now these two $K_{1,k}$s and the 4-cycle xp_hyp_sx form a monochromatic B_k of color 1. This proves that $\mathrm{Tr}(6k + 30) \to B_k$ and thus completes the proof of Theorem 1.2. ∎

References

[1] I. Algor and N. Alon. The star arboricity of graphs, *Discrete Math.* **75** (1989) 11–22.

[2] K. Appel and W. Haken, Every planar map is four colorable. I. Discharging, *Illinois J. Mathematics* **21** (3) (1977) 429–490.

[3] K. Appel, W. Haken, and J. Koch, Every planar map is four colorable. II. Reducibility, *Illinois J. Mathematics* **21** (3) (1977) 491–567.

[4] M. Axenovich, U. Schade C. Thomassen, and T. Ueckerdt, Planar Ramsey graphs, *Electronic J. Combinatorics* **26** (4) (2019) #P4.9.

[5] D. Goncalves, Edge partition of planar graphs into two outerplanar graphs, in STOC?05: Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pages 504–512, 2005.

[6] L. J. Cowen, R. H. Cowen, and D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, *J. Graph Theory* **10** (1986) 187–195.

[7] S. Hakimi, On the degrees of the vertices of a directed graph, *J. Franklin Institute* **279** (1865) 290–308.

[8] S. Hakimi, J. Mitchem, and E. Schmeichel, Star arboricity of graphs, *Discrete Mathematics* **149** (1996) 93–98.

[9] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, The four-colour theorem, *J. Combin. Theory Ser. B* **70** (1) (1997) 2–44.

[10] K. S. Poh, On the linear vertex arboricity of a planar graph, *J. Graph Theory* **14** (1990) 73–75.